

An integrable spin chain for the $SL(2, \mathbb{R})/U(1)$ sigma model

Yacine Ikhlef
LPTHE, Université Paris-6/CNRS

29th May 2013
Saclay

- ▶ YI, J.L. Jacobsen, H. Saleur,
A staggered six-vertex model with non-compact continuum limit,
Nucl. Phys. **B 789**, 483 (2008)
- ▶ YI, J. Jacobsen, H. Saleur,
An integrable spin chain for the $SL(2,R)/U(1)$ black hole sigma model,
Phys. Rev. Lett. **108**, 081601 (2012)
- ▶ C. Candu and YI,
Non-linear integral equations for the $SL(2,R)/U(1)$ black hole sigma model,
in preparation

Outline

1. Motivations
2. The critical spin chain and its continuum limit
3. Non-linear integral equations

1. Motivations

Non-compact sigma models and applications

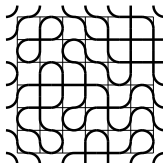
- ▶ For “standard” stat. mech. models (Potts, $O(n)$...):
scaling limit = Coulomb-Gas = **compact** boson
 \Rightarrow conformal spectrum is **discrete** $[h_{e,m} = \frac{1}{4}(m\sqrt{g} + e/\sqrt{g})^2]$

Non-compact sigma models and applications

- ▶ For “standard” stat. mech. models (Potts, $O(n)$...):
scaling limit = Coulomb-Gas = **compact** boson
 \Rightarrow conformal spectrum is **discrete** [$h_{e,m} = \frac{1}{4}(m\sqrt{g} + e/\sqrt{g})^2$]
- ▶ Relax compactification constraint
 \Rightarrow conformal spectrum includes **continua**
e.g. vertex operators of free boson CFT ($V_\alpha = : e^{i\alpha\varphi} :$)

Non-compact sigma models and applications

- ▶ For “standard” stat. mech. models (Potts, $O(n)$...):
scaling limit = Coulomb-Gas = **compact** boson
 \Rightarrow conformal spectrum is **discrete** $[h_{e,m} = \frac{1}{4}(m\sqrt{g} + e/\sqrt{g})^2]$
- ▶ Relax compactification constraint
 \Rightarrow conformal spectrum includes **continua**
e.g. vertex operators of free boson CFT ($V_\alpha = : e^{i\alpha\varphi} :$)
- ▶ Stat. mech. models with non-compact d.o.f.
AF Potts model, loop models with intersections ...



Plateau transition in IQHE (2/2)

[Zirnbauer '99]

- ▶ Re-arrange creation/annihilation ops:

$$\bar{c}_0 = b_+^\dagger, \quad \bar{c}_1 = f_+^\dagger, \quad \bar{c}_2 = -b_-, \quad \bar{c}_3 = f_-$$

$$c_0 = b_+, \quad c_1 = f_+, \quad c_2 = +b_-^\dagger, \quad c_3 = f_-^\dagger$$

⇒ Construct representation of $gl(2|2)$:

$$S_{\alpha\beta} := \bar{c}_\alpha c_\beta \quad \Rightarrow \quad [S_{\alpha\beta}, S_{\gamma\epsilon}] = \delta_{\beta\gamma} S_{\alpha\epsilon} - (-1)^{(\alpha+\beta)(\gamma+\epsilon)} \delta_{\epsilon\alpha} S_{\gamma\beta}$$

Plateau transition in IQHE (2/2)

[Zirnbauer '99]

- ▶ Re-arrange creation/annihilation ops:

$$\begin{aligned}\bar{c}_0 &= b_+^\dagger, & \bar{c}_1 &= f_+^\dagger, & \bar{c}_2 &= -b_-, & \bar{c}_3 &= f_- \\ c_0 &= b_+, & c_1 &= f_+, & c_2 &= +b_+^\dagger, & c_3 &= f_-^\dagger\end{aligned}$$

⇒ Construct representation of $gl(2|2)$:

$$S_{\alpha\beta} := \bar{c}_\alpha c_\beta \quad \Rightarrow \quad [S_{\alpha\beta}, S_{\gamma\epsilon}] = \delta_{\beta\gamma} S_{\alpha\epsilon} - (-1)^{(\alpha+\beta)(\gamma+\epsilon)} \delta_{\epsilon\alpha} S_{\gamma\beta}$$

- ▶ Equivalent spin chain:
$$H = \sum_{j=1}^L S_{\alpha\beta}(j) S_{\alpha\beta}^*(j+1)$$

Plateau transition in IQHE (2/2)

[Zirnbauer '99]

- ▶ Re-arrange creation/annihilation ops:

$$\begin{aligned}\bar{c}_0 &= b_+^\dagger, & \bar{c}_1 &= f_+^\dagger, & \bar{c}_2 &= -b_-, & \bar{c}_3 &= f_- \\ c_0 &= b_+, & c_1 &= f_+, & c_2 &= +b_+^\dagger, & c_3 &= f_-^\dagger\end{aligned}$$

⇒ Construct representation of $gl(2|2)$:

$$S_{\alpha\beta} := \bar{c}_\alpha c_\beta \quad \Rightarrow \quad [S_{\alpha\beta}, S_{\gamma\epsilon}] = \delta_{\beta\gamma} S_{\alpha\epsilon} - (-1)^{(\alpha+\beta)(\gamma+\epsilon)} \delta_{\epsilon\alpha} S_{\gamma\beta}$$

- ▶ Equivalent spin chain:
$$H = \sum_{j=1}^L S_{\alpha\beta}(j) S_{\alpha\beta}^*(j+1)$$

- ▶ Scaling limit = σ -model on non-compact $\frac{U(1,1|2)}{U(1|1) \times U(1|1)}$
consistent CFT for fixed point = ?

Solvable lattice models with non-compact limit

- ▶ Known examples:
 - ▶ finite-dimensional $sl(2|1)$ spin chains
[Essler-Frahm-Saleur '05] [Frahm-Martins '10 '12]
 - ▶ staggered vertex model
[Jacobsen-Saleur '06] [YI-Jacobsen-Saleur '08 '12]

Solvable lattice models with non-compact limit

- ▶ Known examples:
 - ▶ finite-dimensional $sl(2|1)$ spin chains
[Essler-Frahm-Saleur '05] [Frahm-Martins '10 '12]
 - ▶ staggered vertex model
[Jacobsen-Saleur '06] [YI-Jacobsen-Saleur '08 '12]

- ▶ General argument

Solvable lattice models with non-compact limit

- ▶ Known examples:

- ▶ finite-dimensional $\mathfrak{sl}(2|1)$ spin chains
[Essler-Frahm-Saleur '05] [Frahm-Martins '10 '12]
- ▶ staggered vertex model
[Jacobsen-Saleur '06] [YI-Jacobsen-Saleur '08 '12]

- ▶ General argument

- ▶ Bethe-Ansatz equations:
$$L\rho(\lambda_j) = 2\pi l_j - \sum_{\ell} \theta(\lambda_j - \lambda_{\ell})$$

Solvable lattice models with non-compact limit

▶ Known examples:

- ▶ finite-dimensional $\mathfrak{sl}(2|1)$ spin chains
[Essler-Frahm-Saleur '05] [Frahm-Martins '10 '12]
- ▶ staggered vertex model
[Jacobsen-Saleur '06] [YI-Jacobsen-Saleur '08 '12]

▶ General argument

- ▶ Bethe-Ansatz equations: $L\rho(\lambda_j) = 2\pi I_j - \sum_{\ell} \theta(\lambda_j - \lambda_{\ell})$
- ▶ From “naive” analysis of continuum limit ...

$$\rho'(\lambda) = 2\pi\rho(\lambda) - \int d\mu K(\lambda - \mu)\rho(\mu), \quad K := \theta'$$

FT of kernel $(2\pi - \widehat{K})$ is singular \Rightarrow continuous CFT spectrum

General program

- ▶ Construct/classify integrable vertex models with singular Bethe-Ansatz kernel

General program

- ▶ Construct/classify integrable vertex models with singular Bethe-Ansatz kernel
- ▶ Study finite-size effects (with Wiener-Hopf or NLIE)
 - ▶ compute conformal spectrum + density of states
 - ▶ identify CFT / non-compact σ -model

General program

- ▶ Construct/classify integrable vertex models with singular Bethe-Ansatz kernel
- ▶ Study finite-size effects (with Wiener-Hopf or NLIE)
 - ▶ compute conformal spectrum + density of states
 - ▶ identify CFT / non-compact σ -model
- ▶ Use DDV approach on NLIE
 - ▶ Construct+solve integrable perturbations
 - ▶ Describe integrable scattering th. on non-compact target space

General program

- ▶ Construct/classify integrable vertex models with singular Bethe-Ansatz kernel
- ▶ Study finite-size effects (with Wiener-Hopf or NLIE)
 - ▶ compute conformal spectrum + density of states
 - ▶ identify CFT / non-compact σ -model
- ▶ Use DDV approach on NLIE
 - ▶ Construct+solve integrable perturbations
 - ▶ Describe integrable scattering th. on non-compact target space
- ▶ Apply to physical problems: disordered phase transitions, AdS/CFT ...

2. The critical spin chain and its continuum limit

Staggered six-vertex model

► R -matrix:

$$R(u) = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & b(u) & e^{-iu}c & 0 \\ 0 & e^{+iu}c & b(u) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix}, \quad \begin{cases} a(u)=\sin(\gamma-u) \\ b(u)=\sin u \\ c=\sin \gamma \end{cases}$$

Staggered six-vertex model

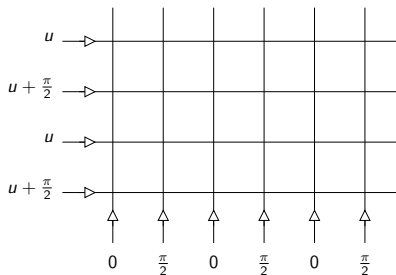
- ▶ R -matrix:

$$R(u) = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & b(u) & e^{-iu}c & 0 \\ 0 & e^{iu}c & b(u) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix}, \quad \begin{cases} a(u) = \sin(\gamma - u) \\ b(u) = \sin u \\ c = \sin \gamma \end{cases}$$

- ▶ Rapidity lines:

$$u \rightarrow \begin{array}{c} | \\ \rightarrow \\ | \\ \uparrow \\ | \end{array} := R(u - v)$$

- ▶ Staggered spectral parameters



Integrable structure

- ▶ Family of commuting transfer matrices

$$t(u) = \text{Tr}_0 [R_{0,2L}(u - \pi/2)R_{0,2L-1}(u) \dots R_{02}(u - \pi/2)R_{01}(u)]$$

Integrable structure

- ▶ Family of commuting transfer matrices

$$t(u) = \text{Tr}_0 [R_{0,2L}(u - \pi/2)R_{0,2L-1}(u) \dots R_{02}(u - \pi/2)R_{01}(u)]$$

- ▶ Conserved quantities

$$Q_{\text{ev}}^{(n)} = \left(\frac{d}{du}\right)^n \left[\log t(u)t\left(u + \frac{\pi}{2}\right) \right] \Big|_{u=0}$$

$$Q_{\text{odd}}^{(n)} = \left(\frac{d}{du}\right)^n \left[\log t^{-1}(u)t\left(u + \frac{\pi}{2}\right) \right] \Big|_{u=0}$$

Integrable structure

- ▶ Family of commuting transfer matrices

$$t(u) = \text{Tr}_0 [R_{0,2L}(u - \pi/2)R_{0,2L-1}(u) \dots R_{02}(u - \pi/2)R_{01}(u)]$$

- ▶ Conserved quantities

$$Q_{\text{ev}}^{(n)} = \left(\frac{d}{du}\right)^n \left[\log t(u)t\left(u + \frac{\pi}{2}\right) \right] \Big|_{u=0}$$

$$Q_{\text{odd}}^{(n)} = \left(\frac{d}{du}\right)^n \left[\log t^{-1}(u)t\left(u + \frac{\pi}{2}\right) \right] \Big|_{u=0}$$

- ▶ Important examples

Integrable structure

- ▶ Family of commuting transfer matrices

$$t(u) = \text{Tr}_0 [R_{0,2L}(u - \pi/2)R_{0,2L-1}(u) \dots R_{02}(u - \pi/2)R_{01}(u)]$$

- ▶ Conserved quantities

$$Q_{\text{ev}}^{(n)} = \left(\frac{d}{du}\right)^n \left[\log t(u)t\left(u + \frac{\pi}{2}\right) \right] \Big|_{u=0}$$

$$Q_{\text{odd}}^{(n)} = \left(\frac{d}{du}\right)^n \left[\log t^{-1}(u)t\left(u + \frac{\pi}{2}\right) \right] \Big|_{u=0}$$

- ▶ Important examples

- ▶ Hamiltonian $H = \frac{d}{du} \left[\log t(u)t\left(u + \frac{\pi}{2}\right) \right] \Big|_{u=0}$

$$H = \sum_{j=1}^{2L} \left[-\frac{1}{2} \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+2} + \sin^2 \gamma \sigma_j^z \sigma_{j+1}^z \right. \\ \left. + i \sin \gamma (\sigma_{j-1}^z - \sigma_{j+2}^z)(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) \right]$$

Integrable structure

- ▶ Family of commuting transfer matrices

$$t(u) = \text{Tr}_0 [R_{0,2L}(u - \pi/2)R_{0,2L-1}(u) \dots R_{02}(u - \pi/2)R_{01}(u)]$$

- ▶ Conserved quantities

$$Q_{\text{ev}}^{(n)} = \left(\frac{d}{du}\right)^n \left[\log t(u)t\left(u + \frac{\pi}{2}\right) \right] \Big|_{u=0}$$

$$Q_{\text{odd}}^{(n)} = \left(\frac{d}{du}\right)^n \left[\log t^{-1}(u)t\left(u + \frac{\pi}{2}\right) \right] \Big|_{u=0}$$

- ▶ Important examples

- ▶ Hamiltonian $H = \frac{d}{du} \left[\log t(u)t\left(u + \frac{\pi}{2}\right) \right] \Big|_{u=0}$

$$H = \sum_{j=1}^{2L} \left[-\frac{1}{2} \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+2} + \sin^2 \gamma \sigma_j^z \sigma_{j+1}^z \right. \\ \left. + i \sin \gamma (\sigma_{j-1}^z - \sigma_{j+2}^z)(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) \right]$$

- ▶ Momentum $e^{-iP} \propto t(0)t(\pi/2)$

Integrable structure

- ▶ Family of commuting transfer matrices

$$t(u) = \text{Tr}_0 [R_{0,2L}(u - \pi/2)R_{0,2L-1}(u) \dots R_{02}(u - \pi/2)R_{01}(u)]$$

- ▶ Conserved quantities

$$Q_{\text{ev}}^{(n)} = \left(\frac{d}{du}\right)^n \left[\log t(u)t\left(u + \frac{\pi}{2}\right) \right] \Big|_{u=0}$$

$$Q_{\text{odd}}^{(n)} = \left(\frac{d}{du}\right)^n \left[\log t^{-1}(u)t\left(u + \frac{\pi}{2}\right) \right] \Big|_{u=0}$$

- ▶ Important examples

- ▶ Hamiltonian $H = \frac{d}{du} \left[\log t(u)t\left(u + \frac{\pi}{2}\right) \right] \Big|_{u=0}$

$$H = \sum_{j=1}^{2L} \left[-\frac{1}{2} \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+2} + \sin^2 \gamma \sigma_j^z \sigma_{j+1}^z \right. \\ \left. + i \sin \gamma (\sigma_{j-1}^z - \sigma_{j+2}^z)(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) \right]$$

- ▶ Momentum $e^{-iP} \propto t(0)t(\pi/2)$

- ▶ “Quasi-momentum” $S := \log \left[t^{-1}(0)t\left(\frac{\pi}{2}\right) \right]$

Bethe-Ansatz solution (1/2)

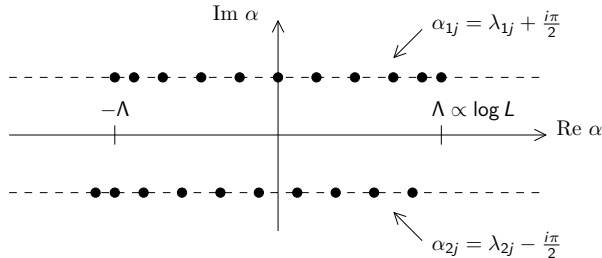
► BA equations:
$$\left[\frac{\sinh(\alpha_j + i\gamma)}{\sinh(\alpha_j - i\gamma)} \right]^L = - \prod_{\ell} \frac{\sinh(\frac{\alpha_j - \alpha_{\ell}}{2} + i\gamma)}{\sinh(\frac{\alpha_j - \alpha_{\ell}}{2} - i\gamma)}$$

Bethe-Ansatz solution (1/2)

- ▶ BA equations:
$$\left[\frac{\sinh(\alpha_j + i\gamma)}{\sinh(\alpha_j - i\gamma)} \right]^L = - \prod_{\ell} \frac{\sinh(\frac{\alpha_j - \alpha_{\ell}}{2} + i\gamma)}{\sinh(\frac{\alpha_j - \alpha_{\ell}}{2} - i\gamma)}$$
- ▶ Energy:
$$E = \sum_j \frac{2 \sin^2 2\gamma}{\cosh 2\alpha_j - \cos 2\gamma}$$

Bethe-Ansatz solution (1/2)

- ▶ BA equations:
$$\left[\frac{\sinh(\alpha_j + i\gamma)}{\sinh(\alpha_j - i\gamma)} \right]^L = - \prod_{\ell} \frac{\sinh(\frac{\alpha_j - \alpha_{\ell}}{2} + i\gamma)}{\sinh(\frac{\alpha_j - \alpha_{\ell}}{2} - i\gamma)}$$
- ▶ Energy:
$$E = \sum_j \frac{2 \sin^2 2\gamma}{\cosh 2\alpha_j - \cos 2\gamma}$$
- ▶ Low-energy states:



Bethe-Ansatz solution (2/2)

- ▶ BA equations and conserved quantities:

$$Lp(\lambda_{aj}) = 2\pi l_{aj} - \sum_{b,\ell} \theta[\lambda_{aj} - \lambda_{b\ell} + i\pi(a-b)]$$

$$E = \sum_{aj} \epsilon(\lambda_{aj}), \quad P = \sum_{aj} p(\lambda_{aj}), \quad S = \sum_{aj} (-1)^a s(\lambda_{aj})$$

Bethe-Ansatz solution (2/2)

- ▶ BA equations and conserved quantities:

$$Lp(\lambda_{aj}) = 2\pi I_{aj} - \sum_{b,\ell} \theta[\lambda_{aj} - \lambda_{b\ell} + i\pi(a-b)]$$

$$E = \sum_{a,j} \epsilon(\lambda_{aj}), \quad P = \sum_{a,j} p(\lambda_{aj}), \quad S = \sum_{a,j} (-1)^a s(\lambda_{aj})$$

- ▶ Continuum approximation:

Bethe-Ansatz solution (2/2)

- ▶ BA equations and conserved quantities:

$$Lp(\lambda_{aj}) = 2\pi I_{aj} - \sum_{b,\ell} \theta[\lambda_{aj} - \lambda_{b\ell} + i\pi(a-b)]$$

$$E = \sum_{a,j} \epsilon(\lambda_{aj}), \quad P = \sum_{a,j} p(\lambda_{aj}), \quad S = \sum_{a,j} (-1)^a s(\lambda_{aj})$$

- ▶ Continuum approximation:

- ▶ $\rho_a(\lambda) :=$ density of roots of type a

Bethe-Ansatz solution (2/2)

- ▶ BA equations and conserved quantities:

$$L\rho(\lambda_{aj}) = 2\pi l_{aj} - \sum_{b,\ell} \theta[\lambda_{aj} - \lambda_{b\ell} + i\pi(a-b)]$$

$$E = \sum_{a,j} \epsilon(\lambda_{aj}), \quad P = \sum_{a,j} p(\lambda_{aj}), \quad S = \sum_{a,j} (-1)^a s(\lambda_{aj})$$

- ▶ Continuum approximation:

- ▶ $\rho_a(\lambda) :=$ density of roots of type a

- ▶ BAE: $\rho'_a(\lambda) = 2\pi\rho_a(\lambda) - \sum_b \int d\mu K_{ab}(\lambda - \mu)\rho_b(\mu)$

Bethe-Ansatz solution (2/2)

- ▶ BA equations and conserved quantities:

$$L\rho(\lambda_{aj}) = 2\pi I_{aj} - \sum_{b,\ell} \theta[\lambda_{aj} - \lambda_{b\ell} + i\pi(a-b)]$$

$$E = \sum_{a,j} \epsilon(\lambda_{aj}), \quad P = \sum_{a,j} p(\lambda_{aj}), \quad S = \sum_{a,j} (-1)^a s(\lambda_{aj})$$

- ▶ Continuum approximation:

- ▶ $\rho_a(\lambda) :=$ density of roots of type a

- ▶ BAE: $\rho'_a(\lambda) = 2\pi\rho_a(\lambda) - \sum_b \int d\mu K_{ab}(\lambda - \mu)\rho_b(\mu)$

- ▶ Kernel matrix: $K_{ab}(\lambda) := \theta'[\lambda + i\pi(a-b)]$

Conformal spectrum

finite-size lattice BA solution



finite-size effects on energies



Wiener-Hopf method [Yang-Yang '66]



Conformal dimensions: $\{x_{\vec{e}, \vec{m}} \mid \vec{m} \in \mathbb{Z}^2, \vec{e} \in \mathbb{Z}^2\}$
 $x_{\vec{e}, \vec{m}} = \frac{1}{2} \vec{m} \cdot \hat{g}(0) \cdot \vec{m} + \frac{1}{2} \vec{e} \cdot \hat{g}^{-1}(0) \cdot \vec{e}$

“Coupling constants” : $\hat{g}_{ab}(\omega) = 2\pi\delta_{ab} - \hat{K}_{ab}(\omega)$

Effect of singularity

- Eigenvalues of matrix $\hat{g}(\omega)$:

$$\hat{g}_{\text{ev}}(\omega) = \frac{2 \sinh \gamma \omega \cosh(\pi/2 - \gamma)\omega}{\sinh \pi \omega / 2}, \quad \hat{g}_{\text{odd}}(\omega) = \frac{2 \sinh \gamma \omega \sinh(\pi/2 - \gamma)\omega}{\cosh \pi \omega / 2}$$

$$\Rightarrow \hat{g}_{\text{ev}}(0) = \frac{4\gamma}{\pi} := g,$$

$$\hat{g}_{\text{odd}}(0) = 0$$

Effect of singularity

- ▶ Eigenvalues of matrix $\widehat{g}(\omega)$:

$$\widehat{g}_{\text{ev}}(\omega) = \frac{2 \sinh \gamma \omega \cosh(\pi/2 - \gamma)\omega}{\sinh \pi \omega/2}, \quad \widehat{g}_{\text{odd}}(\omega) = \frac{2 \sinh \gamma \omega \sinh(\pi/2 - \gamma)\omega}{\cosh \pi \omega/2}$$

$$\Rightarrow \widehat{g}_{\text{ev}}(0) = \frac{4\gamma}{\pi} := g,$$

$$\widehat{g}_{\text{odd}}(0) = 0$$

- ▶ Conformal dimensions for finite \vec{e}, \vec{m} :

$$x_{e,m,\tilde{m}} = \frac{gm^2}{2} + \frac{e^2}{2g} + 0 \times \tilde{m}^2, \quad \begin{cases} e_1 = e_2 = e \\ m = m_1 + m_2 \\ \tilde{m} = m_1 - m_2 \end{cases}$$

Effect of singularity

- ▶ Eigenvalues of matrix $\widehat{g}(\omega)$:

$$\widehat{g}_{\text{ev}}(\omega) = \frac{2 \sinh \gamma \omega \cosh(\pi/2 - \gamma)\omega}{\sinh \pi\omega/2}, \quad \widehat{g}_{\text{odd}}(\omega) = \frac{2 \sinh \gamma \omega \sinh(\pi/2 - \gamma)\omega}{\cosh \pi\omega/2}$$

$$\Rightarrow \widehat{g}_{\text{ev}}(0) = \frac{4\gamma}{\pi} := g,$$

$$\widehat{g}_{\text{odd}}(0) = 0$$

- ▶ Conformal dimensions for finite \vec{e}, \vec{m} :

$$x_{e,m,\tilde{m}} = \frac{gm^2}{2} + \frac{e^2}{2g} + 0 \times \tilde{m}^2, \quad \begin{cases} e_1 = e_2 = e \\ m = m_1 + m_2 \\ \tilde{m} = m_1 - m_2 \end{cases}$$

- ▶ Correct scaling limit:

Effect of singularity

- ▶ Eigenvalues of matrix $\widehat{g}(\omega)$:

$$\widehat{g}_{\text{ev}}(\omega) = \frac{2 \sinh \gamma \omega \cosh(\pi/2 - \gamma)\omega}{\sinh \pi\omega/2}, \quad \widehat{g}_{\text{odd}}(\omega) = \frac{2 \sinh \gamma \omega \sinh(\pi/2 - \gamma)\omega}{\cosh \pi\omega/2}$$

$$\Rightarrow \widehat{g}_{\text{ev}}(0) = \frac{4\gamma}{\pi} := g,$$

$$\widehat{g}_{\text{odd}}(0) = 0$$

- ▶ Conformal dimensions for finite \vec{e}, \vec{m} :

$$x_{e,m,\tilde{m}} = \frac{gm^2}{2} + \frac{e^2}{2g} + 0 \times \tilde{m}^2, \quad \begin{cases} e_1 = e_2 = e \\ m = m_1 + m_2 \\ \tilde{m} = m_1 - m_2 \end{cases}$$

- ▶ Correct scaling limit:

- ▶ $L \rightarrow \infty$, (e, m) finite, $\tilde{m} \rightarrow \infty$, s finite

Effect of singularity

- ▶ Eigenvalues of matrix $\widehat{g}(\omega)$:

$$\widehat{g}_{\text{ev}}(\omega) = \frac{2 \sinh \gamma \omega \cosh(\pi/2 - \gamma)\omega}{\sinh \pi \omega / 2}, \quad \widehat{g}_{\text{odd}}(\omega) = \frac{2 \sinh \gamma \omega \sinh(\pi/2 - \gamma)\omega}{\cosh \pi \omega / 2}$$

$$\Rightarrow \widehat{g}_{\text{ev}}(0) = \frac{4\gamma}{\pi} := g,$$

$$\widehat{g}_{\text{odd}}(0) = 0$$

- ▶ Conformal dimensions for finite \vec{e}, \vec{m} :

$$x_{e,m,\tilde{m}} = \frac{gm^2}{2} + \frac{e^2}{2g} + 0 \times \tilde{m}^2, \quad \begin{cases} e_1 = e_2 = e \\ m = m_1 + m_2 \\ \tilde{m} = m_1 - m_2 \end{cases}$$

- ▶ Correct scaling limit:

- ▶ $L \rightarrow \infty$, (e, m) finite, $\tilde{m} \rightarrow \infty$, s finite

- ▶ Careful WH or NLIE $\Rightarrow x_{e,m,s} = \frac{gm^2}{2} + \frac{e^2}{2g} + \frac{\gamma s^2}{\pi - 2\gamma}$

Effect of singularity

- ▶ Eigenvalues of matrix $\widehat{g}(\omega)$:

$$\widehat{g}_{\text{ev}}(\omega) = \frac{2 \sinh \gamma \omega \cosh(\pi/2 - \gamma)\omega}{\sinh \pi\omega/2}, \quad \widehat{g}_{\text{odd}}(\omega) = \frac{2 \sinh \gamma \omega \sinh(\pi/2 - \gamma)\omega}{\cosh \pi\omega/2}$$

$$\Rightarrow \widehat{g}_{\text{ev}}(0) = \frac{4\gamma}{\pi} := g,$$

$$\widehat{g}_{\text{odd}}(0) = 0$$

- ▶ Conformal dimensions for finite \vec{e}, \vec{m} :

$$x_{e,m,\tilde{m}} = \frac{gm^2}{2} + \frac{e^2}{2g} + 0 \times \tilde{m}^2, \quad \begin{cases} e_1 = e_2 = e \\ m = m_1 + m_2 \\ \tilde{m} = m_1 - m_2 \end{cases}$$

- ▶ Correct scaling limit:

- ▶ $L \rightarrow \infty$, (e, m) finite, $\tilde{m} \rightarrow \infty$, s finite

- ▶ Careful WH or NLIE $\Rightarrow x_{e,m,s} = \frac{gm^2}{2} + \frac{e^2}{2g} + \frac{\gamma s^2}{\pi - 2\gamma}$

- ▶ $\tilde{m} = \frac{4s}{\pi}(\log L + B)$, $\tilde{m} \in 2\mathbb{Z} + m$

Identification of the CFT

- Conformal dimensions ($\gamma = \pi/k$, $k \in]2, +\infty[$):

$$x_{e,m,s} = \frac{m^2}{2k} + \frac{ke^2}{2} + \frac{s^2}{k-2}$$

Same spectrum as WZW model on $SL(2, \mathbb{R})/U(1)$!

Identification of the CFT

- ▶ Conformal dimensions ($\gamma = \pi/k$, $k \in]2, +\infty[$):

$$x_{e,m,s} = \frac{m^2}{2k} + \frac{ke^2}{2} + \frac{s^2}{k-2}$$

Same spectrum as WZW model on $SL(2, \mathbb{R})/U(1)$!

- ▶ k =level, (e, m) =exc. in compact direction, s =string momentum

Identification of the CFT

- ▶ Conformal dimensions ($\gamma = \pi/k$, $k \in]2, +\infty[$):

$$x_{e,m,s} = \frac{m^2}{2k} + \frac{ke^2}{2} + \frac{s^2}{k-2}$$

Same spectrum as WZW model on $SL(2, \mathbb{R})/U(1)$!

- ▶ k =level, (e, m) =exc. in compact direction, s =string momentum
- ▶ Density of states: $\rho(s) = \frac{2}{\pi} [\log L + \partial_s(sB)]$

Identification of the CFT

- ▶ Conformal dimensions ($\gamma = \pi/k$, $k \in]2, +\infty[$):

$$x_{e,m,s} = \frac{m^2}{2k} + \frac{ke^2}{2} + \frac{s^2}{k-2}$$

Same spectrum as WZW model on $SL(2, \mathbb{R})/U(1)$!

- ▶ k =level, (e, m) =exc. in compact direction, s =string momentum
- ▶ Density of states: $\rho(s) = \frac{2}{\pi} [\log L + \partial_s(sB)]$
 - ▶ Numerical solution of NLIE $\rightarrow B_{6V}(s)$

Identification of the CFT

- ▶ Conformal dimensions ($\gamma = \pi/k, k \in]2, +\infty[$):

$$x_{e,m,s} = \frac{m^2}{2k} + \frac{ke^2}{2} + \frac{s^2}{k-2}$$

Same spectrum as WZW model on $SL(2, \mathbb{R})/U(1)$!

- ▶ k =level, (e, m) =exc. in compact direction, s =string momentum
- ▶ Density of states: $\rho(s) = \frac{2}{\pi} [\log L + \partial_s(sB)]$
 - ▶ Numerical solution of NLIE $\rightarrow B_{6V}(s)$
 - ▶ Analytic calculation in $SL(2, \mathbb{R})/U(1)$:

$$\rho(s) = \frac{2}{\pi} [-\log \epsilon + \partial_s(sB)]$$

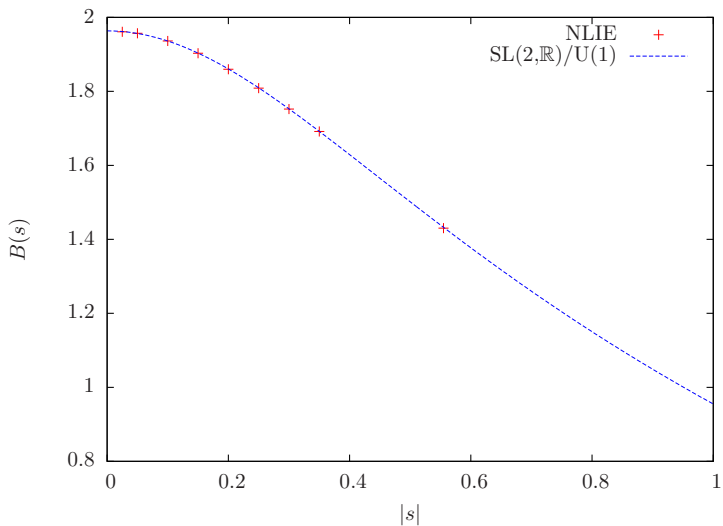
$$B(s) = \frac{1}{2s} \text{Im} \log \left[\Gamma \left(\frac{1-m+ek}{2} - is \right) \Gamma \left(\frac{1-m-ek}{2} - is \right) \right]$$

[Maldacena-Ooguri '01]

[Hanany-Prezas-Troost '02]

[Israel-Pakman-Troost '04]

Density of states



3. Non-linear integral equations

Counting functions

- ▶ BA equations:

$$L\rho(\lambda_{aj}) = 2\pi I_{aj} - \sum_{b,\ell} \theta_{ab}(\lambda_{aj} - \lambda_{b\ell})$$

Counting functions

- ▶ BA equations:

$$L\rho(\lambda_{aj}) = 2\pi I_{aj} - \sum_{b,\ell} \theta_{ab}(\lambda_{aj} - \lambda_{b\ell})$$

- ▶ Bethe integers:

$$I_{aj} \in \frac{n_a+1}{2} + \mathbb{Z} \quad (n_a = \# \text{ roots}_a)$$

Counting functions

- ▶ BA equations:

$$Lp(\lambda_{aj}) = 2\pi I_{aj} - \sum_{b,\ell} \theta_{ab}(\lambda_{aj} - \lambda_{b\ell})$$

- ▶ Bethe integers:

$$I_{aj} \in \frac{n_a+1}{2} + \mathbb{Z} \quad (n_a = \# \text{ roots}_a)$$

- ▶ Counting functions:

$$Z_a(\lambda) := Lp(\lambda) + \sum_{b,\ell} \theta_{ab}(\lambda - \lambda_{b\ell})$$

Counting functions

- ▶ BA equations:
$$Lp(\lambda_{aj}) = 2\pi I_{aj} - \sum_{b,\ell} \theta_{ab}(\lambda_{aj} - \lambda_{b\ell})$$
- ▶ Bethe integers:
$$I_{aj} \in \frac{n_a+1}{2} + \mathbb{Z} \quad (n_a = \# \text{ roots}_a)$$
- ▶ Counting functions:
$$Z_a(\lambda) := Lp(\lambda) + \sum_{b,\ell} \theta_{ab}(\lambda - \lambda_{b\ell})$$
- ▶ BA equations \Leftrightarrow
$$Z_a(\lambda_{aj})/(2\pi) = I_{aj}$$

Counting functions

- ▶ BA equations:
$$L\rho(\lambda_{aj}) = 2\pi I_{aj} - \sum_{b,\ell} \theta_{ab}(\lambda_{aj} - \lambda_{b\ell})$$
- ▶ Bethe integers:
$$I_{aj} \in \frac{n_a+1}{2} + \mathbb{Z} \quad (n_a = \# \text{ roots}_a)$$
- ▶ Counting functions:
$$Z_a(\lambda) := L\rho(\lambda) + \sum_{b,\ell} \theta_{ab}(\lambda - \lambda_{b\ell})$$
- ▶ BA equations $\Leftrightarrow Z_a(\lambda_{aj})/(2\pi) = I_{aj}$
- ▶ Additional (half-)integer values at holes:

$$Z_a(\eta_{aj})/(2\pi) = I_{aj}^h, \quad \eta_{aj} \in \mathbb{R}$$

Finite-size NLIE

- ▶ Inverse kernels

Finite-size NLIE

- ▶ Inverse kernels

- ▶ $H_{\text{ev}}(\lambda) = \int \frac{d\omega}{i\omega} e^{i\lambda\omega} \frac{\widehat{K}_{\text{ev}}(\omega)}{2\pi - \widehat{K}_{\text{ev}}(\omega)}$ $H_{\text{ev}}(\lambda) \underset{\lambda \rightarrow \pm\infty}{\sim} e^{-q|\lambda|}$

Finite-size NLIE

► Inverse kernels

$$\begin{aligned} \text{► } H_{\text{ev}}(\lambda) &= \int \frac{d\omega}{i\omega} e^{i\lambda\omega} \frac{\widehat{K}_{\text{ev}}(\omega)}{2\pi - \widehat{K}_{\text{ev}}(\omega)} & H_{\text{ev}}(\lambda) \underset{\lambda \rightarrow \pm\infty}{\sim} e^{-q|\lambda|} \\ \text{► } H_{\text{odd}}(\lambda) &= \int \frac{d\omega}{i\omega} e^{i\lambda\omega} \frac{\widehat{K}_{\text{odd}}(\omega)}{2\pi - \widehat{K}_{\text{odd}}(\omega)} & H_{\text{odd}}(\lambda) \underset{\lambda \rightarrow \pm\infty}{\sim} \mp \alpha \lambda^2 \end{aligned}$$

Finite-size NLIE

► Inverse kernels

- $H_{\text{ev}}(\lambda) = \int \frac{d\omega}{i\omega} e^{i\lambda\omega} \frac{\widehat{K}_{\text{ev}}(\omega)}{2\pi - \widehat{K}_{\text{ev}}(\omega)}$ $H_{\text{ev}}(\lambda) \underset{\lambda \rightarrow \pm\infty}{\sim} e^{-q|\lambda|}$
- $H_{\text{odd}}(\lambda) = \int \frac{d\omega}{i\omega} e^{i\lambda\omega} \frac{\widehat{K}_{\text{odd}}(\omega)}{2\pi - \widehat{K}_{\text{odd}}(\omega)}$ $H_{\text{odd}}(\lambda) \underset{\lambda \rightarrow \pm\infty}{\sim} \mp \alpha \lambda^2$
- $H_{ab}(\lambda) := H_{\text{ev}}(\lambda) + (-1)^{a-b} H_{\text{odd}}(\lambda)$

Finite-size NLIE

► Inverse kernels

$$\text{► } H_{\text{ev}}(\lambda) = \int \frac{d\omega}{i\omega} e^{i\lambda\omega} \frac{\widehat{K}_{\text{ev}}(\omega)}{2\pi - \widehat{K}_{\text{ev}}(\omega)} \quad H_{\text{ev}}(\lambda) \underset{\lambda \rightarrow \pm\infty}{\sim} e^{-q|\lambda|}$$

$$\text{► } H_{\text{odd}}(\lambda) = \int \frac{d\omega}{i\omega} e^{i\lambda\omega} \frac{\widehat{K}_{\text{odd}}(\omega)}{2\pi - \widehat{K}_{\text{odd}}(\omega)} \quad H_{\text{odd}}(\lambda) \underset{\lambda \rightarrow \pm\infty}{\sim} \mp \alpha \lambda^2$$

$$\text{► } H_{ab}(\lambda) := H_{\text{ev}}(\lambda) + (-1)^{a-b} H_{\text{odd}}(\lambda)$$

- Non-linear functions: $U_a(\lambda) := \log [1 + (-1)^{n_a} e^{iZ_a(\lambda)}]$
defined for $0 < \text{Im}\lambda < \delta$

Finite-size NLIE

► Inverse kernels

$$\text{► } H_{\text{ev}}(\lambda) = \int \frac{d\omega}{i\omega} e^{i\lambda\omega} \frac{\widehat{K}_{\text{ev}}(\omega)}{2\pi - \widehat{K}_{\text{ev}}(\omega)} \quad H_{\text{ev}}(\lambda) \Big|_{\lambda \rightarrow \pm\infty} \sim e^{-q|\lambda|}$$

$$\text{► } H_{\text{odd}}(\lambda) = \int \frac{d\omega}{i\omega} e^{i\lambda\omega} \frac{\widehat{K}_{\text{odd}}(\omega)}{2\pi - \widehat{K}_{\text{odd}}(\omega)} \quad H_{\text{odd}}(\lambda) \Big|_{\lambda \rightarrow \pm\infty} \sim \mp \alpha \lambda^2$$

$$\text{► } H_{ab}(\lambda) := H_{\text{ev}}(\lambda) + (-1)^{a-b} H_{\text{odd}}(\lambda)$$

► Non-linear functions: $U_a(\lambda) := \log [1 + (-1)^{n_a} e^{iZ_a(\lambda)}]$ defined for $0 < \text{Im}\lambda < \delta$

► Exact NLIEs:

$$Z_a(\lambda) = L\sigma(\lambda) + C - \sum_{b,\ell} H_{ab}(\lambda - \eta_{b\ell})$$

$$- \frac{1}{2i\pi} \sum_b \left[\int_{\Gamma_>} d\mu H_{ab}(\lambda - \mu) U'_b(\mu) - \int_{\Gamma_<} d\mu H_{ab}(\lambda - \mu) \overline{U}'_b(\mu) \right]$$

Scaling limit

- ▶ Scaling regime: $\lambda_{aj} \simeq \pm\Lambda + \lambda_{aj}^{\pm}$

Scaling limit

- ▶ Scaling regime: $\lambda_{aj} \simeq \pm\Lambda + \lambda_{aj}^{\pm}$
- ▶ Modified kernel $\tilde{H}_{\text{odd}}(\lambda) := H_{\text{odd}}(\lambda) - \alpha\lambda^2$

Scaling limit

- ▶ Scaling regime: $\lambda_{aj} \simeq \pm\Lambda + \lambda_{aj}^{\pm}$
- ▶ Modified kernel $\tilde{H}_{\text{odd}}(\lambda) := H_{\text{odd}}(\lambda) - \alpha\lambda^2$
- ▶ $\tilde{H}_{\text{odd}}(\lambda) \underset{\lambda \rightarrow -\infty}{\sim} e^{-q|\lambda|}$ $\tilde{H}_{\text{odd}}(\lambda) \underset{\lambda \rightarrow +\infty}{\sim} -2\alpha\lambda^2$

Scaling limit

- ▶ Scaling regime: $\lambda_{aj} \simeq \pm\Lambda + \lambda_{aj}^{\pm}$
- ▶ Modified kernel $\tilde{H}_{\text{odd}}(\lambda) := H_{\text{odd}}(\lambda) - \alpha\lambda^2$
- ▶ $\tilde{H}_{\text{odd}}(\lambda) \underset{\lambda \rightarrow -\infty}{\sim} e^{-q|\lambda|}$ $\tilde{H}_{\text{odd}}(\lambda) \underset{\lambda \rightarrow +\infty}{\sim} -2\alpha\lambda^2$
- ▶ Integrate by parts \Rightarrow extract **divergent part** of $H_{ab}(\lambda - \mu)$

Scaling limit

- ▶ Scaling regime: $\lambda_{aj} \simeq \pm\Lambda + \lambda_{aj}^{\pm}$
- ▶ Modified kernel $\tilde{H}_{\text{odd}}(\lambda) := H_{\text{odd}}(\lambda) - \alpha\lambda^2$
- ▶ $\tilde{H}_{\text{odd}}(\lambda) \underset{\lambda \rightarrow -\infty}{\sim} e^{-q|\lambda|}$ $\tilde{H}_{\text{odd}}(\lambda) \underset{\lambda \rightarrow +\infty}{\sim} -2\alpha\lambda^2$
- ▶ Integrate by parts \Rightarrow extract **divergent part** of $H_{ab}(\lambda - \mu)$
- ▶ Obtain scaling NLIEs:

$$Z_a^+(\lambda) = -2e^{-\lambda} + C + (-1)^a (s\lambda + B) - \sum_{b,\ell} \tilde{H}_{ab}(\lambda - \eta_{b\ell}^+) \\ - \frac{1}{i} \sum_b \left[\int_{\Gamma_>} d\mu \tilde{H}'_{ab}(\lambda - \mu) U_b^+(\mu) - \int_{\Gamma_<} d\mu \tilde{H}'_{ab}(\lambda - \mu) \bar{U}_b^+(\mu) \right]$$

Scaling limit

- ▶ Scaling regime: $\lambda_{aj} \simeq \pm\Lambda + \lambda_{aj}^{\pm}$
- ▶ Modified kernel $\tilde{H}_{\text{odd}}(\lambda) := H_{\text{odd}}(\lambda) - \alpha\lambda^2$
- ▶ $\tilde{H}_{\text{odd}}(\lambda) \underset{\lambda \rightarrow -\infty}{\sim} e^{-q|\lambda|}$ $\tilde{H}_{\text{odd}}(\lambda) \underset{\lambda \rightarrow +\infty}{\sim} -2\alpha\lambda^2$
- ▶ Integrate by parts \Rightarrow extract **divergent part** of $H_{ab}(\lambda - \mu)$
- ▶ Obtain scaling NLIEs:

$$Z_a^+(\lambda) = -2e^{-\lambda} + C + (-1)^a (s\lambda + B) - \sum_{b,l} \tilde{H}_{ab}(\lambda - \eta_{bl}^+) \\ - \frac{1}{i} \sum_b \left[\int_{\Gamma_>} d\mu \tilde{H}'_{ab}(\lambda - \mu) U_b^+(\mu) - \int_{\Gamma_<} d\mu \tilde{H}'_{ab}(\lambda - \mu) \bar{U}_b^+(\mu) \right]$$

- ▶ Density of states $B \equiv$ integration constant in NLIE!

Conclusion

Conclusion

- ▶ Analytical sol. of lattice model \Rightarrow continuous CFT spectrum

Conclusion

- ▶ Analytical sol. of lattice model \Rightarrow continuous CFT spectrum
- ▶ Identify scaling limit $\equiv \text{SL}(2, \mathbb{R})/\text{U}(1)$

Conclusion

- ▶ Analytical sol. of lattice model \Rightarrow continuous CFT spectrum
- ▶ Identify scaling limit $\equiv \text{SL}(2, \mathbb{R})/\text{U}(1)$
- ▶ Setup NLIE in the presence of singular kernel

Conclusion

- ▶ Analytical sol. of lattice model \Rightarrow continuous CFT spectrum
- ▶ Identify scaling limit $\equiv \text{SL}(2, \mathbb{R})/\text{U}(1)$
- ▶ Setup NLIE in the presence of singular kernel
- ▶ Numerics on NLIE \Rightarrow density of states

Conclusion

- ▶ Analytical sol. of lattice model \Rightarrow continuous CFT spectrum
- ▶ Identify scaling limit $\equiv \text{SL}(2, \mathbb{R})/\text{U}(1)$
- ▶ Setup NLIE in the presence of singular kernel
- ▶ Numerics on NLIE \Rightarrow density of states

- ▶ Massive perturbations of non-compact models?
[Lukyanov, Zamolodchikov, Onofri, Fateev ...]

Conclusion

- ▶ Analytical sol. of lattice model \Rightarrow continuous CFT spectrum
- ▶ Identify scaling limit $\equiv \text{SL}(2, \mathbb{R})/\text{U}(1)$
- ▶ Setup NLIE in the presence of singular kernel
- ▶ Numerics on NLIE \Rightarrow density of states

- ▶ Massive perturbations of non-compact models?
[Lukyanov, Zamolodchikov, Onofri, Fateev ...]
- ▶ More lattice models with singular kernels?

Thank you for your attention!