Classical conformal blocks and Painlevé VI

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Introduction: Conformal blocks

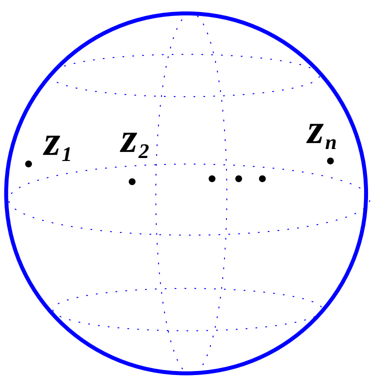
Classical conformal blocks (**Al Zamolodchikov; 1986**) are certain limits of Virasoro conformal blocks

$$\begin{matrix}\text{Virasoro} \\ \text{conformal blocks}\end{matrix} \xrightarrow{c\to\infty,\ \Delta_i\to\infty} \exp\left[\frac{c}{6}\times\left(\begin{matrix}\text{classical} \\ \text{conformal block}\end{matrix}\right)\right]$$

Virasoro conformal blocks are generally associated with the moduli space of Riemann surfaces $\Sigma_{g,n}$ with n punctures

$$\mathcal{F} \quad : \quad \underbrace{\mathcal{M}_{g,n}}_{\text{moduli space}} \to \mathbb{C}$$

I will limit attention to the case of n-punctured Riemann sphere $\Sigma_{0,n}$



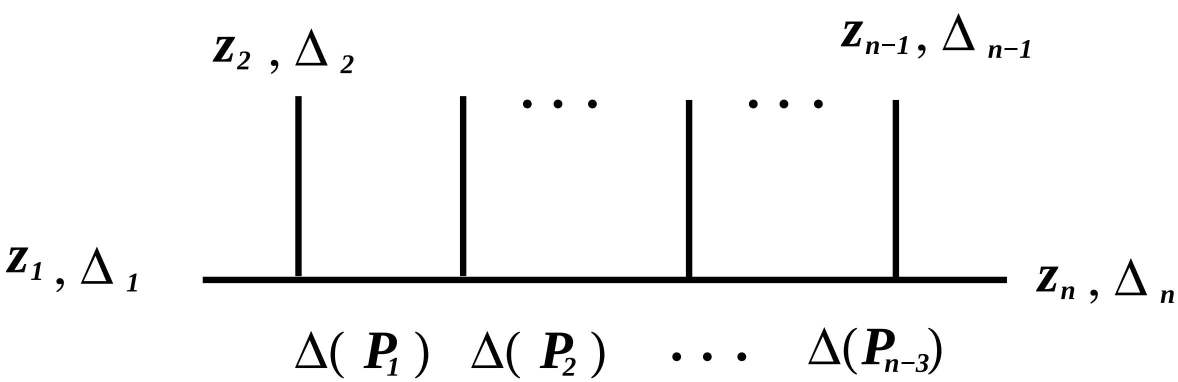
$\{z_i\}$ – locations of the punctures in some coordinate z

The n-point conformal block is, roughly speaking, the n-point correlation function of

$$V_\Delta(z) \; - \; \text{“chiral vertex operators”},$$

the holomorphic primary operators. Since generally $V_\Delta(z)$ are **not** local fields, the notion of the correlation function is ambiguous. To make it precise, one needs to specify a

“Dual diagram” (= **“pant decomposition”** of $\Sigma_{0,n}$)



$$= \mathcal{F}_{P_1\cdots P_{n-3}}(\{z_i\})$$

The dual diagram represents the instruction to include only the states from irreducible representations with conformal weights $\Delta(P_\alpha)$ in the intermediate-state decompositions.

$$\mathcal{F}_{P_1\ldots P_{n-3}}(\{z_i\}) = \langle\, V_{\Delta_1}(z_1)\, V_{\Delta_2}(z_2) \overset{\boxed{\Pi_{\Delta(P_1)}}}{\downarrow} V_{\Delta_3}(z_3) \overset{\boxed{\Pi_{\Delta(P_2)}}}{\downarrow} \quad \ldots \quad \overset{\boxed{\Pi_{\Delta(P_{n-3})}}}{\downarrow} V_{\Delta_{n-1}}(z_{n-1})\, V_{\Delta_n}(z_n)\,\rangle$$

Here I use the Liouville-like parameterization of the intermediate weights

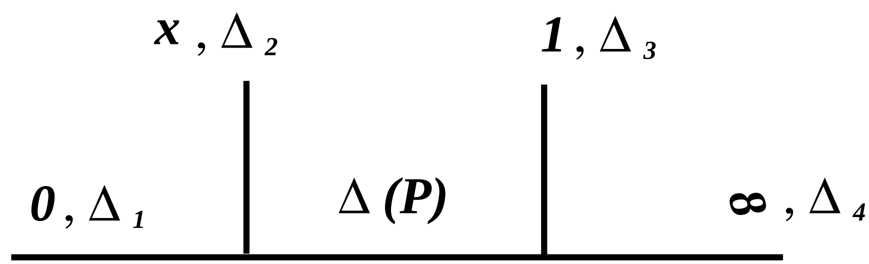
$$\Delta(P_\alpha) = \frac{c-1}{24} + P_\alpha^2$$

Therefore, the n-point conformal block depends on $n-3$ parameters P_α. It also depends on z_i. The later dependence is essentially up to the projective transformations

$$\{z_i\}' = \{z_i\}/ \text{ projective } \ \mathbb{SL}(2,\mathbb{C})$$

(i.e. $\mathcal{F}$ essentially depend on $n-3$ "cross ratios".) The conformal block also depends on n "external" conformal dimensions $\{\Delta_1, \Delta_2, \ldots, \Delta_n\}$ which are omitted in the above notations.

The simplest nontrivial case is $n = 4$



$$= \mathcal{F}_P(x)$$

In the diagram, I have assumed that a projective transformation has been made such that

$$\{z_1,\, z_2,\, z_3,\, z_4\} \to \{0,\, x,\, 1,\, \infty\}$$

so that $x \in \mathbb{C}$ is the coordinate on the moduli space of $\Sigma_{0,4}$.

The Virasoro conformal blocks are fundamental objects in 2D CFT, as correlations of local fields are built from them via holomorphic factorization.

For example in **Liouville CFT**

$$\langle\, \mathrm{e}^{2a_1\phi}(0)\, \mathrm{e}^{2a_2\phi}(x,\bar{x})\, \mathrm{e}^{2a_3\phi}(1)\, \mathrm{e}^{2a_4\phi}(\infty)\, \rangle_{\text{Liouville}} =$$

$$\int_{-\infty}^{\infty} \mathrm{d}P\; \mathbb{C}\big(a_1, a_2, \tfrac{Q}{2} + \mathrm{i}\,P\big)\, \mathbb{C}\big(a_3, a_4, \tfrac{Q}{2} - \mathrm{i}\,P\big)\; |\mathcal{F}_P(x)|^2$$

where a_i relate to Δ_i as

$$\Delta_i = a_i(Q - a_i)$$

and Q is related to the Liouville coupling b as

$$Q = b + b^{-1}, \qquad c_{\text{Liouville}} = 1 + 6\,Q^2$$

and $\mathbb{C}(a, a', a'')$ are known functions - the “Liouville structure constants”.

Lately, the Virasoro conformal blocks attract renewed interest due to the **"AGT correspondence"**:

$$\mathcal{F}_{\{P_\alpha\}}(\{z_i\}') \ \text{ emerge } \ \text{in} \ \ \mathcal{N}=2 \ \ \text{SUSY} \ \ \text{gauge theories}\,.$$

(**Alday, Gaiotto, Tachikawa, 2010**)

Both Liouville correlation functions, and the conformal blocks emerge, depending on the context. In particular, $\mathcal{F}_{\{P_\alpha\}}(\{z_i\}')$ coincide with the instanton part

$$\mathcal{F}_{\{P_\alpha\}}(\{z_i\}') = Z_{\rm inst}$$

of the Nekrasov's partition function of (quiver) SUSY gauge theories in the **"Ω-background"**

$$Z_{\rm Nekrasov} = Z_{\rm classical}\ Z_{1-{\rm loop}}\ Z_{\rm inst}$$

The parameters z_i relate to the gauge coupling constants, and P_α - to the vacuum moduli.

Although the conformal blocks are completely fixed by their conformal properties, in general case no closed form is known. (In special cases they satisfy certain differential equations, and admit closed form representations.)

Using conformal properties, one can evaluate, in principal, any term in the power series expansion* in the moduli $\{z_i\}'$, e.g. any coefficient in

$$\mathcal{F}_P(x) = x^{\Delta_P - \Delta_1 - \Delta_2} \sum_{n=0}^{\infty} \mathcal{F}_P^{(n)} \, x^n$$

The AGT correspondence provides powerful combinatorial representations for the coefficients of such expansions, because such representations exist for the Nekrasov's instanton partition sums.

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* For the 4-point block there is very fast way to generate the expansion coefficients through the recursion relation due to **Alyosha Zamolodchikov, 1984** .

Global analytic control is more limited. The following general properties are generally assumed:

- $\mathcal{F}_P(x)$ is analytic in x on the universal cover of $\mathbb{C}\backslash\{0, 1, \infty\}$.

- $\mathcal{F}_P(x)$ is meromorphic in P.

- $\mathcal{F}_P(x)$ obeys the "cross relations" (= modular transformations), e.g.

$$\mathcal{F}_P(x) = \int B_{P\,P'}\,\mathcal{F}_{P'}(1-x)\;dP'$$

with known $B_{P\,P'}$ (essentially the $6j$-symbols for continuous representations of $U_q(SL(2))$, **Ponsot, Teschner, 1999**)

Classical conformal block

Classical limit: corresponds to the limit

$$c \to \infty$$

(in which case the Virasoro commutators $\longrightarrow$ Poisson brackets)
I will use the Liouville-like parameterization

$$c = 1 + 6\, Q^2 \ , \qquad Q = b + b^{-1} \ ;$$

and take the limit $b \to 0$. Interesting limit emerges if all the dimensions

$$\Delta_i \, , \quad \Delta(P_\alpha)$$

also go to infinity, in such a way that the products

$$\delta_i = b^2 \ \Delta_i \ , \qquad \delta_\alpha = b^2 \ \Delta(P_\alpha)$$

remain fixed. In what follows I use the parameterization

$$\delta_i = \frac{1 - \lambda_i^2}{4} \ , \qquad \delta_\alpha = \frac{1 - \nu_\alpha^2}{4}$$

For real δ the parameters λ_i, ν_α can be real or pure imaginary, but generally are allowed to take complex values.

In this classical limit the conformal blocks exponentiate

$$\mathcal{F}_{\{P_\alpha\}}(\{z_i\}) \sim \exp\left(\frac{1}{b^2}\, f_{\{\nu_\alpha\}}(\{z_i\})\right)$$

where

$$f_{\{\nu_\alpha\}}(\{z_i\}) \quad - \quad \textbf{“classical conformal block”}$$

Mathematically, the status of this statement - the exponentiation - is not completely clear yet (at least to me). There is large body of evidence for it, both from 2D CFT and from the $\mathcal{N} = 2$ SUSY sides of the AGT correspondence *.

The proof likely follows from the combination of the following results:

• Recent proof of the AGT conjecture (**Alba, Fateev, Litvinov, Ternopolsky, 2011**)

• Nekrasov's partition sum = virial expansion for an ensemble of "particles", with the above classical limit corresponding to the thermodynamic limit $\rightarrow$ exponentiation of Z_{inst}.

* $f = W_{\text{inst}}$, the instanton term in "effective twisted superpotential" of 2D sigma model obtained by compactification of gauge theories in Ω-background (**Nekrasov, Shatashvili, 2009**).

Analyticity of $f_\nu(x)$ is much more complicated then that of $\mathcal{F}_P(x)$.

- $f_\nu(x)$ has (generally infinitely many) algebraic singularities in x.
- $f_\nu(x)$ has brunching singularities in ν.

Understanding this analytic structure was one of the motivations of this project

Monodromy of ODE

The classical conformal block admits neat interpretation in terms of monodromies of ordinary differential equations

$$\partial_z^2\,\psi(z) + \underbrace{\sum_{i=1}^{n}\left[\frac{\delta_i}{(z-z_i)^2} + \frac{c_i}{z-z_i}\right]}_{t(z)}\psi(z) = 0$$

The variable z can be regarded as complex coordinate on the Riemann sphere $\mathbb{S}^2 = \mathbb{C}_\infty$.
This is generic second-order differential equation with n regular singularities.
The parameters δ_i will be set equal to the δ_i's in the classical conformal block, and will be regarded as fixed numbers.

The positions $\{z_i\}$ of the singularities, and $\{c_i\}$ (**"accessory parameters"**)

are treated as variables. In fact, there are only $n-3$ independent c_i's, since if we insist that $z=\infty$ is a regular point, then

$$t(z) \sim \frac{1}{z^4} \quad \text{as} \quad z \to \infty \qquad \Longrightarrow \qquad \left\{\begin{array}{l} \sum_i c_i = 0 \\ \sum_i (z_i c_i + \delta_i) = 0 \\ \sum_i (z_i c_i + 2 z_i \delta_i) = 0 \end{array}\right.$$

Since projective transformations preserve the form of the differential equation, there are essentially $n-3$ independent z_i's

$$n-3 \qquad \{z_i'\} = \{z_i\}/\text{projective } \mathbb{SL}(2,\mathbb{C})$$

Therefore the space of the above differential equations is described by

$$t(z) \ : \quad \mathbf{2\,(n-3)} \quad \textbf{complex parameters}$$

Consider

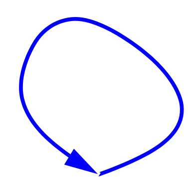
$$\pi_1\big(\mathbb{S}^2\backslash\{z_i\}\big) \ := \ \text{Fundamental group of} \ \ \mathbb{S}^2\backslash\{z_i\}$$

The above differential equation generates the **Monodromy Group**, the homomorphism of the fundamental group into $\mathbb{SL}(2,\mathbb{C})$

$$\boldsymbol{M} \ : \quad \pi_1\big(\mathbb{S}^2\backslash\{z_i\}\big) \longrightarrow \mathbb{SL}(2,\mathbb{C})$$

which to any element $\gamma \in \pi_1\big(\mathbb{S}^2\backslash\{z_i\}\big)$ associates $\mathbb{SL}(2,\mathbb{C})$ matrix $\boldsymbol{M}(\gamma)$:

$$\boldsymbol{\Psi}(\gamma \circ z) = \boldsymbol{\Psi}(z)\boldsymbol{M}(\gamma)\,, \qquad \boldsymbol{\Psi}(z) = \big(\,\psi_1(z)\,,\,\psi_2(z)\,\big)$$

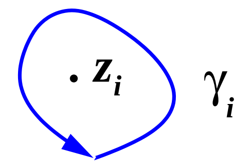


γ

The fact that δ_i are fixed means that the conjugacy classes of "elementary matrices"

$$\delta_i = \frac{1-\lambda_i^2}{4} \quad - \quad \text{fixed} \qquad \Longrightarrow \qquad \mathrm{tr}(\boldsymbol{M}_i) = -2\,\cos(\pi\lambda_i)$$

$$\boldsymbol{M}_i := \boldsymbol{M}(\gamma_i)$$



Finally, since the basis $\big(\psi_1,\,\psi_2\big)$ is not fixed, the matrices

$$\boldsymbol{M}(\gamma), \quad \gamma \in \pi_1\big(\mathbb{S}^2\backslash\{z_i\}\big) \quad \text{are} \quad \text{defined} \quad \text{up to overall conjugation}$$

The space of all homomorphisms

$$\begin{matrix} \pi_1\big(\mathbb{S}^2\backslash\{z_i\}\big) \;\to\; \mathbb{SL}(2,\mathbb{C}) \\ \text{with } \mathrm{tr}(\boldsymbol{M}_i) \text{ fixed, up to conjugation} \end{matrix} \;=\; \begin{matrix} \text{"Moduli space of flat connections"} \\ \text{over} \quad \mathbb{S}^{\mathbf{2}}\backslash\{z_i\} \end{matrix}$$

Generally, the points of this space can be parameterized by invariants like

$$\mathrm{tr}\big(\,\boldsymbol{M}(\gamma_{i_1} * \cdots * \gamma_{i_n})\,\big)$$

which obey certain polynomial relations. There are exactly $\mathbf{2\,(n-3)}$ independent invariants, i.e. the above differential equation does not admit continuous isomonodromic deformations – once δ_i are fixed, any changes in $\{z'_\alpha\}, \{c'_\alpha\}$ change the monodromy.
In other words,

$$\{z'_\alpha\}, \qquad \{c'_\alpha\} \qquad \alpha = 1, \ldots, n-3$$

can be regarded as local coordinates on the moduli space of the flat connections.

The moduli space of flat connections admits natural symplectic form (**Atiyah, Bott; 1982**), in which $\{z'_\alpha\},\ \{c'_\alpha\}$ - Darboux coordinates

$$\Omega = \sum_{\alpha=1}^{n-3} \mathrm{d}c'_\alpha \wedge \mathrm{d}z'_\alpha$$

Classical conformal block and monodromies

Getting back to the quantum case of finite b, let us recall that there are

$$\text{special values of } \Delta \quad \longrightarrow \quad \begin{array}{c}\text{Degenerate representations}\\ \text{(with null} - \text{vectors)}\end{array}$$

$$\Delta_{(1,2)} = -\frac{1}{2} - \frac{3b^2}{4}\,, \qquad \left(\frac{1}{b^2}\,L_{-1}^2 + L_{-2}\right)|\,\Delta_{(1,2)}\,\rangle = \text{null}$$

$$\Delta_{(2,1)} = -\frac{1}{2} - \frac{3}{4b^2}\,, \qquad \left(b^2\,L_{-1}^2 + L_{-2}\right)|\,\Delta_{(1,2)}\,\rangle = \text{null}$$

Then the null-vector decoupling leads to the differential equations

$$\left\{\frac{1}{b^2}\,\frac{\partial^2}{\partial z^2} + \sum_{i=1}^{n}\left[\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i}\,\frac{\partial}{\partial z_i}\right]\right\}\langle\,V_{(1,2)}(z)\,V_{\Delta_1}(z_1)\,\cdots\,V_{\Delta_n}(z_n)\,\rangle = 0$$

where $V_{(1,2)} \equiv V_{\Delta_{(1,2)}}$. The differential equation does not depend on the "dual diagram", which comes through the choice of the solution.

Conformal blocks with $V_{(2,1)}$ satisfy similar equations with $b \to b^{-1}$.

Now we take the classical limit $b \to 0$, with $\Delta_i = \delta_i / b^2$. Since

$$\Delta_{(1,2)} \to -\frac{1}{2}$$

remains finite, by usual semiclassical intuition one expects to have

$$\langle\, V_{(1,2)}(z)\, V_{\Delta_1}(z_1)\, \cdots\, V_{\Delta_n}(z_n)\,\rangle \;\to\; \psi(z, \{z_i\})\; \exp\left(\frac{1}{b^2}\, f_{\{\nu_\alpha\}}(\{z_i\})\right)$$

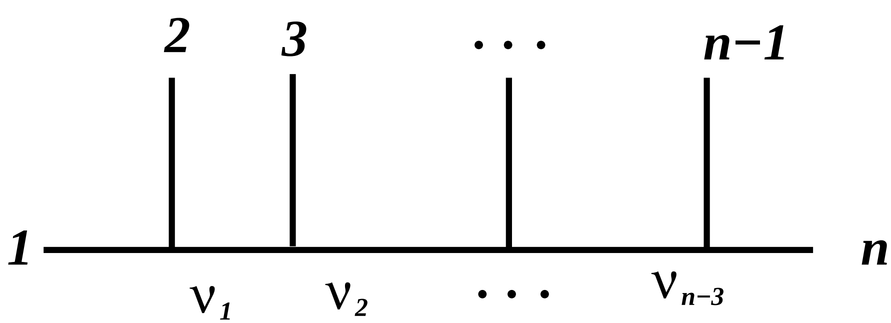
where f is the same classical conformal block as in

$$\langle\, V_{\Delta_1}(z_1)\, \cdots\, V_{\Delta_n}(z_n)\,\rangle \;\sim\; \mathrm{e}^{\frac{1}{b^2}\, f_{\{\nu_\alpha\}}(\{z_i\})}\,.$$

Decoupling equation leads to the above second-order ordinary differential equation with

$$c_i = \frac{\partial}{\partial z_i}\, f_{\nu_1 \ldots \nu_{n-3}}(\{z_i\})$$

Recall that the definition of f involves the "pant decomposition" of $\Sigma_{0,n}$



$$\longrightarrow \quad f_{\nu_1 \ldots \nu_{n-3}}(\{z_i\})$$

We conclude that the classical conformal block $f_{\nu_1 \dots \nu_{n-3}}$ solves the following monodromy problem:

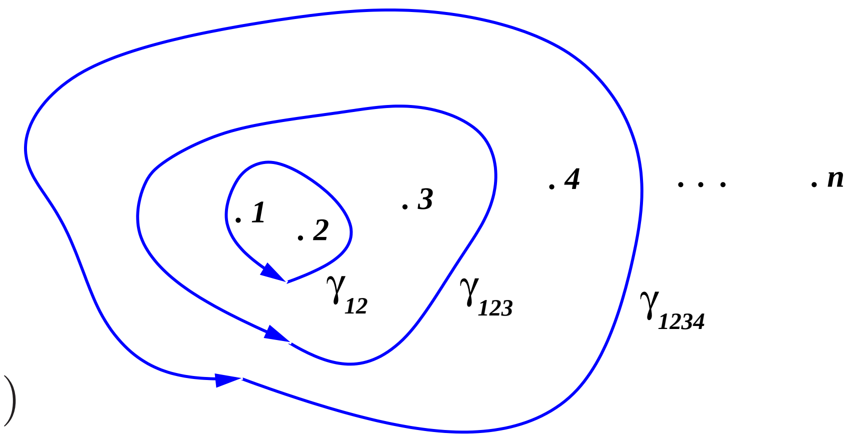
Fix $\{z_i\}$, and adjust the accessory parameters $c_i = c_i(\{z_i\}, \{\nu_\alpha\})$ such that

$$\mathrm{tr}(\boldsymbol{M}(\gamma_{12})) = -2\cos(\pi\nu_1)$$

$$\mathrm{tr}(\boldsymbol{M}(\gamma_{123})) = -2\cos(\pi\nu_2)$$

$$\vdots$$

$$\mathrm{tr}(\boldsymbol{M}(\gamma_{12\dots n-2})) = -2\cos(\pi\nu_{n-3})$$



Then

$$c_i = \frac{\partial f}{\partial z_i}$$

gives the solution to this monodromy problem.

More complete and elegant way to make this statement is due to **Nekrasov, Rosly, Shatashvilli; 2011**:

$$\{\nu_1,\,\nu_2,\dots\,\nu_{n-3}\}\quad-\quad\begin{array}{l}\text{Poisson}-\text{commuting coordinates w.r.t.}\\ \Omega=\sum_{\alpha=1}^{n-3}\mathrm{d}c'_\alpha\wedge\mathrm{d}z'_\alpha=\sum_{\alpha=1}^{n-3}\mathrm{d}\mu_\alpha\wedge\mathrm{d}\nu_\alpha\end{array}$$

i.e. $\{\nu_\alpha,\mu_\alpha\}$-another Darboux coordinates on the moduli space of the flat connections.

The classical conformal block f with some term added

$$W(\{\nu_\alpha,z'_\alpha\})=\underbrace{\sigma(\{\nu_\alpha\})}_{\text{Explicitly known}}+f_{\{\nu_\alpha\}}(\{z'_\alpha\})$$

is the generating function of the canonical transformation

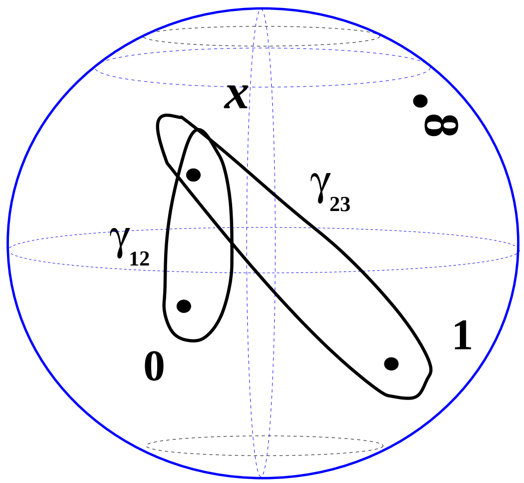
$$\{c'_\alpha,z'_\alpha\}\;\leftrightarrow\;\{\mu_\alpha,\nu_\alpha\}\qquad\qquad(\alpha=1,\dots\,n-3)\,,$$

so that

$$c_\alpha=\frac{\partial W}{\partial z_\alpha}\,,\qquad\mu_\alpha=-\frac{\partial W}{\partial\nu_\alpha}\,.$$

Remarkably, W has direct interpretation in SUSY gauge theories (Nekrasov-Shatashvili “twisted superpotential”).

There are interesting things to say about canonical transformations between $(\mu,\,\nu)$-coordinates associated with different dual diagrams, say



$$\mathrm{Tr}\big(\boldsymbol{M}(\gamma_{12})\big) = -2\ \cos(\pi\nu)\ ,\ \mathrm{Tr}\big(\boldsymbol{M}(\gamma_{23})\big) = -2\ \cos(\pi\nu')$$

$$(\mu,\nu)\to(\mu',\nu')\quad:\quad \mu = \tfrac{\partial}{\partial\nu}\,\hat{\Sigma}(\nu,\nu')\ ,\qquad \mu' = -\tfrac{\partial}{\partial\nu'}\,\hat{\Sigma}(\nu,\nu')$$

The generating function $\Sigma(\nu,\nu')$ can be understood in terms of Conformal blocks, as follows. At finite b, consider the "crossing relation"

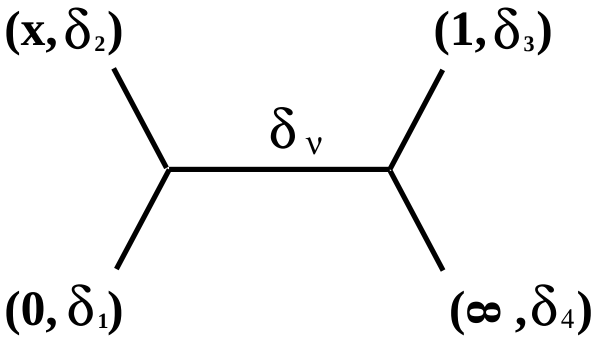
$$\mathcal{F}_P(x) = \int_{P'} B(P,P')\,\mathcal{F}_{P'}(1-x)$$

In the limit $b\to 0$ both the Ponsot-Teschner crossing coefficient and the Conformal block exponentiate

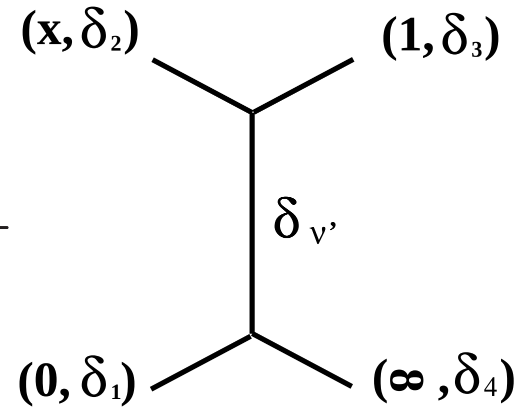
$$B(P,P')\ \simeq\ \exp\left(\frac{1}{b^2}\,\Sigma(\nu,\nu')\right)\ ,\qquad \mathcal{F}_P(x)\ \simeq\ \exp\left(\frac{1}{b^2}\,f_\nu(x)\right)\ ,$$

and the integral is dominated by the saddle point.

As the result one finds the classical version of the crossing relation



$$= \overbrace{\Sigma(\nu,\nu')}^{\textit{Explicitly known}} +$$



where ν' and ν are related by the saddle point equation

$$\frac{\partial}{\partial\nu'}\left(\Sigma(\nu,\nu') + f_{\nu'}(1-x)\right) = 0\,.$$

The function $\Sigma(\nu,\nu')$ can be computed

(i) Directly from the geometric construction of (μ,ν) and (μ',ν') (**Nekrasov, Rosly, Shatashvili, 2011**),

or

(ii) By taking $b \to 0$ limit of the Ponsot-Teschner $B(P,P')$ (**Teschner, Vartanov, 2012**),

with the same result.

Hamilton-Jacobi equation

Consider another level-2 degenerate operator $V_{(2,1)}$,

$$V_{(2,1)} \quad : \qquad \Delta_{(2,1)} = -\frac{1}{2} - \frac{3}{4b^2} \approx -\frac{3}{4b^2}$$

The null vector decoupling equation has the form

$$\left\{ b^2 \frac{\partial^2}{\partial y^2} + \sum_{i=1}^{n} \left[\frac{\Delta_i}{(y-z_i)^2} + \frac{1}{y-z_i} \frac{\partial}{\partial z_i} \right] \right\} \langle V_{(2,1)}(y) \, V_{\Delta_1}(z_1) \cdots V_{\Delta_n}(z_n) \rangle = 0$$

In the classical limit $\Delta_{(2,1)} \to -\frac{3}{4b^2} \to \infty$. Assuming $\Delta_i = \delta_i/b^2$, $\Delta(P_\alpha) = \delta_{\nu_\alpha}/b^2$,

$$\langle \quad \rangle_{b \to 0} \quad \simeq \quad \mathrm{e}^{\frac{1}{b^2} \, S(y,\{z_i\})}$$

The above decoupling equation then reduces to the Hamilton-Jacobi-like equation

$$\left(\frac{\partial S}{\partial y}\right)^2 + \sum_{i=1}^{n} \left[\frac{\delta_i}{(y-z_i)^2} + \frac{1}{y-z_i} \frac{\partial S}{\partial z_i} \right] = 0 \, .$$

In the case $n = 4$ it becomes literally the Hamilton-Jacobi equation for certain 1D system.

Let us fix the projective transformation freedom by sending three of the points z_1, z_2, z_3, z_4 to the standard positions 0, 1, ∞, i.e. consider

$$\langle\, V_{(2,1)}(y)\, V_{\Delta_1}(0)\, V_{\Delta_2}(t)\, V_{\Delta_3}(1)\, V_{\Delta_5}(\infty)\,\rangle$$

Here I denote the remaining cross-ratio t, because it is going to play the rôle of time. Also, I denote the fourth dimension Δ_5, not Δ_4, for reasons which will become clear soon.

In the classical limit

$$\langle \quad \rangle_{b\to 0} \quad \simeq \quad \mathrm{e}^{\frac{1}{b^2}\ S}$$

one obtains the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H\Big(y,\, \frac{\partial S}{\partial y},\, t\Big) = 0$$

with the Hamiltonian $H(y,p,t)$ of the form,

$$H(y,p,t) = \frac{(y-t)y(1-y)}{t(1-t)}\; p^2 - \frac{\delta_1 - \frac{1}{4}}{y(1-t)} - \frac{\delta_2}{t-y} - \frac{\delta_3 - \frac{1}{4}}{t(y-1)} - \frac{(\delta_5 - \frac{1}{4})\, y}{t(1-t)}$$

Painlevé VI

The associated equation of motion

$$\begin{aligned}\ddot{y} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\dot{y}^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\dot{y}\\ &+ \frac{2y(y-1)(y-t)}{t^2(1-t)^2}\left[\left(\delta_1-\tfrac{1}{4}\right)\frac{t}{y^2}+\delta_2\frac{t(t-1)}{(y-t)^2}+\left(\delta_3-\tfrac{1}{4}\right)\frac{1-t}{(y-1)^2}+\left(\tfrac{1}{4}-\delta_5\right)\right]\end{aligned}$$

is the **Painlevé VI equation**. It is one (and the most general) of six ordinary differential equations of the form

$$\ddot{y} = R(y, \dot{y}, t)$$

which have no **"movable singularities"** but simple poles. By "movable singularities" one understands singularities of a solution $y(t)$, as the function of complex t, whose positions depend on the choice of the solution (i.e. on the initial conditions).

As in our case it is natural to regard y as living on the Riemann sphere

$$y \in \mathbb{C}_\infty$$

simple poles are not singularities, and thus it is fair to say that the solutions $y(t)$ have no movable singularities at all. Of course, there are **"fixed"** singularities of $y(t)$ at

$$t = 0,\ 1,\ \infty$$

How this relation to the Painlevé VI can be useful in the analysis of the classical conformal blocks?

Let $y(t)$ be some solution of the Painlevé VI equation such that

$$y(t) \quad : \qquad \begin{aligned} y(t_1) &= y_1 \\ y(t_2) &= y_2 \end{aligned}$$

Then, by the definition of the classical action, we have

$$\begin{aligned} \langle\, V_{(2,1)}(y_2)\, V_{\Delta_1}(0)\, V_{\Delta_2}(t_2)\, V_{\Delta_3}(1)\, V_{\Delta_5}(\infty)\,\rangle_{b\to 0} &= \exp\left[\,\frac{1}{b^2}\,\int_{t_1}^{t_2} \mathrm{d}t\ L(y,\dot{y},t)\right] \\ &\times\,\langle\, V_{(2,1)}(y_1)\, V_{\Delta_1}(0)\, V_{\Delta_2}(t_1)\, V_{\Delta_3}(1)\, V_{\Delta_5}(\infty)\,\rangle_{b\to 0} \end{aligned}$$

where $L(y,\dot{y},t)$ is the Lagrangian associated with the above Hamiltonian.

Note that if $y = 0,\ t,\ 1,\ \infty$, the insertion $V_{(2,1)}(y)$ hits one of the other insertions. Suppose at some point $t = x$, $y(t)$ hits one of those points, say $y = \infty$:

$$t = x\ ; \qquad y(t = x) = \infty$$

At this point the 5-points block reduces to 4-point one

$$\langle\, V_{\Delta_1}(0)\, V_{\Delta_2}(x)\, V_{\Delta_3}(1)\, V_{\Delta_4}(\infty)\,\rangle|_{b\to 0} = \exp\Big(\frac{1}{b^2}\ f_\nu(x)\Big)\,.$$

What is Δ_4 here?

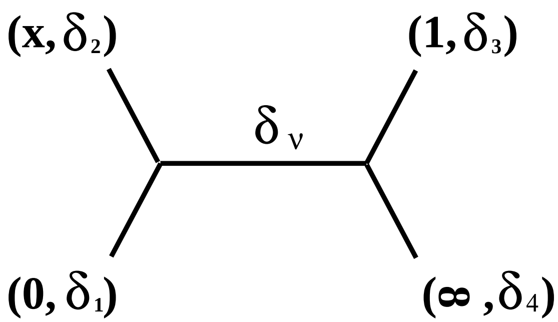
- The vertex operator $V_{(2,1)}(y)$ obeys the **"fusion rules"** which can be written as

$$V_{(2,1)}\, V_{\Delta_a} = \left[\, V_{\Delta_{a+\frac{1}{b}}} \right] + \left[\, V_{\Delta_{a-\frac{1}{b}}} \right]$$

where I use again the Liouville like parameterization

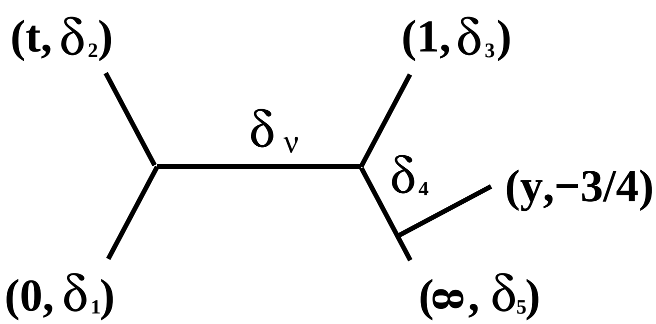
$$\Delta_a = a(Q-a)$$

In the classical limit it means that if we want to end up with the classical block



with $\quad \delta_4 = \frac{1}{4}\,(1-\lambda_4^2)$

we need to start with the 5-point function



with, say $\quad \delta_5 = \frac{1}{4}\left(1-(\lambda_4-1)^2\right)$

- For a generic trajectory $y(t)$, the points $y = 0,\ 1,\ t,\ \infty$ are regular.
  E.g. if $y(t = x) = \infty$, then

$$y(t) = \pm \frac{1}{\lambda_4 - 1} \frac{x(1-x)}{t-x} + \text{regular}$$

- $t = 0,\ 1,\ \infty$ are singular points. E.g. near $t = 0$ generic solution behaves as

$$y(t) \to -\kappa\ t^{\nu} \qquad \text{as} \qquad t \to 0$$

  where $\nu,\ \kappa$ are parameters of the solution, $0 < \Re e\, \nu < 1$.
  More precisely, the solution admits $t \to 0$ expansion

$$\frac{1}{y(t)} = \sum_{n=0}^{\infty} t^n\ U_n(t)$$

  with

$$U_n(t) = \sum_{m=-n-1}^{n+1} t^{-\nu m}\ U_{n,m}$$

  where the coefficients $U_{n,m}$ can be developed order by order once the constants ν, κ are given. In particular

$$U_0(t) = C_-\ t^{-\nu} + C_0 + C_+\ t^{\nu}$$

  where $C_{(\pm,0)}$ are certain expressions in terms of κ, ν and δ_i.

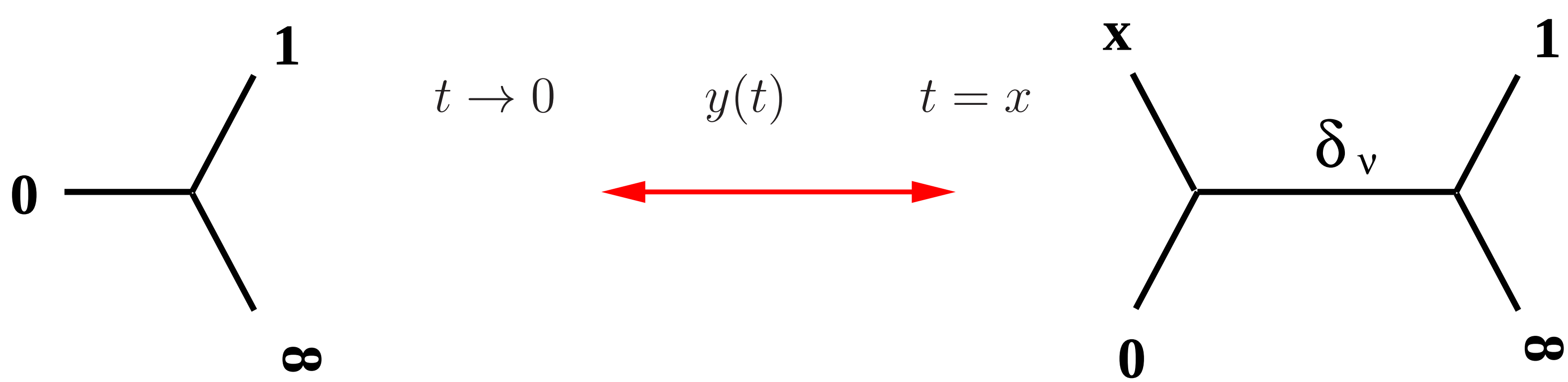
Note that

$$y(t) \to 0 \qquad \text{as} \quad t \to 0$$

so that in this singular limit **three** of the five points merge, resulting in a three-point function, which can be regarded as the normalization constant. At certain $t = x$ this trajectory hits one of the points 0, 1, t, ∞, say

$$y(x) = \infty$$

Then this trajectory interpolates between three-point and four point blocks



It is possible to show that if $y(t) \simeq -\kappa\, t^{\nu}$, the exponent ν exactly corresponds to the parameter of the intermediate

$$\delta_{\nu} = \frac{1-\nu^2}{4}$$

(Direct relation exists only for $|\Re e\, \nu| < 1$, but probably can be extended by analytic continuation.)

We arrive at the following prescription:

- Find (the solution) $y(t)$ of the Painlevé VI

$$y(t) \to \begin{cases} -\kappa\, t^{\nu} & \text{as} \qquad t \to 0 \\ \infty & \text{as} \qquad t \to x \end{cases}$$

  where κ should be understood as a function of (x, ν)

$$\kappa = \kappa(x, \nu)$$

- Calculate the (regularized) action

$$f_{\nu}(x) = (\delta_{\nu} - \delta_1 - \delta_2)\ \log x + (\delta_1 - \delta_2 - \delta_3 + \delta_4)\ \log(1-x) \\ + \frac{\nu}{2}\left(\log\left(\kappa/\kappa_0\right) + \nu\ \log(x)\right) + \int_0^x \mathrm{d}t \left[L(y, \dot{y}, t) - \frac{\nu^2}{4t} - \frac{\lambda_4 - 1}{2(x-t)}\right]$$

  where

$$\kappa_0 = \lim_{x \to 0} \kappa(x, \nu) = \frac{4\nu^2}{(\nu - 1 - \lambda_3 + \lambda_4)(\nu - 1 + \lambda_3 + \lambda_4)}\ , \qquad \delta_i = \tfrac{1}{4}\ (1 - \lambda_i^2)$$

Accessory parameters and connection problem for Painlevé VI

$(x, c) \to (\nu, \mu)$ is the canonical transformation

$$\mu = -\frac{\partial W}{\partial \nu}\,, \qquad c = \frac{\partial W}{\partial x}$$

with $W(\nu, x) = \sigma(\nu) + f_\nu(x)$. Then one can show that

$$\begin{aligned} \mu &= \log\kappa + \overbrace{F(\nu)}^{\text{Explicitly known}} \\ c &= -\frac{(\lambda_4 - 1)^2}{x(1-x)}\,(y_0 - x) + \frac{\delta_3 - \delta_1 + \delta_2 - \delta_4}{x} + \frac{\delta_2 + \delta_3 + \delta_4 - \delta_1}{1-x} \end{aligned}$$

$$y(t) \to \begin{cases} -\kappa\, t^\nu + \dots & \text{as} \qquad t \to 0 \\ \frac{x(1-x)}{\lambda_4 - 1}\,\frac{1}{t-x} + y_0 + \sum_{n=1}^\infty y_n(y_0, x)\,(t-x)^n & \text{as} \qquad t \to x \end{cases}$$

where the coefficients $y_n(y_0, x)$ $(n = 1,\, 2,\, \dots)$ are uniquely determined by Painlevé VI.

Recall that the solution $y = y(t)$ admits $t \to 0$ expansion of the form

$$U(t) := \frac{1}{y(t)} = \sum_{n=0}^{\infty} t^n \; U_n(t) \; , \quad U_n(t) = \sum_{m=-n-1}^{n+1} t^{-\nu m} \; U_{n,m}$$

where the coefficients $U_{n,m}$ can be developed order by order once the constants ν, κ and δ_i are given. Using the conditions

$$U(t)|_{t=x} = 0 \; , \qquad U'(t)|_{t=x} = \pm \frac{\lambda_4 - 1}{(1-x)x}$$

$$c = \frac{1}{2}\, x(1-x) \left[U''(t) + 2t\big(U'(t)\big)^2 \right]_{t=x} + \frac{\delta_3 - \delta_1 + \delta_2 - \delta_4}{x} + \frac{\delta_2 + \delta_3 + \delta_4 - \delta_1}{1-x}$$

we can exclude κ and find the accessory parameter $c = c(\nu, x)$. This gives a systematic procedure to determine the expansion coefficients

$$c = (\delta_\nu - \delta_1 - \delta_2)\; x^{-1} + \frac{(\delta_\nu - \delta_1 + \delta_2)(\delta - \delta_4 + \delta_3)}{2\,\delta_\nu} + \sum_{n=2}^{\infty} c_n \; x^{n-1}$$

($\delta_\nu = \frac{1-\nu^2}{4}$) and then, using $c = \frac{\mathrm{d} f_\nu(x)}{\mathrm{d}x}$,

$$f_\nu(x) = (\delta - \delta_1 - \delta_2)\; \log(x) + \sum_{n=1}^{\infty} \frac{c_n}{n} \; x^n$$

Conclusion

- Null-vector decoupling for "heavy" degenerate vertex $V_{(2,1)}$ in 5-point function $\xrightarrow{b\to 0}$ Hamilton-Jacobi equation for Painlevé VI
- There are solutions $y(t)$ interpolating between 3- and 4-point functions $\Longrightarrow$

$$4-\text{point classical conformal block} = \text{Action of Painlevé VI computed on } y(t)$$

Remarks

- There is close relationship to the Hamiltonian structure of the monodromy preserving mations of order 2 Fuchsian systems

$$\left[\partial_z + \sum_{i=1}^{n} \frac{A_i}{z - z_i}\right] \Psi = 0$$

i.e. the Schlesinger systems (**Jimbo, Miwa, 1982**).

- Painlevé VI has appeared before in connection with conformal blocks.

Recently, **Gamayun, Iorgov, Lisovyy (2012)** have found that the tau-function for Painlevé VI generates **Quantum conformal block with** $c = 1$

$$\tau(t) = \sum_{n=-\infty}^{\infty} C_n \, t^{(\frac{1-\nu}{2}+n)^2-\Delta_1-\Delta_2} \, \mathcal{F}_{\Delta(\nu-2n)}(t)$$