Near-BPS Cusp Anomalous Dimension at Any Coupling

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Outline

- Introduction and the setup.
- Calculation of the cusp anomalous dimension.
- Matrix model reformulation and the classical limit.
- Conclusions and further directions.

Introduction

Exact calculations in supersymmetric gauge theories

- Non-perturbative methods in $\mathcal{N}=4$ SYM have been developing rapidly
- In particular, two efficient approaches are known:

Localization

- BPS, non-planar
- Example: $\langle W_{circle}\rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$

  [Erickson, Semenoff, Zarembo'00], [Drukker, Gross'00], [Pestun'12]

Integrability

- Planar, Non-BPS
- Example: tr $\left[Z D^S Z\right] \to \frac{I_1(\sqrt{\lambda})}{I_2(\sqrt{\lambda})}$

  [Basso'11]

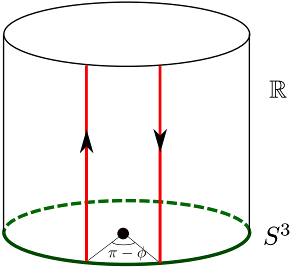
$$?$$

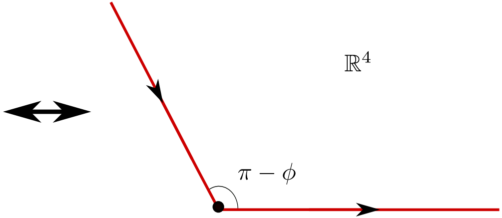
Is there any non-trivial observable accessible from both approaches?

Quark-Antiquark Potential/Cusped Wilson Line.

$$W = \mathrm{Tr}\,\mathcal{P}\exp\int dt\left[iA\cdot\dot{x}_q + \vec{\Phi}\cdot\vec{n}\,|\dot{x}_q|\right]$$

Two Wilson line configurations related by a conformal map



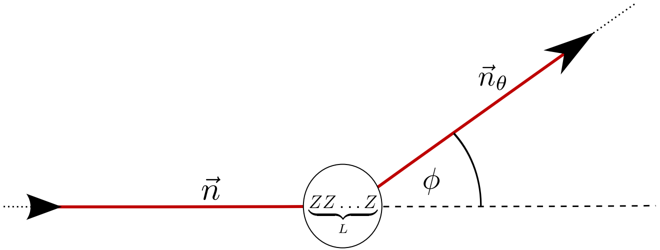


$$\langle W\rangle = e^{-TV}$$

$$\langle W\rangle \sim \left(\frac{\Lambda_{IR}}{\Lambda_{UV}}\right)^{\Gamma_{\text{cusp}}}$$

Conformal invariance $\Rightarrow V = \Gamma_{\text{cusp}}$

The Cusped Wilson Line: Turning on more parameters



- Cusp angle ϕ
- Angle θ between the couplings to scalars on two rays
- R-charge L of a local operator inserted at the cusp
- 't Hooft coupling λ

For $\theta^2 - \phi^2 = 0$ this observable is protected. We will be working in the near-BPS limit $\phi \approx \theta$.

Relation to other physical quantities

Cusp anomalous dimension is related to a variety of physical quantities, as

- IR divergences of scattering amplitudes, $i\phi$ is a boost angle for massive particles and $i\phi \to \infty$ for massless.
- Bremsstrahlung function — radiation of a moving particle $(\phi \to 0)$
- The quark-antiquark potential in the flat space $(\phi \to \pi)$

Known results

- For $L = 0$ the Γ_{cusp} is known from localization

  [Correa at al.'12],[Fiol, Garolera, Lewkowycz'12]

$$\Gamma_{\text{cusp}}(\lambda) = -\frac{1}{4\pi^2}(\phi^2 - \theta^2)\frac{\sqrt{\lambda} I_2\left(\sqrt{\lambda}\sqrt{1-\frac{\theta^2}{\pi^2}}\right)}{I_1\left(\sqrt{\lambda}\sqrt{1-\frac{\theta^2}{\pi^2}}\right)}$$

- arbitrary L, $\theta = 0$, $\phi \ll 1$ — solved in [Gromov, Sever'12] using integrability.

  Γ_{cusp} is expressed through determinants made of $I_n\left(\sqrt{\lambda}\right)$.
- We will get the result for finite $\theta \approx \phi$, arbitrary L and λ from integrability.

Calculation of the cusp anomalous dimension.

Thermodynamical Bethe Ansatz

The standard method to attack the problem from integrability point of view is TBA

[Bombardelli, Fioravanti, Tateo'09], [Gromov et al'09], [Arutyunov, Frolov'09], [Gromov, Kazakov, Vieira'09], [Correa et al'12],

[Gromov, Sever'12].

$$\log \frac{Y_{1,1}}{\mathbf{Y}_{1,1}} = K_{m-1} * \log \frac{1+\overline{Y}_{1,m}}{1+\overline{\mathbf{Y}}_{1,m}} \frac{1+\mathbf{Y}_{m,1}}{1+Y_{m,1}} + \mathcal{R}^{(01)}_{1a} * \log(1+Y_{a,0})$$

$$\log \frac{\overline{Y}_{2,2}}{\overline{\mathbf{Y}}_{2,2}} = K_{m-1} * \log \frac{1+\overline{Y}_{1,m}}{1+\overline{\mathbf{Y}}_{1,m}} \frac{1+\mathbf{Y}_{m,1}}{1+Y_{m,1}} + \mathcal{B}^{(01)}_{1a} * \log(1+Y_{a,0})$$

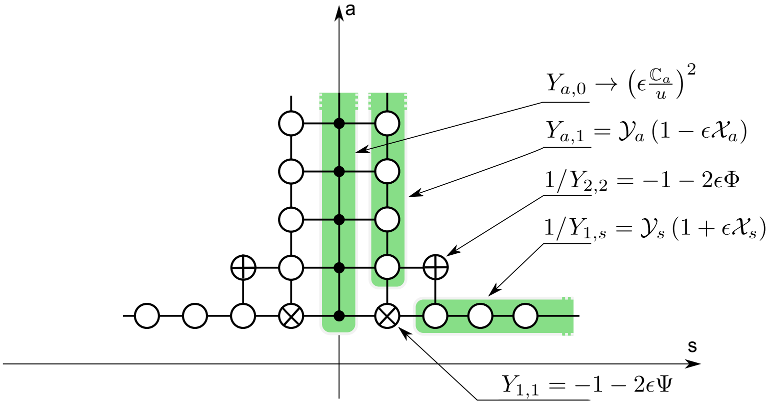
$$\log \frac{\overline{Y}_{1,s}}{\overline{\mathbf{Y}}_{1,s}} = -K_{s-1,t-1} * \log \frac{1+\overline{Y}_{1,t}}{1+\overline{\mathbf{Y}}_{1,t}} - K_{s-1} \hat{*} \log \frac{1+Y_{1,1}}{1+\overline{Y}_{2,2}}$$

$$\log \frac{Y_{a,1}}{\mathbf{Y}_{a,1}} = -K_{a-1,b-1} * \log \frac{1+Y_{b,1}}{1+\mathbf{Y}_{b,1}} - K_{a-1} \hat{*} \log \frac{1+Y_{1,1}}{1+\overline{Y}_{2,2}} + \left[\mathcal{R}^{(01)}_{ab} + \mathcal{B}^{(01)}_{a-2,b}\right] * \log(1+Y_{b,0})$$

- Infinite system of nlin integral equations for $Y_{a,s}(u)$
- The indices of (a, s) of Y-functions live on a T-shaped hook.
- The energy can be expressed through $Y_{a,0}$

The Y-functions and their near-BPS expansion

In near-BSP limit we expand Y-functions in $\epsilon = (\phi - \theta)\tan\frac{\phi+\theta}{2}$



Simplified TBA

A system for the coefficients of expansion $\Phi, \Psi, \mathcal{Y}_a, \mathcal{X}_a, \mathbb{C}_a$

$$\Phi - \Psi = \pi \mathbb{C}_a \hat{K}_a(u),$$

$$\Phi + \Psi = \mathbf{s} * \left[-2\frac{\mathcal{X}_2}{1+\mathcal{Y}_2} + \pi(\hat{K}_a^+ - \hat{K}_a^-)\mathbb{C}_a - \pi\delta(u)\mathbb{C}_1 \right],$$

$$\log Y_{1,m} = \mathbf{s} * I_{m,n} \log\left(1 + Y_{1,n}\right) - \delta_{m,2}\mathbf{s}\hat{*}\left(\log\frac{\Phi}{\Psi} + \epsilon\left(\Phi - \Psi\right)\right) - \epsilon\pi\mathbf{s}\mathbb{C}$$

$$\Delta_a = [\mathcal{R}_{ab}^{(10)} + \mathcal{B}_{a,b-2}^{(10)}]\hat{*}\log\frac{1+\mathcal{Y}_b}{1+A_b} + \mathcal{R}_{a1}^{(10)}\hat{*}\log\left(\frac{\Psi}{1/2}\right) -$$

$$-\mathcal{B}_{a1}^{(10)}\hat{*}\log\left(\frac{\Phi}{1/2}\right),$$

$$\mathbb{C}_a = (-1)^{a+1} a \frac{\sin a\theta}{\tan\theta}\left(\sqrt{1+\frac{a^2}{16g^2}} - \frac{a}{4g}\right)^{2+2L} F(a,g)e^{\Delta_a},$$

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- From localization we know the result at $L = 0$ is extremely simple $\Gamma_L(g) \sim \tilde{\lambda}^{-1/2} I_2\left(\sqrt{\tilde{\lambda}}\right)/I_1\left(\sqrt{\tilde{\lambda}}\right)$

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- Hope for a drastic simplification?

Step1: From TBA to FiNLIE

Thermodynamical Bethe Ansatz

∞ system of nlin integral eqs

$$
\begin{aligned}
\log\frac{Y_{1,1}}{\mathbf{Y}_{1,1}} &= K_{m-1}*\log\frac{1+\overline{Y}_{1,m}}{1+\overline{\mathbf{Y}}_{1,m}}\frac{1+\mathbf{Y}_{m,1}}{1+Y_{m,1}} + \mathcal{R}^{(01)}_{1a}*\log(1+Y_{a,0})\\
\log\frac{\overline{Y}_{2,2}}{\overline{\mathbf{Y}}_{2,2}} &= K_{m-1}*\log\frac{1+\overline{Y}_{1,m}}{1+\overline{\mathbf{Y}}_{1,m}}\frac{1+\mathbf{Y}_{m,1}}{1+Y_{m,1}} + \mathcal{B}^{(01)}_{1a}*\log(1+Y_{a,0})\\
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\log\frac{Y_{a,1}}{\mathbf{Y}_{a,1}} &= -K_{a-1,b-1}*\log\frac{1+Y_{b,1}}{1+\mathbf{Y}_{b,1}} - K_{a-1}\hat{*}\log\frac{1+Y_{1,1}}{1+\overline{Y}_{2,2}}\\
&\quad + \left[\mathcal{R}^{(01)}_{ab}+\mathcal{B}^{(01)}_{a-2,b}\right]*\log(1+Y_{b,0})
\end{aligned}
$$

$\Downarrow$

Using the relation between Y-system and integrable Hirota dynamics
[Gromov, Kazakov, Vieira'09]

Finite system of nlin integral equations (FiNLIE)

Ansatz for Y-functions through T-functions

$$Y_{1,m} = \frac{T^+_{1,m} T^-_{1,m}}{T_{1,m+1} T_{1,m-1}} - 1.$$

The general solution for T is given by

$$T_{1,s} = C \begin{vmatrix} Q_1^{[s]} & \bar{Q}_1^{[-s]} \\ Q_2^{[s]} & \bar{Q}_2^{[-s]} \end{vmatrix} .$$

The non-trivial part is finding $Q_{1,2}$

The "twisted" ansatz

Our ansatz for $Q_{1,2}$ is

$$Q_1 = \bar{Q}_1 = e^{+\theta(u - iG(u))},$$
$$Q_2 = \bar{Q}_2 = e^{-\theta(u - iG(u))},$$

The resolvent G has a short cut and a series of poles

$$G(u) = \frac{1}{2\pi i} \int_{-2g}^{2g} dv \frac{\rho(v)}{u - v} + \epsilon \sum_{a \neq 0} \frac{b_a}{u - ia/2} .$$

This generates for T-functions

$$T_s = \frac{\sin\left(s - G^{[s]} + G^{[-s]}\right)\theta}{\sin\theta}.$$

Everything is expressed in terms of $\rho(u), \eta(u), \mathbb{C}_a$

$$\eta \frac{\sin\theta\rho}{\sin\theta} = -\sum_a \pi \mathbb{C}_a \hat{K}_a,$$

$$\eta \frac{\cos\theta\rho\cos(2 - G^+ + G^-)\theta - \cos(2\not{G} - G^+ - G^-)\theta}{\sin\theta\sin(2 - G^+ + G^-)\theta} =$$

$$= \mathbf{s} * \left[-2\frac{\mathcal{X}_2}{1+\mathcal{Y}_2} + \pi(\hat{K}_a^+ - \hat{K}_a^-)\mathbb{C}_a - \pi\delta(u)\mathbb{C}_1 \right],$$

$$\mathbb{C}_a = (-1)^a a \mathcal{T}_a(0) \left(\sqrt{1 + \frac{a^2}{16g^2}} - \frac{a}{4g} \right)^{2+2L} \exp\left[\tilde{K}_a \hat{*} \log\left(\eta \frac{\sinh 2\pi u}{2\pi u} \right) \right]$$

Step 2: Analytical ansatz for FiNLIE quantities

The way to solve FiNLIE is to make certain assumptions about its analytical properties

Assumptions:

- $\eta(u)^2$ is meromorphic in the whole complex plane
- η has simple poles at $ia/2$

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Then the goal is to express the FiNLIE quantities in terms of zeros of η. Introduce a bookkeeping function

$$\mathbf{Q}_{\pm}(x) = \prod_{k\neq 0} \frac{x_{k,\pm} - x}{x_{k,\pm}}, \ \tilde{\mathbf{Q}}_{\pm}(x) = \mathbf{Q}_{\pm}(1/x)$$

where we use Zhoukovsky transform of u: $u/g = x + 1/x$.

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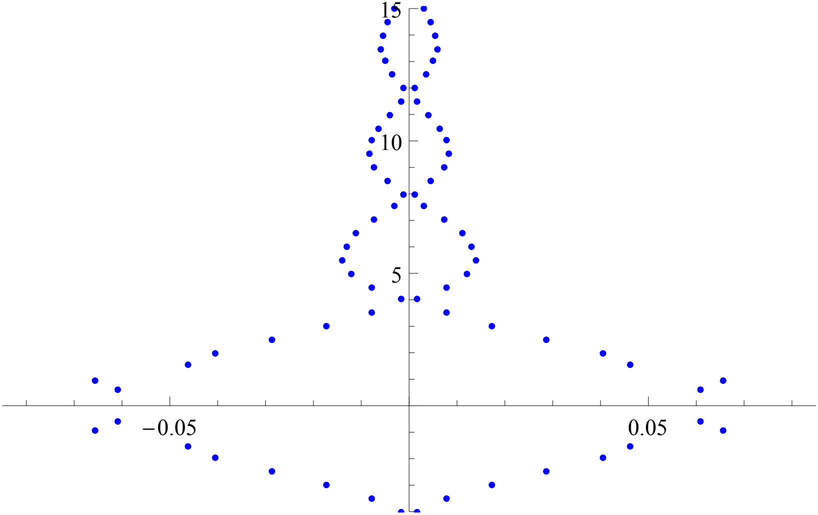
Consequences:

$$e^{i\theta\rho} = \sqrt{\frac{\mathbf{Q}_+ \tilde{\mathbf{Q}}_-}{\mathbf{Q}_- \tilde{\mathbf{Q}}_+}}, \quad \eta = \cos\theta \frac{\sqrt{\mathbf{Q}_+ \mathbf{Q}_- \tilde{\mathbf{Q}}_+ \tilde{\mathbf{Q}}_-}}{\tilde{C} \frac{\sinh 2\pi u}{2\pi u}} .$$

The zeros



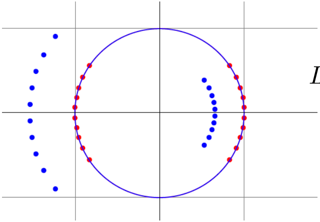
Effective Baxter equation

FiNLIE+Analyticity assumptions

$$\Downarrow$$

The zeros satisfy effective "crossing" Bethe equations, which can be solved by introducing a Baxter polynomial.

Roots of the Baxter polynomial

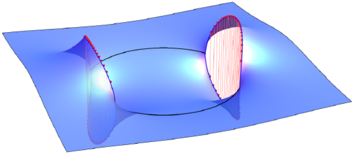


$$L \sim \sqrt{\lambda} \gg 1$$

$$\Longrightarrow$$

Classical limit

The algebraic curve



The result

Cusp anomalous dimension for arbitrary L, finite $\theta \approx \phi$ and any value of 't Hooft coupling

$$\Gamma_L(\lambda) = \frac{\phi - \theta}{4} \partial_\theta \log \frac{\det \mathcal{M}_{2L+1}}{\det \mathcal{M}_{2L-1}}$$

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$$\mathcal{M}_N = \begin{pmatrix} I_1^\theta & I_0^\theta & \cdots & I_{2-N}^\theta & I_{1-N}^\theta \\ I_2^\theta & I_1^\theta & \cdots & I_{3-N}^\theta & I_{2-N}^\theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_N^\theta & I_{N-1}^\theta & \cdots & I_1^\theta & I_0^\theta \\ I_{N+1}^\theta & I_N^\theta & \cdots & I_2^\theta & I_1^\theta \end{pmatrix}$$

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$$I_n^\theta = \tfrac{1}{2} I_n \left(\sqrt{\lambda} \sqrt{1 - \tfrac{\theta^2}{\pi^2}} \right) \left[\left(\sqrt{\tfrac{\pi+\theta}{\pi-\theta}} \right)^n - (-1)^n \left(\sqrt{\tfrac{\pi-\theta}{\pi+\theta}} \right)^n \right].$$

Tests: $L = 0$

$$\Gamma_L(\lambda) = \frac{\phi - \theta}{4} \partial_\theta \log \frac{\det \mathcal{M}_{2L+1}}{\det \mathcal{M}_{2L-1}}$$

$$\Big\downarrow \; L = 0$$

$$\Gamma_0(\lambda) = -\frac{1}{2\pi}(\phi - \theta)\theta \frac{\sqrt{\lambda}}{\sqrt{\pi^2 - \theta^2}} \frac{I_2\left(\sqrt{\tilde{\lambda}}\right)}{I_1\left(\sqrt{\tilde{\lambda}}\right)}$$

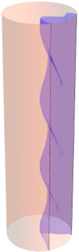
The localization result is reproduced!

Tests: Strong coupling

$$\Gamma_L(\lambda) = \frac{\phi - \theta}{4}\partial_\theta \log \frac{\det \mathcal{M}_{2L+1}}{\det \mathcal{M}_{2L-1}}$$

The limit $L \sim \sqrt{\lambda} \to \infty$ matches perfectly with the energy of a classical open string

$$\begin{aligned}
\frac{\Gamma_L}{2(\phi-\theta)\theta} = &\left(-\frac{g}{\pi} + \frac{3L}{4\pi^2} - \frac{9L^2}{64g\pi^3} - \frac{5L^3}{256g^2\pi^4} + \frac{45L^4}{16384g^3\pi^5}\right) \\
&+ \theta^2\left(-\frac{g}{2\pi^3} + \frac{3L}{4\pi^4} - \frac{21L^2}{128g\pi^5} - \frac{L^3}{16g^2\pi^6} - \frac{105L^4}{32768g^3\pi^7}\right) \\
&+ \theta^4\left(-\frac{3g}{8\pi^5} + \frac{3L}{4\pi^6} - \frac{99L^2}{512g\pi^7} - \frac{3L^3}{32g^2\pi^8} - \frac{2085L^4}{131072g^3\pi^9}\right) \\
&+ \theta^6\left(-\frac{5g}{16\pi^7} + \frac{3L}{4\pi^8} - \frac{225L^2}{1024g\pi^9} - \frac{L^3}{8g^2\pi^{10}} - \frac{7905L^4}{262144g^3\pi^{11}}\right) \\
&+ \theta^8\left(-\frac{35g}{128\pi^9} + \frac{3L}{4\pi^{10}} - \frac{1995L^2}{8192g\pi^{11}} - \frac{5L^3}{32g^2\pi^{12}} - \frac{97425L^4}{2097152g^3\pi^{13}}\right)
\end{aligned}$$



$$g = \frac{\sqrt{\lambda}}{4\pi}$$

Matrix model reformulation and the classical limit.

Classical Limit

- Γ_{cusp} is related to the energy of a classical open string in the limit $L \sim g \to \infty$

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- How to take the $L \to \infty$ limit of $\det \mathcal{M}_{2L+1}$?

Classical Limit

- Γ_{cusp} is related to the energy of a classical open string in the limit $L \sim g \to \infty$
- How to take the $L \to \infty$ limit of $\det \mathcal{M}_{2L+1}$?
- The technique of expansion in 1 over the size of the matrix is well developed in matrix models

Matrix Model reformulation

Using

$$I_n^\theta = \frac{1}{2\pi i}\oint \frac{dx}{x^{n+1}} \sinh(2\pi g\,(x+1/x))\, e^{2g\theta(x-1/x)}$$

for every element of

$$\mathcal{M}_N = \begin{pmatrix} I_1^\theta & I_0^\theta & \cdots & I_{2-N}^\theta & I_{1-N}^\theta \\ I_2^\theta & I_1^\theta & \cdots & I_{3-N}^\theta & I_{2-N}^\theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_N^\theta & I_{N-1}^\theta & \cdots & I_1^\theta & I_0^\theta \\ I_{N+1}^\theta & I_N^\theta & \cdots & I_2^\theta & I_1^\theta \end{pmatrix}$$

we obtain

$$\det \mathcal{M}_N = \oint \prod_{i=1}^{N+1} \frac{dx_i}{2\pi i x_i^{N+2}} \frac{\Delta^2(x_i)}{(N+1)!} \sinh\left[2\pi g\left(x_i + \frac{1}{x_i}\right)\right] e^{2g\theta\left(x_i - \frac{1}{x_i}\right)}$$

Saddle-point approximation

- In the quasi-classical approximation the value of the integral

$$\int dx_i e^{-S[x_i]}$$

is given by

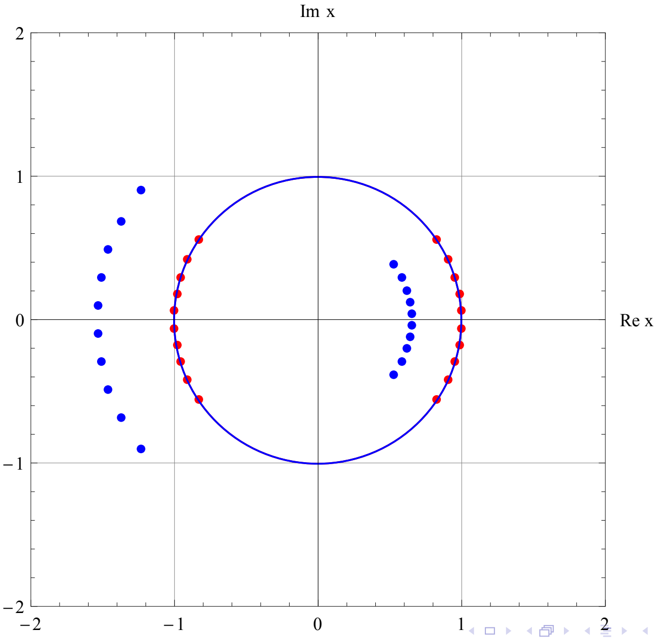
$$\left(S''[x_i^*]\right)^{-1/2} e^{-S[x_i^*]},$$

where x^* is a solution of a saddle-point equation $\frac{\partial S}{\partial x_i} = 0$.

- In the case of $L \sim g \to \infty$ limit of $\det \mathcal{M}_{2L+1}$ the saddle-point equation is

$$-\theta \frac{x_j^2+1}{x_j^2-1} + \frac{L}{g} \frac{x_j}{x_j^2-1} - \frac{1}{g} \frac{x_j^2}{x_j^2-1} \sum_{i \neq j}^{2L+1} \frac{1}{x_j - x_i} = \pi \, \mathrm{sgn}(\mathrm{Re}(x_j)).$$

The distribution of the roots at $L \gg 1$



The classical quasimomentum

Introduce the quantum quasimomentum $p(x)$

$$p(x) = -\theta\,\frac{x^2+1}{x^2-1} + \frac{L}{g}\,\frac{x}{x^2-1} - \frac{2L}{g}\,\frac{x^2}{x^2-1}\,G_L^{cl}(x),$$

where

$$G_L^{cl}(x) = \frac{1}{2L}\sum_{k=1}^{2L+1}\frac{1}{x-x_k}.$$

The saddle-point equation then is

$$\frac{1}{2}\left(p(x_i+i\epsilon)+p(x_i-i\epsilon)\right) = \pi\,\mathrm{sgn}(\mathrm{Re}(x_i)).$$

As $L \to \infty$, the roots aggregate into two cuts and $p(x)$ becomes a classical algebraic curve with two cuts.

The classical algebraic curve

In the classical limit $p(x)$ becomes the classical algebraic curve.

[V.A.Kazakov, A.Marshakov, J.A.Minahan, K.Zarembo, hep-th/0402207]

Properties:

- $p(x) = -p(-1/x)$
- $p(0) = -p(\infty) = \theta$
- Two cuts with branch-points parametrized by $\{-re^{i\phi}, -re^{-i\phi}, 1/re^{i\phi}, 1/re^{-i\phi}\}$
- $p(x_{bp}) = \pm\pi$
- Simple poles at $x = \pm 1$

Start with an ansatz for p':

$$p'(x) = \frac{A_1 x^4 + A_2 x^3 + A_3 x^2 + A_4 x + A_5}{(x^2-1)^2 \sqrt{x + re^{i\phi}}\sqrt{x + re^{-i\phi}}\sqrt{x - \frac{1}{r}e^{i\phi}}\sqrt{x - \frac{1}{r}e^{-i\phi}}}.$$

To get $p(x)$ we integrate and fix A_i and the integration constant using the properties above.

The classical algebraic curve

$$p(x) = \pi - 4\,i\,\mathbb{E}(a^2\sin^2(\phi))\,\mathbb{F}_1 + 4\,i\,\mathbb{K}(a^2\sin^2(\phi))\,\mathbb{F}_2$$
$$- a\left(\frac{x+\frac{1}{r}e^{-i\phi}}{x+re^{i\phi}}\right)\left(\frac{2\,r\,e^{i\phi}}{x^2-1}\right)y(x)\,\mathbb{K}(a^2\sin^2(\phi)),$$

[Valatka&Sizov, to appear]

where

$$\mathbb{F}_1 = \mathbb{F}\left(\sin^{-1}\sqrt{a\left(\frac{x-\frac{1}{r}e^{-i\phi}}{x+re^{i\phi}}\right)\left(\frac{2\,r\,e^{2i\phi}}{e^{2i\phi}-1}\right)}\,\middle|\,a^2\sin^2(\phi)\right),$$

$$\mathbb{F}_2 = \mathbb{E}\left(\sin^{-1}\sqrt{a\left(\frac{x-\frac{1}{r}e^{-i\phi}}{x+re^{i\phi}}\right)\left(\frac{2\,r\,e^{2i\phi}}{e^{2i\phi}-1}\right)}\,\middle|\,a^2\sin^2(\phi)\right),$$

and

$$a = \frac{2\,r}{r^2+1}.$$

The parameters of the cuts

The parameters of the curve r, ϕ are related to $L/g, \theta$ by

$$\begin{aligned}
\frac{L}{g} &= \frac{4}{a}\left(\mathbb{K}(a^2\sin^2(\phi)) - \mathbb{E}(a^2\sin^2(\phi))\right) \\
\theta &= -\pi + \frac{4\,r^2 e^{i\phi}\,\mathbb{K}\left(a^2\sin^2(\phi)\right)}{r^2+1} \\
&- 4i\mathbb{K}\left(a^2\sin^2(\phi)\right)\mathbb{E}\left(\sin^{-1}\left(\sqrt{\frac{e^{2i\phi}r}{-1+e^{2i\phi}}}\sqrt{r+\frac{1}{r}}\right)\middle|\, a^2\sin^2(\phi)\right) \\
&+ 4i\mathbb{E}\left(a^2\sin^2(\phi)\right)\mathbb{F}\left(\sin^{-1}\left(\sqrt{\frac{e^{2i\phi}r}{-1+e^{2i\phi}}}\sqrt{r+\frac{1}{r}}\right)\middle|\, a^2\sin^2(\phi)\right)
\end{aligned}$$

The classical energy

The energy can be expressed as an expectation value in the matrix model

$$\partial_\theta \log \det \mathcal{M}_L = \left\langle 2g \sum_{i=1}^{2L} (x_i - 1/x_i) \right\rangle$$

In the saddle-point approximation due to the symmetry $x \to -1/x$ only one term matters. We can express it through $G(0) \propto \sum_i \frac{1}{x_i}$,

so $\Gamma_{cusp} = -(\phi - \theta)\frac{g^2}{2}\partial_L p_L''(0)$. Using the explicit formula for p we get

$$\frac{\Gamma_{cusp}}{\phi_{cusp} - \theta_{cusp}} = g\left(r - \frac{1}{r}\right)\cos\phi$$

Notice: all the elliptic functions in $p(x)$ got cancelled out when expressed through r and ϕ.
The same result we get considering the conserved charge of the corresponding classical string solution.

Expansion around the classical solution

Expansion in $L, g \to \infty$ with L/g fixed

$$\Gamma_L(g) = (\phi - \theta) \sum_{k=0} g^{-k} b_k \left(L/g\right)$$

- We checked that the leading terms are reproduced by our solution
- The symmetry $\Gamma_L(g) = -\Gamma_{-L-1}(-g)$ of the large L expansion [Beccaria&Macorini 1305.4839] implies that $b_1 = \frac{g}{2}\partial_L b_0$. Thus we found

$$b_1/b_0 = \frac{g}{4} \frac{\left|r^2 e^{2i\phi} + 1\right|^2 K_1 - r^2 \left|r + \frac{1}{r} + e^{i\phi} - e^{-i\phi}\right|^2 E_1,}{\left|\left(r + \frac{1}{r}\right)\left(r^2 e^{2i\phi} - 1\right) E_1 - \left(r - \frac{1}{r}\right)\left(r^2 e^{2i\phi} + 1\right) K_1\right|^2}$$

where $E_1 = \mathbb{E}\left(\frac{4r^2 \sin^2\phi}{(r^2+1)^2}\right),\ K_1 = \mathbb{K}\left(\frac{4r^2 \sin^2\phi}{(r^2+1)^2}\right)$.

- Next corrections can be generated by topological recursion — work in progress by I.Kostov,N.Gromov,S.Valatka,G.S.

Conclusions

Results

- We calculated the cusp anomalous dimension in the near-BSP limit at any coupling
- Using matrix model reformulation we have found the corresponding algebraic curve and the classical limit of the cusp anomalous dimension

Remarks

- The result in a general near-BPS case $\phi \approx \theta$ is simpler than the degenerate $\theta = 0$ case
- Analyticity assumption gives a key to solving TBA
- This analyticity can be derived from the novel $P - \mu$ system[Gromov et al 1305.1935].
- The strong coupling expansion has a form of a matrix model expectation value
- The form of the result hints it can be derived from localization.

Further directions

- The curve is related to the eigenvalues of monodromy of a flat connection around the worldsheet, though the exact procedure for an open string is not yet established
- Knowing the classical algebraic curve allows to calculate the corrections to the classical energy
- Topological recursion can be implemented for these purposes
- The relation to $P-\mu$ system allows to calculate further corrections in $\phi-\theta$, which may help to see a structure of the full result.