Enumeration of planar maps: an extension of Tutte's formula 60 years later

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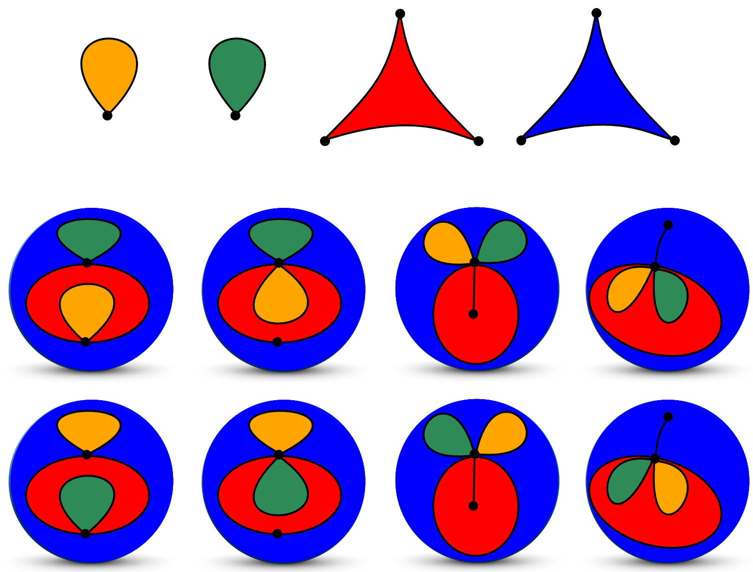
The study of random maps constitutes one of the major parts in the construction of two-dimensional random geometry, in particular in all areas where fluctuating surfaces appear. Let us recall that a map, in the mathematical sense, is a connected graph made of vertices connected by edges and drawn on a two-dimensional surface without edge crossings. These edges then cut the surface into elementary domains, the faces of the map, characterized by their degree, i.e. their number of sides. A face of degree d is therefore a polygon with d sides and a map can be equivalently defined as the gluing of such polygons along their edges to produce a given surface. Maps are considered up to continuous deformations, i.e. only the mutual arrangement of the faces matters and one does not change the map by deforming its constitutive polygons. Given a set of faces numbered from 1 to n whose respective degrees $d_1, d_2, \ldots, d_n$ are prescribed, there exists a finite number $N(d_1, d_2, \cdots, d_n)$ of so-called "planar" maps, i.e. maps such that the obtained surface is a sphere (the map can then be drawn without edge crossings in the plane). The study of random maps therefore starts as of combinatorial problem, consisting in counting them. It is the British-born mathematician William Tutte (also known for his contribution to the deciphering of German secret codes at Bletchley Park during World War II) who, in his attempt to prove the famous 4-color theorem, obtained the first significant results on the combinatorics of maps in the 1960's. In particular, in a famous article published in 1962, "A Census of Slices", he gives a particularly simple explicit formula for the number $N(d_1, d_2, \cdots, d_n)$ in the case where all the d_i are even, except at most two of them, but fails to conclude in the case where more than two faces have odd degrees, case which, he admits, "seems to be more difficult". Since these first works, the combinatorics of maps, but also the study of their metric properties and their continuous limits, have undergone many developments, both thanks to the techniques of matrix integrals developped by physicists and thanks to the discovery of bijective codings of the maps by recursive, and therefore easier to enumerate, objets. Despite all these efforts, giving an explicit formula for $N(d_1, d_2, \cdots, d_n)$ in the case of arbitrary parities remained an open problem. In a very recent article (pre-publication IPhT-t22/009, arXiv:2203.14796 (2022)), two researchers from IPhT, Jérémie Bouttier and Emmanuel Guitter, in collaboration with Grégory Miermont (ENS de Lyon) used a slicing technique along geodesic paths on the map, a method which they had developed in the 2010's, to finally, sixty years after Tutte's original paper, give an explicit expression for $N(d_1, d_2, \cdots, d_n)$ in the case of arbitrary parities of the degrees of the faces. Even though the general formula is, as W. Tutte foresaw, much more complex, it nevertheless remains particularly elegant and conceals many hidden symmetries.



The $N(1, 1, 3, 3) = 8$ ways to assemble two faces of degree 1 and two faces of degree 3 into a sphere.