Correlation functions of Twist-2 operators in $\mathcal{N} = 4$ Super Yang-Mills Theory

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Based on

- V.Kazakov, ES arxiv: 1212.6563
- I.Balitsky, V.Kazakov, ES arxiv: 1306.xxxx

Outline

Part 1 V.Kazakov, ES arxiv: 1212.6563

- Introduction.
- Leading Twist-2 operators.
- Two and three-point correlators of twist-2 operators in leading order g^0. Large-spin asymptotics.
- Three-point correlator of two twist-2 and one Konishi operator in Born approximation.

Part 2 I.Balitsky, V.Kazakov, ES arxiv: 1306.xxxx

- Generalization of twist-2 operators on case of principal series of $sl(2, R)$.
- Sketch of derivation of two-point correlation function in BFKL regime.

Overview of $\mathcal{N} = 4$ SYM.

$\mathcal{N} = 4$ SYM is a maximal supesymmetric extension of Yang-Mills theory with gauge group $SU(N_c)$ in 4D. The group of symmetry is $PSU(2,2|4)$ and it contains a conformal group $SO(4,2)$ as subgroup. There is no conformal anomaly, and, thus, conformal symmetry preserves even on quantum level. All these things make $\mathcal{N} = 4$ SYM interesting by itself, but in 1997 Maldacena made it one of the main heroes of string theory by discovering AdS/CFT duality, which establish correspondence between type 2B superstrings on $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ SYM in 't Hooft limit :

$$N_c \to \infty,\ \lambda = g_{YM}^2 N_c - \text{fixed}$$

Notations

The Lagrangian of $\mathcal{N} = 4$ SYM with the $SU(N_c)$ gauge group has the following form:

$$\mathfrak{L} = \mathrm{Tr}\Big\{ -\frac{1}{2} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu \phi^{AB})(D^\mu \bar{\phi}_{AB}) + \frac{1}{8} g^2 [\phi^{AB}, \phi^{CD}][\bar{\phi}_{AB}, \bar{\phi}_{CD}] +$$
$$+ 2i \bar{\lambda}_{\dot{\alpha}A} \sigma_\mu^{\dot{\alpha}\beta} D^\mu \lambda_\beta^A - \sqrt{2} g \lambda^{\alpha A} [\bar{\phi}_{AB}, \lambda_\alpha^B] + \sqrt{2} g \bar{\lambda}_{\dot{\alpha}A} [\phi^{AB}, \bar{\lambda}_B^{\dot{\alpha}}] \Big\},$$

Scalars form $SU(4)$ multiplet of R-symmetry:

$$[\phi^{AB}] = \begin{pmatrix} 0 & Z & -Y & \bar{X} \\ -Z & 0 & X & \bar{Y} \\ Y & -X & 0 & \bar{Z} \\ -\bar{X} & -\bar{Y} & -\bar{Z} & 0 \end{pmatrix},$$

fermions are realized as a two-component Weyl spinors λ_α^A and covariant derivative is defined as $D_\mu = \partial_\mu - ig[A_\mu, ...]$

Leading Twist-2 operators in QCD.

Leading twist operators are very important objects in YM theory. They naturally appear in QCD for describing deep inelastic scattering.For example cross-section for electron-proton collision reads as follows:

$$\sigma(ep \to eX) = \frac{1}{2s}\int \frac{d^3k'}{(2\pi)^3}\frac{1}{2k'}e^4\frac{1}{2}\sum_{spins}[\bar{u}(k)\gamma_\mu u(k')\bar{u}(k')\gamma_\nu u(k)](\frac{1}{Q^2})^2 2\mathrm{Im}W^{\mu\nu} \quad (1)$$

where Q^2 - momentum transfer, $k,\ k'$ are initial and final momentum of electron and $W^{\mu\nu}$ is forward Compton scattering amplitude:

$$W^{\mu\nu} = i\int d^4x e^{iqx}\langle P|T\{J^\mu(x)J^\nu(0)\}|P\rangle \quad (2)$$

where $|P\rangle$ - proton's wave-function and $J^\mu(x) = \bar{q}(x)\gamma^\mu q(x)$ is quark electromagnetic current.

The main contribution comes from the small x and we can apply OPE:

$$J^\mu(x)J^\nu(0) = \sum_j C_{JJj}(x)O_j(0) \quad (3)$$

Operators which appear in OPE have a form $O_j^{\tilde{\mu}_1\cdots\tilde{\mu}_j} = \Pi_j[\bar{q}\gamma^{\mu_1}D^{\mu_2}...D^{\mu_n}q]$, where Π_j - projection on representation with spin j. Structure constant $C_{JJj}(x)$ also has a tensor structure and for amplitude $W^{\mu\nu}$ we get:

$$W^{\mu\nu} \sim \langle P|O_j(0)|P\rangle \int d^4 x e^{iqx} C_{JJj}(x) \tag{4}$$

In DIS kinematic $\frac{2P\cdot q}{Q^2} \gg 1,\ \frac{1}{Q^2} \ll 1$ the matrix element of operator reads as $\langle P|O_j^{\tilde{\mu}_1\cdots\tilde{\mu}_j}|P\rangle \sim P^{\tilde{\mu}_1}...P^{\tilde{\mu}_j}$ and

$$W^{\mu\nu} \sim \left(\frac{2P\cdot q}{Q^2}\right)^j \left(\frac{1}{Q^2}\right)^{\frac{\Delta-j-2}{2}} \tag{5}$$

So we see that the main contribution comes from the operators with minimal (leading) twist $t = \Delta - j$. Minimal twist corresponds to the maximal spin. For example for quarks with j derivatives , maximal spin projection corresponds to the symmetric and traceless component:

$$O_j^{\mu_1\cdots\mu_j} = \bar{q}\gamma^{\{\mu_1}D^{\mu_2}...D^{\mu_j\}}q - \text{traces} \tag{6}$$

Leading Twist-2 operators in $\mathcal{N} = 4$ SYM.

Based on [Belitsky,Derkachov,Korchemsky,Manashov]

One-trace gauge-invariant operator has the following form:

$$\mathcal{W}^N_{\mu_1\ldots\mu_j} = \mathrm{tr}\{(D_{\mu_1}\ldots D_{\mu_k}X_1(0))(D_{\mu_{k+1}}\ldots D_{\mu_l}X_2(0))\ldots(D_{\mu_{m+1}}\ldots D_{\mu_j}X_N(0))\} \quad (7)$$

Maximal spin component can be obtained by projection on the light-like vector n_+:

$$\mathcal{W}^N_{+\ldots+} = \mathrm{tr}\{(D_+\ldots D_+X_1(0))(D_+\ldots D_+X_2(0))\ldots(D_+\ldots D_+X_N(0))\} \quad (8)$$

Twist of $\mathcal{W}^N_{+\ldots+}$ equal to the sum of twists of components. Then we should choose field components which have a maximal spin. It restricts the fields X_i on the subset $\mathbf{X} = \{F^{+\mu}_{\perp}, \lambda^A_{+\alpha}, \bar{\lambda}^{\dot{\alpha}}_{+A}, \phi^{AB}\}$. . Where gluon field $F^{+\mu}_{\perp}$ is obtained by projection of one of the indices of the field strength tensor $F^{\mu\nu}$ on n^+ direction where as the second index is automatically restricted to the transverse plane. Weyl spinors $\lambda_{+\alpha}$ and $\bar{\lambda}^{\dot{\alpha}}_+$ correspond to the states with definite helicity $1, -1$, respectively and they are parameterized as $\lambda_{+\alpha} = \frac{1}{2}\bar{\sigma}^-_{\alpha\dot{\beta}}\sigma^{+\dot{\beta}\gamma}\lambda_\gamma$ and $\bar{\lambda}^{\dot{\alpha}}_+ = \frac{1}{2}\sigma^{-\dot{\alpha}\beta}\bar{\sigma}^+_{\beta\dot{\gamma}}\bar{\lambda}^{\dot{\gamma}}$.

These operators naturally appear in Taylor expansion of nonlocal light-ray:

$$O_{i_1,\ldots,i_N}(\zeta_1,\ldots,\zeta_N) = \mathrm{tr}X_{i_1}(\zeta_1 n_+)X_{i_2}(\zeta_2 n_+)\ldots X_{i_N}(\zeta_N n_+) \quad (9)$$

Operators $\mathcal{W}^N_{+\dots+}$ have a set of important properties:

- The twist of $\mathcal{W}^N_{+\dots+}$ equal to N
- Operators $\mathcal{W}^N_{+\dots+}$ with fixed N (but different distribution of derivatives) form a closed sector under renormalization

Today we foxus our attention on the case of twist-2 operators.

Since the fields entering $O_{i_1,\dots,i_N}(\zeta_1,\dots,\zeta_N)$ live on the light-ray we can restrict full conformal group $SO(4,2)$ on the collinear conformal subgroup $SO(2,1) \sim SL(2,R)$ with action $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,R)$ on a line $\mathbb{R}$ as $\zeta \to \frac{a\zeta+b}{c\zeta+d}$

$sl(2,R)$ algebra can be constructed from 4-dim generators:

$$J^+ = \frac{i}{\sqrt{2}}P_+, \quad J^- = \frac{i}{\sqrt{2}}K_+, \quad J^3 = \frac{i}{2}(D + M^{\pm}) \tag{10}$$

Primary fields transforms under the $SL(2,R)$ transformations as

$$X(\zeta) \to X'(\zeta) = \frac{1}{(c\zeta+d)^{2j}} X\left(\frac{a\zeta+b}{c\zeta+d}\right) \tag{11}$$

where $j = \frac{d+s}{2}$ - conformal spin. For scalars, fermions and gluons from $\mathbf{X}$ conformal spins has values

$$j_{sc} = \frac{1}{2}, \; j_f = 1 \;\; \text{and} \;\; j_{gl} = \frac{3}{2} \tag{12}$$

correspondingly.

The representation of $sl(2,R)$ with conformal spin j reads as

$$J_+ = \frac{1}{\sqrt{2}}\frac{d}{dx}, \quad J_- = \sqrt{2}(2jx + x^2\frac{d}{dx}), \quad J_3 = j + x\frac{d}{dx}.$$

Now let us introduce conformal operator $O_J^{j_1 j_2}$ with conformal spin J, which is constructed from two fields belonging to the **X** with conformal spins j_1 and j_2. It satisfy two conditions:

$$J_{12}^- \circ O_J^{j_1 j_2}(0) = 0, \quad J_{12}^2 \circ O_J(0) = J(J-1)O_J^{j_1 j_2}(0) \tag{13}$$

where $J_{12}^- = J_1^- + J_2^-$ and $J_{12}^2 = (\vec{J}_1 + \vec{J}_2)^2$ is Casimir. It is easy to demonstrate that $O_J^{j_1 j_2}$ has the following form:

[Makeenko,Ohrndorf]

$$O_J^{j_1,j_2}(x) = X_{j_1}(x) i^J (\overleftarrow{D_+} + \overrightarrow{D_+})^J P_J^{(2j_1-1,2j_2-1)}\left(\frac{\overleftarrow{D_+} - \overrightarrow{D_+}}{\overleftarrow{D_+} + \overrightarrow{D_+}}\right) X_{j_2}(x) \tag{14}$$

where $P_J^{(2j_1-1,2j_2-1)}$ is a Jacobi polynomial. In particular case of equal spins $j_1 = j_2 = j$ Jacobi polynomial turns to be out a Gegenbauer polynomial $C_J^j(x)$:

$$C_J^j(x) \sim P_J^{(2j-1,2j-1)}(x) \tag{15}$$

In $\mathcal{N}=4$ SYM there is a mixing between operators with the same spin, and, thus, they are combined in the components of supermultiplet. For example one of singlet w.r.t $SU(4)$ components is:

$$\mathcal{S}^{1}_{jl} = 6\mathcal{O}^{gg}_{jl} + \frac{j}{4}\mathcal{O}^{qq}_{jl} + \frac{j(j+1)}{4}\mathcal{O}^{ss}_{jl} \tag{16}$$

where operators

$$\mathcal{O}^{gg}_{jl} = \frac{1}{2}\sigma_j \mathrm{tr} i^{l-1}(D_{x_2}+D_{x_1})^{l-1} C^{5/2}_{j-1}\left(\frac{D_{x_2}-D_{x_1}}{D_{x_2}+D_{x_1}}\right) F^{+\mu}_{\perp}(x_1) g^{\perp}_{\mu\nu} F^{\nu+}_{\perp}(x_2)|_{x_1=x_2} \tag{17}$$

$$\mathcal{O}^{qq}_{jl} = \sigma_j \mathrm{tr} i^{l}(D_{x_2}+D_{x_1})^{l} C^{3/2}_{j}\left(\frac{D_{x_2}-D_{x_1}}{D_{x_2}+D_{x_1}}\right) \bar{\lambda}_{+\dot{\alpha}A}\sigma^{+\dot{\alpha}\beta}(x_1)\lambda^{A}_{+\beta(x_2)}|_{x_1=x_2} \tag{18}$$

$$\mathcal{O}^{ss}_{jl} = \frac{1}{2}\sigma_j \mathrm{tr} i^{l+1}(D_{x_2}+D_{x_1})^{l+1} C^{1/2}_{j+1}\left(\frac{D_{x_2}-D_{x_1}}{D_{x_2}+D_{x_1}}\right) \bar{\phi}_{AB}(x_1)\phi^{AB}(x_2)|_{x_1=x_2} \tag{19}$$

Correlation functions.

As it was noticed before, $\mathcal{N} = 4$ SYM is conformal theory. Conformal symmetry imposes quite strong restrictions on correlation functions. For example, one can show that any 2-point correlation function (of scalars) has the following form:

$$\langle \mathcal{O}_I(x)\mathcal{O}_J(y)\rangle = \frac{\delta_{IJ}}{|x-y|^{2\Delta_I}} \tag{20}$$

where Δ_I is a dimension of operator $\mathcal{O}_I(x)$. For three corelators we have:

$$\langle \mathcal{O}_I(x)\mathcal{O}_J(y)\mathcal{O}_K(z)\rangle = \frac{C_{IJK}}{|x-y|^{\Delta_I+\Delta_J-\Delta_K}|y-z|^{\Delta_J+\Delta_K-\Delta_I}|z-x|^{\Delta_I+\Delta_K-\Delta_J}} \tag{21}$$

The same structure constant C_{IJK} appears in operator product expansion (OPE):

$$\mathcal{O}_I(x)\mathcal{O}_J(y) = \sum_K \frac{C_{IJK}}{|x-y|^{\Delta_I+\Delta_J-\Delta_K}}\mathcal{O}_K(x) \tag{22}$$

which means that any n-point correlation functions can be reduced to the two and three-point case.

2-point correlator

Situation is more reach for operators with spin . For example the two point correlation function of conformal primary operators $O^{j_1,\cdots,j_k}$ has the following tensor structure:

$$\langle O^{\mu_1,\cdots,\mu_k}(x)\bar{O}^{\nu_1,\cdots,\nu_k}(y)\rangle = C_{O\bar{O}}\frac{I^{\mu_1\nu_1}\ldots I^{\mu_k\nu_k}}{|x-y|^{2\Delta}}, \tag{23}$$

$$I^{\mu\nu} = g^{\mu\nu} - \frac{2(x-y)^\mu(x-y)^\nu}{|x-y|^2}. \tag{24}$$

Due to the fact that all indexes are contracted with the light-like vector n_+ in our case, we get the following formula for the operator with k $+$indices:

$$\langle O_{+,\ldots,+}(x)\bar{O}_{+,\ldots,+}(y)\rangle = C_{O\bar{O}}\frac{(-2(x-y)_+^2)^k}{|x-y|^{2(\Delta+k)}}. \tag{25}$$

For singlet operators $\mathcal{S}^1_{jl}$ we get in tree-level:

$$\langle \mathcal{S}^1_{j_1}(x)\mathcal{S}^1_{j_2}(y)\rangle = \delta_{j_1j_2}\sigma^2_{j_1}(N_c^2-1)H(j_1)\frac{(x-y)_+^{2j_1+2}}{((x-y)^2)^{2j_1+4}},$$

$$H(j_1) = \mathcal{N}^2 j_1(j_1+1)(96+115j_1+35j_1^2)2^{2j_1-4}(2j_1+2)!.$$

3-point correlation function of Twist-2 operators.

In g^0 order calculation we should contract only the fields of the same type, and the calculation of correlator $\langle \mathcal{S}^1_{j_1j_1}(x)\mathcal{S}^1_{j_2j_2}(y)\mathcal{S}^1_{j_3j_3}(z)\rangle$ reduces to the calculation of the sum of three independent correlators:

$$\langle \mathcal{S}^1_{j_1}(x)\mathcal{S}^1_{j_2}(y)\mathcal{S}^1_{j_3}(z)\rangle = 6^3\langle \mathcal{O}^{gg}_{j_1}(x)\mathcal{O}^{gg}_{j_2}(y)\mathcal{O}^{gg}_{j_3}(z)\rangle + \frac{j_1j_2j_3}{4^3}\langle \mathcal{O}^{qq}_{j_1}(x)\mathcal{O}^{qq}_{j_2}(y)\mathcal{O}^{qq}_{j_3}(z)\rangle + \\ + \frac{j_1(j_1+1)j_2(j_2+1)j_3(j_3+1)}{4^3}\langle \mathcal{O}^{ss}_{j_1}(x)\mathcal{O}^{ss}_{j_2}(y)\mathcal{O}^{ss}_{j_3}(z)\rangle, \quad (26)$$

Formally calculation is rather straightforward but answer looks very large. For example for scalars it reeds as:

$$\langle \mathcal{O}^{ss}_{j_1}(x)\mathcal{O}^{ss}_{j_2}(y)\mathcal{O}^{ss}_{j_3}(z)\rangle = \mathcal{N}_{j_1j_2j_3}2^7 3\sum_{k_1=0}^{j_1+1}\sum_{k_2=0}^{j_2+1}\sum_{k_3=0}^{j_3+1}\eta_s(k_1,k_2,k_3)\theta(k_1,k_2,k_3), \quad (27)$$

where

$$\begin{aligned}\eta_s(k_1,k_2,k_3) \quad &= \quad \binom{j_1+1}{k_1}^2\binom{j_2+1}{k_2}^2\binom{j_3+1}{k_3}^2 \times \\ &\times \quad (j_1+1-k_1+k_2)!(j_2+1-k_2+k_3)!(j_3+1-k_3+k_1)!\end{aligned}$$

and

$$\theta(k_1,k_2,k_3) = \frac{(x-y)_+^{j_1+1-k_1+k_2}}{(|x-y|^2)^{j_1+2-k_1+k_2}}\frac{(y-z)_+^{j_2+1-k_2+k_3}}{(|y-z|^2)^{j_2+2-k_2+k_3}}\frac{(z-x)_+^{j_3+1-k_3+k_1}}{(|z-x|^2)^{j_3+2-k_3+k_1}}. \quad (28)$$

Reason for that is a nonzero spin of operators. In general case , 3-point correlator of operators with spins $j_1,\ j_2,\ j_3$ and dimensions $\Delta_1,\ \Delta_2,\ \Delta_3$ is a sum over different tensor structures:

[Costa, Penedones, Poland, Rychkov]

$$\langle \mathcal{O}_{\Delta_1,j_1}\mathcal{O}_{\Delta_2,j_2}\mathcal{O}_{\Delta_3,j_3}\rangle = \sum_{n_{12},n_{13},n_{23}\geq 0} \lambda_{n_{12},n_{13},n_{23}} \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ j_1 & j_2 & j_3 \\ n_{23} & n_{13} & n_{12} \end{bmatrix}, \tag{29}$$

The coefficients $\lambda_{n_{12},n_{13},n_{23}}$ are labeled by the set $\{n_{12}, n_{13}, n_{23}\}$ of integers satisfying the following inequalities $l_1 - n_{12} - n_{13} \geq 0,\ l_2 - n_{12} - n_{23} \geq 0,\ l_3 - n_{13} - n_{23} \geq 0$
Turned out that in the case when the coordinates are restricted to the 2-dimensional plane $\{n_+, n_-\}$ all tensor structures is collapsed in one:

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ j_1 & j_2 & j_3 \\ n_{23} & n_{13} & n_{12} \end{bmatrix} \sim \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ j_1 & j_2 & j_3 \\ n'_{23} & n'_{13} & n'_{12} \end{bmatrix} \tag{30}$$

and we can introduce only one structure constant. In case of twist-2 operators it looks as follows:

$$\langle O_{j_1}(x_1) O_{j_2}(x_2) O_{j_3}(x_3)\rangle = \frac{B_{j_1 j_2 j_3}}{x_{12+}x_{13+}x_{23+}x_{12-}^{j_1+j_2-j_3+2}\, x_{13-}^{j_1+j_3-j_2+2}\, x_{23-}^{j_2+j_3-j_1+2}}. \tag{31}$$

Also we introduce normalized structure constant, which roughly reads as :

$$C_{j_1 j_2 j_3} = \frac{\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle}{\sqrt{\langle \mathcal{O}\mathcal{O} \rangle \langle \mathcal{O}\mathcal{O} \rangle \langle \mathcal{O}\mathcal{O} \rangle}} \tag{32}$$

Setting $x_{1+} = -1, \; x_{2+} = 0, \; x_{3+} = 1$ we can get $B_{j_1 j_2 j_3}$ as a triple sum. There are few situations when this sum can be simplified. For example in case when three spins are large $j_1 \sim j_2 \sim j_3 \gg 1$. In this case we can make continuous limit and represent the sum as an integral:

$$B_{j_1, j_2, j_3} = \sum \sum \sum \simeq \int \int \int dk_1 dk_2 dk_3 f(j_1, j_2, j_3, k_1, k_2, k_3) e^S, \tag{33}$$

where

$$\begin{gathered} S = S(j_1, j_2, j_3, k_1, k_2, k_3) = (-k_1 + k_3)\ln(-2) - 2k_1 \ln k_1 - \\ -2(j_1 - k_1)\ln(j_1 - k_1) - 2k_2 \ln k_2 - 2(j_2 - k_2)\ln(j_2 - k_2) - 2k_3 \ln k_3 - \\ -2(j_3 - k_3)\ln(j_3 - k_3) + (j_1 - k_1 + k_2)\ln(j_1 - k_1 + k_2) + \\ +(j_2 - k_2 + k_3)\ln(j_2 - k_2 + k_3) + (j_3 - k_3 + k_1)\ln(j_3 - k_3 + k_1). \end{gathered} \tag{34}$$

The 3 saddle-point equations $\frac{\partial S}{\partial k_j}$ =0, k=1,2,3, read as follows

$$\begin{cases} -\dfrac{(j_1-k_1)^2(j_3-k_3+k_1)}{2k_1^2(j_1-k_1+k_2)}=1, \\ \dfrac{(j_2-k_2)^2(j_1-k_1+k_2)}{k_2^2(j_2-k_2+k_3)}=1, \\ -\dfrac{2(j_3-k_3)^2(j_2-k_2+k_3)}{k_3^2(j_3-k_3+k_1)}=1. \end{cases} \tag{35}$$

The solution rendering the main contribution can be obtained explicitly

$$\begin{cases} k_1=\dfrac{j_1(j_3-j_1)}{j_1+j_2+j_3}, \\ k_2=\dfrac{j_2(2j_1+j_2)}{2(j_1+j_2+j_3)}, \\ k_3=\dfrac{j_3(j_2+2j_3)}{j_1+j_2+j_3}. \end{cases} \tag{36}$$

Finally structure constant for large spins reads as follows

$$C_{\natural j_1 j_2 j_3} \simeq \sigma_{j_1}\sigma_{j_2}\sigma_{j_3}\frac{1}{\sqrt{N_c^2-1}}\frac{i^{j_1+j_2+j_3+3}}{\pi^{\frac{1}{2}}5^{\frac{3}{2}}7^{\frac{1}{2}}2(j_1j_2j_3)^{\frac{3}{2}}}\frac{(j_1+j_2+j_3+4)!}{\sqrt{(2j_1)!(2j_2)!(2j_3)!}}(1+O(j_k^{-1})). \tag{37}$$

The similar analysis can be done for the case when only two or one spin is large.

Comparing with strong coupling

Y.Kazama and S.Komatsu had calculated 3-point correlator of twist -2 operators in case of strong coupling. In our case of large spin j dimension has asymptotic

$$\Delta = j + f(g) log(j) + O(j^0) \tag{38}$$

And two expressions of $\ln C_{j_1 j_2 j_3}$ in weak (left) and strong (rigft) cases are different:

$$(j_1 + j_2 + j_3) \ln(j_1 + j_2 + j_3) - \sum_{k=1}^{3} j_k \ln(j_k) - (j_1 + j_2 + j_3) \ln 2 \quad \neq \quad -(j_1 + j_2 + j_3) \ln 2 \tag{39}$$

Twist-2 Twist-2 Konishi

Now we are going to calculate the three-point correlation function of one Konishi operator

$$\mathcal{K} = \mathrm{tr}[\bar{X}, \bar{Z}]^2 = 2\mathrm{tr}(\bar{X}\bar{Z}\bar{X}\bar{Z}) - 2\mathrm{tr}(\bar{X}^2\bar{Z}^2) \tag{40}$$

and two scalar twist-2 operators of spins $j_1 + 1$ and $j_2 + 1$:

$$O^X_{j_1}(\alpha) = 6\sigma_{j_1}\mathrm{tr}[i^{j_1+1}(D_{\alpha_2} + D_{\alpha_1})^{j_1+1} C^{1/2}_{j_1+1}(\frac{D_{\alpha_2} - D_{\alpha_1}}{D_{\alpha_2} + D_{\alpha_1}})X(\alpha_1)X(\alpha_2)]|_{\alpha=\alpha_1=\alpha_2},$$
$$O^Z_{j_2}(\beta) = 6\sigma_{j_2}\mathrm{tr}[i^{j_2+1}(D_{\beta_2} + D_{\beta_1})^{j_2+1} C^{1/2}_{j_2+1}(\frac{D_{\beta_2} - D_{\beta_1}}{D_{\beta_2} + D_{\beta_1}})Z(\beta_1)Z(\beta_2)]|_{\beta=\beta_1=\beta_2}. \tag{41}$$

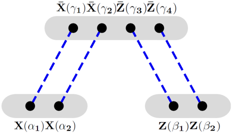
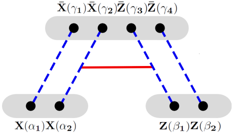
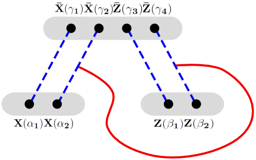


Figure: Leading diagram.

$$\lim_{\alpha_{1,2}\to\alpha}(\partial_{\alpha_1} + \partial_{\alpha_2})^n C^{\frac{1}{2}}_n\left(\frac{\partial_{\alpha_1} - \partial_{\alpha_2}}{\partial_{\alpha_1} + \partial_{\alpha_2}}\right)\frac{1}{|\alpha_2 - \gamma_3|^2|\alpha_1 - \gamma_4|^2} = 0. \tag{42}$$

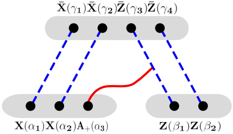


(a)

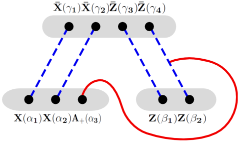


(b)

(a)+(b)=0



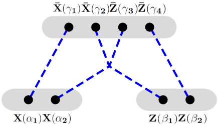
(c)



(d)

$$(c) + (d) = 0 \quad (44)$$

Nonzero contribution comes only from the 4-scalar vertex:



$$K_{j_1 j_2}(\alpha,\beta,\gamma) \equiv \langle O^X_{j_1}(\alpha) O^Z_{j_2}(\beta)\ \mathrm{tr}[X,Z]^2(\gamma)\rangle =$$

$$= -\sigma_{j_1}\sigma_{j_2} 3^3 2^6 \pi^2 \mathcal{N}^6 g^2 (N_c^4 - N_c^2) \mathcal{G}^{\frac{1}{2}}_{j_1+1,\alpha_1,\alpha_2} \mathcal{G}^{\frac{1}{2}}_{j_2+1,\beta_1,\beta_2} \Psi|_{\substack{\alpha_{1,2}=\alpha,\\ \beta_{1,2}=\beta}}, \tag{45}$$

where $\mathcal{G}^{\frac{1}{2}}_{j_1+1,\alpha_1,\alpha_2} = i^{j_1+1}(\partial_{\alpha_2}+\partial_{\alpha_1})^{j_1+1} C^{1/2}_{j_1+1}(\frac{\partial_{\alpha_2}-\partial_{\alpha_1}}{\partial_{\alpha_2}+\partial_{\alpha_1}})$ and Ψ is defined as

$$\Psi = \frac{\log \frac{|\alpha_2-\gamma|^2|\beta_2-\gamma|^2}{|\alpha_2-\beta_2|^2|\epsilon|^2}}{|\alpha_1-\gamma|^2|\alpha_2-\gamma|^2|\beta_1-\gamma|^2|\beta_2-\gamma|^2} + \left(\substack{\alpha_1\leftrightarrow\alpha_2,\\ \beta_1\leftrightarrow\beta_2}\right) \tag{46}$$

If we restrict the coordinates of the points to the 2-d plane (n_+, n_-) all tensor structures factorize again into a single one:

$$C_{j_1 j_2} = g^2 \sigma_{j_1}\sigma_{j_2} 3^{\frac{1}{2}} 2^{-4} \pi^{-2} \frac{N_c}{\sqrt{N_c^2-1}} i^{j_1+j_2} \frac{\Gamma(j_1+j_2+2)}{(\Gamma(2j_1+3)\Gamma(2j_2+3))^{\frac{1}{2}}}. \tag{47}$$

Open questions of Part 1

- 3-point function of operators with arbitrary Twist in tree level, using integrability. Work in progress.
- 1-loop diagrams for 3-point correlators of arbitrary twist and interpretation them in terms of insertation of Hamiltonian like $SU(2)$ case. Work in progress in collaboration with P. Laskos-Grabowski

2-point correlation of Twist-2 operators in BFKL regime

Now I would back to the 2-point correlator. In High-Energy Physics there is a very deep topic with long history - BFKL approach. Roughly speaking BFKL is about summation of diagrams proportional to $(g^2)^k(g^2 \ln s)^m$ in two-hadron scattering, where $g^2 \to 0$, $\ln s \to \infty$ and $g^2 \ln s = \text{fix}$. LO BFKL is a sum of leading terms $\sum_m (g^2 \ln s)^m$, NLO BFKL is a sum of subleading terms $\sum_m g^2(g^2 \ln s)^m$ etc. Turned out that all dynamics is reduced to the two-dimensional orthogonal subspace and amplitude satisfy an equation a la Bethe-Salpeter equation which is a BFKL equation.

All this stuff was generalized to the case of N=4 SYM. Moreover using intuition from DGLAP Lipatov had calculated the anomalous dimension of twist-2 operator $\text{tr} Z D_+^{-1+\omega} Z$ in the limit $\omega \to 0$, $\frac{g^2}{\omega} = \text{fix}$. In LO equation on $\gamma_{-1+\omega}$ reads as follows:

$$-\frac{\omega}{4g^2} = \psi\left(\frac{\gamma_{-1+\omega}}{2}\right) + \psi\left(1 + \frac{\gamma_{-1+\omega}}{2}\right) - 2\psi(1), \quad (48)$$

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) \quad (49)$$

or solving

$$\gamma_{-1+\omega} = 2\left(\frac{-4g^2}{\omega}\right) - 0\left(\frac{-4g^2}{\omega}\right)^2 + 0\left(\frac{-4g^2}{\omega}\right)^3 - 4\zeta(3)\left(\frac{-4g^2}{\omega}\right)^4 + \ldots \quad (50)$$

So $\frac{1}{\omega}$ plays here the role of $\log s$

This result coincides with calculation from Bethe ansatz and Luscher formula up to 5 loops. But still there is the main question

What is $\mathrm{tr} Z D_+^{-1+\omega} Z$?

Namely how one can make analitycal continuation of $\sum_{k=0}^{j} c_k \mathrm{tr} D^k Z D^{j-k} Z$ on j? Also important to stress that analytical continuation of anomalous dimension from Bethe ansatz implies principle of maximal trancendality which hasn't proof thus far.
With a view to clarify these questions let us revisit derivation of Twist-2 operators. Operators with integer spin can be expressed through the Gegenbauer polynomials and they realize a discreet representation of $sl(2, R)$. But the last one has also principal series representation with label (J, ϵ) , where $\epsilon \in \{0, 1\}$ is parity and $J = \frac{1}{2} + i\nu, \quad \nu \in \mathbb{R}$ is conformal spin.
Let us introduce a general light-ray operator:

$$S(x_1, x_2) = \int \int dx_1 dx_2 \phi(x_1, x_2) \Phi(x_1) [x_1, x_2]_{Adj} \Phi(x_2) \tag{51}$$

Diagonalization of Casimir reads as follows:

$$\vec{J}^2 S(x_1, x_2) = J(J-1) S(x_1, x_2) \tag{52}$$

The last equation can be rewritten as a partial differential equation for function $\phi(x_1, x_2)$:

$$\left[\beta^2\left(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2}\right) - 2s\beta\frac{\partial}{\partial\beta} + s(s+1)\right]\phi(\alpha,\beta) = J(J-1)\phi(\alpha,\beta) \tag{53}$$

where $\alpha = x_1 + x_2,\ \beta = x_1 - x_2$. Factorising $\phi(\alpha,\beta) = f(\alpha)g(\beta)$ we get:

$$\begin{cases} \frac{\partial^2}{\partial\alpha^2}f(\alpha) = -k^2 f(\alpha), \\ \left(\beta^2\frac{\partial}{\partial\beta^2} - 2s\beta\frac{\partial}{\partial\beta} + s(s+1) + k^2\beta^2\right)g(\beta) = J(J-1)g(\beta) \end{cases} \tag{54}$$

The general solution reads as follows:

$$f(x_1, x_2) = \int dk\eta(k)e^{ik(x_1+x_2)}(x_1 - x_2)^{2s-\frac{3}{2}}(C_1 J_{-\frac{1}{2}+J}(k(x_1-x_2)) + C_2 J_{\frac{1}{2}-J}(k(x_1-x_2))) \tag{55}$$

We should choose solution which satisfy two conditions:

- Analiticity in J
- Coincidence of anomalous dimension with local operators in case of integer J

They unequally leads to the Hankel function of second order:

$$C_1 J_{-\frac{1}{2}+J}(k(x_1-x_2)) + C_2 J_{\frac{1}{2}-J}(k(x_1-x_2)) \to H^2_{J-\frac{1}{2}}(k(x_1-x_2)). \tag{56}$$

Now we can choose $\eta(k) = \delta(k)(\frac{k}{2})^{J-\frac{1}{2}}$ and using the asymptotic of Hankel function at $k \to 0$:

$$H^2_{J-\frac{1}{2}}(k(x_1 - x_2)) \to -(\frac{k(x_1 - x_2)}{2})^{-J+\frac{1}{2}} \frac{\Gamma(J - \frac{1}{2})}{\pi} \tag{57}$$

we get the form of light-ray operators for scalars, fermions and gluons:

$$S^J_{sc} = \int\int dx_1 dx_2 (x_1 - x_2)^{-J} \mathrm{tr}\bar{\phi}_{AB}(x_1)[x_1, x_2]_{Adj}\phi^{AB}(x_2), \tag{58}$$

$$S^J_f = i\int\int dx_1 dx_2 (x_1 - x_2)^{-J+1} \text{fermions}, \tag{59}$$

$$S^J_{gl} = \int\int dx_1 dx_2 (x_1 - x_2)^{-J+2} \text{gluons} \tag{60}$$

with respect to supersymmetry we can construct generalisation of $\mathcal{S}^1_j$:

$$\mathcal{S}^J_{nloc} = -(J-1)(J-2)S^J_{sc} + 2(J-2)S^J_f + 2S^J_{gl} \tag{61}$$

Such defined operator is very singular object in BFKL regime and we should regularise one . Namely we define our nonlocal operators as a limit of Wilson rectangle loop with two fields on the opposite corners, when small side belong to the transverse plane and goes to zero. For example for scalars we get:

$$S_{sc}^{J} = \lim_{(x_{1\perp}-x_{2\perp})^2\to 0} \int\int dx_{1-} dx_{2-} (x_{1-}-x_{2-})^{-J} \mathrm{tr}\bar{\phi}_{AB}(x_1)[x_1, x_2]\phi^{AB}(x_2), \tag{62}$$

$$x_1 = (0, x_{1-}, x_{1\perp}), \quad x_2 = (0, x_{2-}, x_{2\perp}) \tag{63}$$

where x_1, x_2 is a 4-dimensional coordinates with components $(x_+, x_-, x_\perp)$, and operation "lim" is understood in generalized sense. The last means that we should firstly carry out all calculations with rectangle and only in the end take the limit.

BFKL regime corresponds to the $J = 2 + \omega$, $\omega \to 0$. For sake of brevity I will talk about $S_{gl}^{2+\omega}$ Moreover we can express the Wilson rectangle with $F_{\mu+}$ in corners as differentiation of pure Wilson rectangle. So then we should calculate correlator of two such rectangles:

$$W_{\omega_1}^{+}(x_{1\perp}, x_{2\perp}) = \int_0^\infty dL_1 L_1^{-2-\omega_1} \int_{-\infty}^{\infty} dx^+ \text{Pexp}\left[(x^+, 0, x_{1\perp}), (x^+ + L_1, 0, x_{2\perp})\right],$$

$$W_{\omega_2}^{-}(y_{1\perp}, y_{2\perp}) = \int_0^\infty dL_2 L_2^{-2-\omega_2} \int_{-\infty}^{\infty} dy^- \text{Pexp}\left[(0, y^-, y_{1\perp}), (0, y^- + L_2, y_{2\perp})\right] \tag{64}$$

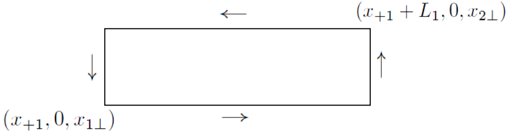


Figure: Wilson frame $[(x_{1+}, 0, x_{1\perp}), (x_+ + L_1, 0, x_{2\perp})]$

As was established by Balitsky, objects which is stretched along n_+ and n_- direction posses a decomposition over so called "colour dipole" - pair of infinite Wilson lines:

$$\mathcal{U}^{\sigma U}(x_{1\perp}, x_{2\perp}) = 1 - \frac{1}{N_c^2 - 1} Tr[U^{\sigma U}_{x_{1\perp}} U^{\sigma U \dagger}_{x_{2\perp}}] \quad (65)$$

where

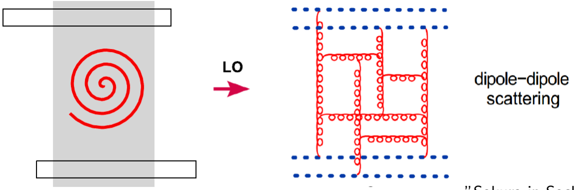
$$U^{\sigma}_{x} = \mathrm{Pexp}[-ig \int_{-\infty}^{\infty} dx^+ A^{\sigma -}(x_+, 0, x_\perp)], \quad (66)$$

$$A^{\sigma}_{\mu}(x) = \int d^4k \theta(\sigma - |\alpha_k|) e^{ikx} A_\mu(k) \quad (67)$$

Rectangles for example can be simply replaced by such color dipole with some cut-off σ:

$$\mathrm{Pexp}\left[(x^+, 0, x_{1\perp}), (x^+ + L_1, 0, x_{2\perp})\right] \to \mathcal{U}^{\sigma U}(x_{1\perp}, x_{2\perp}) \quad (68)$$

Symbolically it looks like



Evolution of dipoles gives by BFKL equation (in LO):

$$\sigma \frac{d}{d\sigma} \mathcal{U}(z_1, z_2) = \mathcal{K}_{\text{BFKL}} * \mathcal{U}(z_1, z_2) \tag{69}$$

where the BFKL kernel is defined by

$$\mathcal{K}_{\text{BFKL}} * \mathcal{U}(z_1, z_2) = 4g^2 \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[\mathcal{U}(z_1, z_3) + \mathcal{U}(z_3, z_2) - \mathcal{U}(z_1, z_2)\right] \tag{70}$$

The BFKL equation has $SL(2, C)$ symmetry and thus solution can be decomposed over irrep of $SL(2, C)$. They are numerated by $h = \frac{1+n}{2} + i\nu$, $\bar{h} = \frac{1-n}{2} + i\nu$ and 2-dim coordinate z_0: $E_{\nu,n}(z_{10}, z_{20}) = \left[\frac{z_{12}}{z_{10} z_{20}}\right]^{\frac{1+n}{2}+i\nu} \left[\frac{\bar{z}_{12}}{\bar{z}_{10}\bar{z}_{20}}\right]^{\frac{1-n}{2}+i\nu}$ So the solution of BFKL is easily formulated in terms of $\mathcal{U}^{\sigma}(\nu, z_0)$ - projection of the dipole on the $E_{\nu,n}(z_{10}, z_{20})$ (keep only $n = 0$ component):

$$\mathcal{U}^{\sigma}(\nu, z_0) = (\frac{\sigma}{\sigma_0})^{\aleph(\nu)} \mathcal{U}^{\sigma_0}(\nu, z_0), \tag{71}$$

$$\mathcal{U}^{\sigma}(\nu, z_0) = \frac{1}{\pi^2} \int \frac{d^2 z_1 d^2 z_2}{z_{12}^4} \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^{\frac{1}{2} - i\nu} \mathcal{U}^{\sigma}(z_1, z_2) \tag{72}$$

$$\aleph(h) = 4g^2 (2\psi(1) - \psi(h) - \psi(1-h)) \tag{73}$$

Inverse formula for $\mathcal{U}^\sigma(z_1, z_2)$ reads as:

$$\mathcal{U}^\sigma(z_1, z_2) = \int d\nu \int d^2 z_0 \frac{\nu^2}{\pi} \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^{\frac{1}{2}+i\nu} \mathcal{U}^\sigma(\nu, z_0) \qquad (74)$$

Now we can calculate correlation function of two dipoles $\mathcal{U}^{\sigma_U}(\nu, z_0)$ and $\mathcal{V}^{\sigma_V}(\nu', z_0')$ for small $\sigma_{U0},\ \sigma_{V0}$ (in LO)

$$\langle \mathcal{U}^{\sigma_{U0}}(\nu, z_0) \mathcal{V}^{\sigma_{V0}}(\nu', z_0') \rangle \sim \frac{16\pi^2}{\nu^2(1+4\nu^2)^2} \left[\delta(z_0 - z_0')\delta(\nu + \nu') + \right.$$
$$\left. + \frac{2^{1-4i\nu}\delta(\nu - \nu')}{\pi |z_0 - z_0'|^{2-4i\nu}} \frac{\Gamma(\frac{1}{2}+i\nu)\Gamma(1-i\nu)}{\Gamma(i\nu)\Gamma(\frac{1}{2}-i\nu)} \right] \qquad (75)$$

and evolve from the initial $\sigma_{U0},\ \sigma_{V0}$ to the $\sigma_U,\ \sigma_V$ using BFKL evolution.

After evolution it looks as:

$$\langle \mathcal{U}^{\sigma_U}(\nu, z_0)\mathcal{V}^{\sigma_V}(\nu', z_0')\rangle \sim$$
$$\sim \frac{16\pi^2}{\nu^2(1+4\nu^2)^2}(\sigma_U\sigma_V)^{\aleph(\nu)}\left[\delta(z_0-z_0')\delta(\nu+\nu') + \frac{2^{1-4i\nu}\delta(\nu-\nu')}{\pi|z_0-z_0'|^{2-4i\nu}}\frac{\Gamma(\frac{1}{2}+i\nu)\Gamma(1-i\nu)}{\Gamma(i\nu)\Gamma(\frac{1}{2}-i\nu)}\right] \tag{76}$$

Now we should fix $(\sigma_U\sigma_V)^{\aleph(\nu)}$. It can be done using a formula similar to Cornalba's expression for 4-point correlation function. In our limit it looks as

$$A \sim \int d\nu \left(\left(\frac{(x_1-y_2)^2(x_2-y_1)^2}{x_{12\perp}^2 y_{12\perp}^2}\right)^{\frac{\aleph(\nu)}{2}} - \left(\frac{(x_1-y_1)^2(x_2-y_2)^2}{x_{12\perp}^2 y_{12\perp}^2}\right)^{\frac{\aleph(\nu)}{2}}\right) F(\nu, x_1, x_2, y_1, y_2) \tag{77}$$

and we can uniquely identify $(\sigma_U\sigma_V)^{\aleph(\nu)}$ as the red brackets

Now we can back from $\langle \mathcal{U}^{\sigma_U}(\nu, z_0)\mathcal{V}^{\sigma_V}(\nu', z_0')\rangle$ to the $\langle \mathcal{U}^{\sigma_U}(x_{1\perp}, x_{2\perp})\mathcal{V}^{\sigma_V}(y_{1\perp}, y_{2\perp})\rangle$:

$$\langle \mathcal{U}^{\sigma_U}(x_{1\perp}, x_{2\perp})\mathcal{V}^{\sigma_V}(y_{1\perp}, y_{2\perp})\rangle = \int d\nu_1 \int d^2 z_0 \frac{\nu_1^2}{\pi}\left(\frac{x_{12}^2}{(x_1 - z_0)^2(x_2 - z_0)^2}\right)^{\frac{1}{2}+i\nu_1} \times$$

$$\times \int d\nu_2 \int d^2 z_0' \frac{\nu_2^2}{\pi}\left(\frac{y_{12}^2}{(y_1 - z_0')^2(y_2 - z_0')^2}\right)^{\frac{1}{2}+i\nu_2} \langle \mathcal{U}^{\sigma_U}(\nu, z_0)\mathcal{V}^{\sigma_V}(\nu', z_0')\rangle \qquad (78)$$

This expression equal to the correlator of two Wilson rectangles with fixed positions and sizes. Now we should integrate over all configurations:

$$\langle W^+_{\omega_1}(x_{1\perp}, x_{2\perp}) W^-_{\omega_2}(y_{1\perp}, y_{2\perp})\rangle =$$

$$\int_0^\infty dL_1 L_1^{-2-\omega_1} \int_{-L_1}^0 dx^+ \int_0^\infty dL_2 L_2^{-2-\omega_2} \int_{-L_2}^0 dy^- \langle \mathcal{U}^{\sigma_U}(x_{1\perp}, x_{2\perp})\mathcal{V}^{\sigma_V}(y_{1\perp}, y_{2\perp})\rangle \sim \qquad (79)$$

$$\sim (x_{12\perp}^2)^{1+\gamma}(y_{12\perp}^2)^{1+\gamma}\frac{1}{((x_\perp - y_\perp)^2)^{2+\gamma+\omega}} \qquad (80)$$

$$\text{where} \;\; \omega = \aleph(\gamma) = 4g^2(2\psi(1) - \psi(-\frac{\gamma}{2}) - \psi(1 + \frac{\gamma}{2})) \qquad (81)$$

We made the similar calculation in NLO BFKL.

Open questions

- Three-point correlator of $F_{\mu+}D^{-1+\omega}F_{+}{}^{\mu}$ using definition through the Wilson rectangles
- Strong coupling regime, connection with GKP etc

Thank you!