Random maps and their scaling limits

Grégory Miermont

Abstract. We review some aspects of scaling limits of random planar maps, which can be considered as a model of a continuous random surface, and have driven much interest in the recent years. As a start, we will treat in a relatively detailed fashion the well-known convergence of uniform plane trees to the Brownian Continuum Random Tree. We will put a special emphasis on the fractal properties of the random metric spaces that are involved, by giving a detailed proof of the calculation of the Hausdorff dimension of the scaling limits.

Mathematics Subject Classification (2000). 60C05, 60F17.

Keywords. Random maps, random trees, scaling limits, Brownian CRT, random metric spaces, Hausdorff dimension.

1. Introduction

A planar map is an embedding of a finite, connected graph (loops and multiple edges are allowed) into the 2-dimensional sphere. A planar map determines *faces*, which are the connected components of the complementary of the union of edges. The set of edges, vertices, and faces of the map $\mathbf{m}$ are denoted by $E(\mathbf{m}), V(\mathbf{m}), F(\mathbf{m})$. The Euler Formula asserts that

$$\#V(\mathbf{m}) - \#E(\mathbf{m}) + \#F(\mathbf{m}) = 2\,. \tag{1.1}$$

A map is said to be *rooted* if one of its oriented edges, called the root edge, is distinguished. The origin of the root edge is called the root vertex. Two (rooted) maps are systematically identified if there exists an orientation-preserving homeomorphism of the sphere that corresponds the two embedded graphs (and the roots). With these identifications, maps are combinatorial objects, which can be understood as the non-equivalent ways of gluing by pairs the edges of a finite set of polygons, so that the resulting surface is the sphere. See [50, Chapter 3] for a detailed discussion. The number of edges of a polygon is called the *degree* of the corresponding face in the map: it is the number of edges incident to the face, where edges incident to only one face are counted twice (once for each orientation).

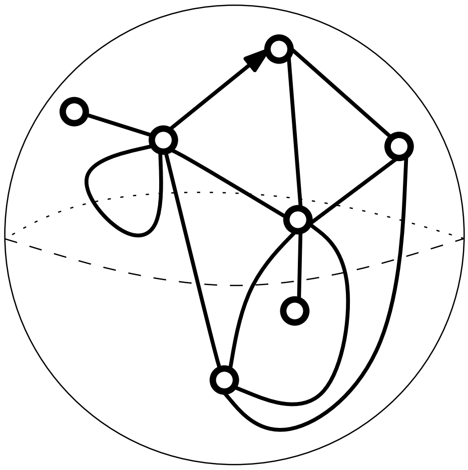

FIGURE 1. A rooted planar map with faces of degrees $1, 3, 4, 7$

Familiar examples of maps are triangulations, where all faces have degree 3, and quadrangulations, where faces have degree 4.

The theory of (planar) maps takes its roots in graph theory, with the 4-color theorem, and has developed considerably in other branches of mathematics. Tutte [54] founded the combinatorial study of planar maps by developing methods to solve the equations satisfied by the associated generating functions. It was then noticed by theoretical physicists, starting from 't Hooft [32] and Brézin, Parisi, Itzykson and Zuber [16], that the generating functions of maps can be interpreted as certain matrix integrals. This initiated an extremely abundant literature on (colored) graph enumeration, and has deep connections with statistical physics, representation theory and algebraic geometry, see the book by Lando and Zvonkin [35] and the recent article [30].

In the last few years, there has been also a growing interest in understanding the geometric structure of a randomly chosen map. This was partly motivated by the so-called 2-dimensional *quantum gravity theory* arising in theoretical physics, in which ill-defined measures over spaces of surfaces are considered. There are several attempts to make such theories mathematically rigorous. One of them, called Liouville theory, is to extend the language of Riemannian geometry to (very irregular) random fields, see for instance [24] for motivation. It is however not understood at present how to obtain well-defined random metric spaces with this approach.

Another approach [5, 4] is to consider discrete versions of surfaces, a role that is naturally performed by maps, and to take *scaling limits*. Scaling limits are relatively common when dealing with combinatorial aspects of probability theory: one chooses an object at random inside a class of discrete objects, and as the size of the object grows, the latter approaches, once suitably normalized, a continuous structure. The most familiar situation is to consider the Wiener measure, i.e. the law of Brownian motion, as the scaling limit of the law of the simple random walk. This legitimates viewing the Wiener measure as the "uniform measure over continuous paths". The latter has a *universal* character, in the sense that any centered random walk whose i.i.d. steps have a finite variance also admit Brownian motion as a scaling limit, according to the Donsker invariance principle. Another well-known study of scaling limits of discrete structures is that of random trees, for instance uniform random plane trees, which are known to converge to Aldous' Brownian Continuum Random Tree (CRT) [1, 2, 3]. In turn, this is a universal limit for many models of trees, e.g. arising from branching processes.

One can attempt to follow the same approach for maps: consider a large random planar map, say with uniform distribution among the set of quadrangulations of the sphere with n faces, and endow the set of its vertices with the graph distance. This means that the distance between two vertices is the minimal number of edges needed to link them. This yields a random, finite metric space. As the number of faces grows larger, the typical distances in the map are expected to grow like a power of n, which turns out to be $n^{1/4}$ as we will see below (Theorem 4.1). Therefore, one tries to understand the limiting behavior of the map where graph distances are all multiplied by $n^{-1/4}$. In some sense, it is expected that these random metric spaces converge to a limiting random surface, the so-called *Brownian map*.

This last approach turns out to be mathematically tractable thanks to powerful bijective encodings of maps, initiated by Schaeffer [52], and taking their roots in the work of Cori and Vauquelin [22] and Arquès [8]. With this method, the (hard) study of maps is amenable to the (simpler) study of certain decorated trees, in such a way that crucial geometric information of the maps, like graph distances between vertices, are encoded in a convenient way in the underlying tree structure. The extensive study of random trees in the probabilistic literature allows to understand in a detailed way the structure of large random planar maps.

This line of reasoning was first explored by Chassaing and Schaeffer [21]. This was pursued by Marckert and Mokkadem [43], who introduced a natural scaling limit for random quadrangulations, while Marckert, Miermont and Weill [42, 55, 46, 49] studied the universal aspects of these results, building on the powerful generalization of Schaeffer's bijection due to Bouttier, Di Francesco and Guitter [13]. In two important papers, Le Gall [38] and Le Gall and Paulin [39] studied in detail the fractal and topological structure of the scaling limits of random quadrangulations (and more generally 2κ-angulations). Further properties on the geodesics in the limit are known [45, 40]. Hence, at present, there is a relatively good understanding of the problem of scaling limits of planar (and non-planar) maps, even though many open questions remain, the most important (Conjecture 4.5) being a problem of identification of the limiting space. More precisely, the best that can be shown at present is, using a relative compactness argument, that scaling limits exist *up to taking extractions*, and the properties that are known up to now are valid for any scaling limit. The question of uniqueness for the scaling limit appears in Schramm [53].

In the present survey, we will introduce some of the basic and more elaborate results in this vein, focusing essentially on the particular case of random uniform quadrangulations of the sphere with n faces. It turns out to be the simplest model that can be shown to converge, up to extraction, to a limiting continuous structure. Like Brownian motion, these "Brownian map" limits have a singular, fractal structure, and we will put a special emphasis on the derivation of the Hausdorff dimension.

We also mention that a related, but different approach of random maps exists in the literature. One can also consider *local limits*, in which the convergence of arbitrary large but finite neighborhoods of the root of a large random planar map is studied. Here, the lengths are not normalized, and neighboring vertices remain at distance 1, so that the objects arising in the limit are still discrete (but infinite) structures. These infinite limiting random graphs have driven much interest [7, 6, 20, 34]. However, we will not cover this approach here.

The rest of the paper is organized as follows. Section 2 gathers the combinatorial tools that are required in the study of quadrangulations, and we give a detailed construction of the Schaeffer bijection. In Sect. 3, we will focus on the scaling limits of labeled trees, which are the key tools to understand scaling limits of quadrangulations. In Sect. 4, we discuss the construction of the scaling limits of random quadrangulations, discuss some of its most important properties and give a detailed computation of the Hausdorff dimension. Finally, Sect. 5 gathers developments on more general situations (more general families of maps, higher genera), some recent results, and open questions.

2. Coding quadrangulations with trees

2.1. Notations

Let

$$\mathcal{U} = \bigcup_{n\geq 0} \mathbb{N}^n \,,$$

where $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}^0 = \{\varnothing\}$ by convention. We denote by $u = u_1 \ldots u_n$ a word in $\mathbb{N}^n$, and call $|u| = n$ the *height* of u. The concatenation of the words u, v is denoted by $uv = u_1 \ldots u_{|u|} v_1 \ldots v_{|v|}$. We say that u is a prefix of v, if there exists w such that $v = uw$. For $u \neq \varnothing$, the maximal strict prefix of u is called the *parent* of u. For $u, v \in \mathcal{U}$, the common prefix of u, v with maximal length is denoted by $u \wedge v$, and called the most recent common ancestor to u, v.

A *rooted plane tree* is a finite subset $\mathbf{t}$ of $\mathcal{U}$ such that

- $\varnothing \in \mathbf{t}$
- if $u \in \mathbf{t}$ and $u \neq \varnothing$, then the parent of u is in $\mathbf{t}$
- if $u \in \mathbf{t}$ and $i \geq 1$ is such that $ui \in \mathbf{t}$, then $uj \in \mathbf{t}$ for every $1 \leq j \leq i$.

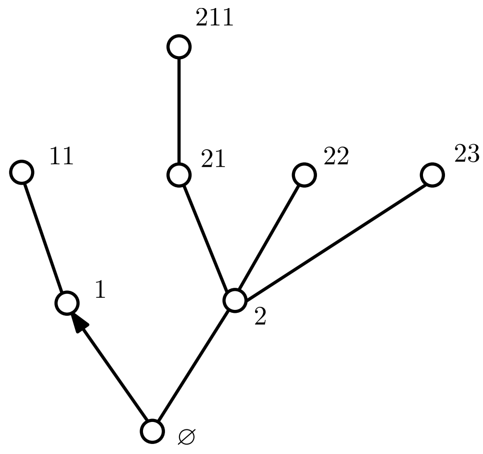

FIGURE 2. The rooted plane tree $\{\varnothing, 1, 11, 2, 21, 211, 22, 23\}$

The elements of $\mathbf{t}$ are called *vertices*. The maximal $i \geq 1$ such that $ui \in \mathbf{t}$ is called the number of children of u, and denoted by $c_u(\mathbf{t}) \geq 0$. The word $\varnothing$ is called the root vertex of $\mathbf{t}$. We let $\mathbf{T}_n$ be the set of rooted plane trees with $n+1$ vertices.

If a rooted plane tree $\mathbf{t} \in \mathbf{T}_n, n \geq 1$ is given, one can turn it into a planar map by drawing edges between each $u \in \mathbf{t}$ and its children, so that the edges between u and $u1, \dots, uc_u(\mathbf{t})$ appear in this order when turning clockwise around u, and are followed by the edge between u and its parent whenever $u \neq \varnothing$. This embedding is naturally rooted at the oriented edge pointing from $\varnothing$ to 1, and yields a rooted planar map with one face by the Jordan Curve Theorem, since the underlying graph is connected and has no loops. These two points of view turn out to be equivalent, so that we will also refer to $\mathbf{T}_n$ as the set of trees with n edges. This is summed up in Figure 2.

A *labeled* plane tree (with n edges) is a pair of the form $(\mathbf{t}, \ell)$, where $\mathbf{t} \in \mathbf{T}_n$ and ℓ is a labeling function defined on the set of vertices of $\mathbf{t}$, with values in $\mathbb{Z}$, and such that $\ell(\varnothing) = 0$ and $|\ell(u) - \ell(ui)| \leq 1$ whenever $u \in \mathbf{t}$ and $1 \leq i \leq c_u(\mathbf{t})$. Let $\mathbb{T}_n$ be the set of labeled plane trees with n edges. The cardinality of this set equals

$$|\mathbb{T}_n| = 3^n |\mathbf{T}_n| = 3^n \frac{1}{n+1}\binom{2n}{n}. \tag{2.1}$$

Indeed, since $\ell(\varnothing) = 0$ is fixed, choosing a labeling for the tree $\mathbf{t}$ is equivalent to choosing the increments of ℓ along the n edges of $\mathbf{t}$, i.e. the quantities $\ell(ui) - \ell(u) \in \{-1, 0, 1\}$, where $u \in \mathbf{t}$ is a vertex of $\mathbf{t}$ and $1 \leq i \leq c_u(\mathbf{t})$.

2.2. The Schaeffer bijection

With every labeled tree $(\mathbf{t}, \ell)$, we want to associate a planar quadrangulation. To this end, we introduce the so-called *contour* exploration of $\mathbf{t}$. Let $\varphi(0) = \varnothing$ be the root vertex of $\mathbf{t}$, and given $\varphi(0), \dots, \varphi(i)$ have been constructed, let $\varphi(i+1)$ be the first child of $\varphi(i)$ that does not belong to $\{\varphi(0), \dots, \varphi(i)\}$, if any, otherwise, $\varphi(i+1)$ is the parent of $\varphi(i)$. At step $i = 2n$, all vertices have been visited and $\varphi(2n) = \varnothing$. To see this, note that the oriented edges e_i pointing from $\varphi(i)$ to $\varphi(i+1)$, for $0 \le i \le 2n-1$, are an enumeration of the $2n$ oriented edges of $\mathbf{t}$. They form a path that "wraps around" $\mathbf{t}$ in clockwise order, starting from the root. It is convenient to extend the sequence $(\varphi(i), 0 \le i \le 2n)$ by periodicity, by letting $\varphi(i) = \varphi(i-2n)$ whenever $i > 2n$.

Take a particular planar representation of the tree $\mathbf{t}$, as in Figure 3, and add an extra vertex v_*, not belonging to the union of edges of $\mathbf{t}$. This extra vertex is assigned label $\ell(v_*) = \min_{u \in \mathbf{t}} \ell(u) - 1$.

Now, for every i such that $0 \le i \le 2n-1$, we let

$$s(i) = \inf\{j \ge i : \ell(\varphi(j)) = \ell(\varphi(i)) - 1\} \in \mathbb{Z}_+ \cup \{\infty\},$$

and call it the *successor* of i. Similarly, $\varphi(s(i))$ is called a successor of the vertex $\varphi(i)$. For every $0 \le i \le 2n-1$, we then draw an *arch*, i.e. an edge between $\varphi(i)$ and the successor $\varphi(s(i))$, where by convention, $\varphi(\infty) = v_*$. Note that the number of arches drawn from a particular vertex equals the number of times when this vertex is visited in the contour exploration, which equals its degree.

We claim that it is possible to draw the arches in such a way that they do not cross other arches nor edges of $\mathbf{t}$, i.e., such that the resulting graph is a map. When we delete the interior of the edges of $\mathbf{t}$, it holds that the resulting embedded graph $\mathbf{q}$ is still a map, and in fact, a quadrangulation. We adopt a rooting convention for this map, as follows. Choose $\epsilon \in \{-1, 1\}$, and consider the arch between $\varnothing = \varphi(0)$ and $\varphi(s(0))$. If $\epsilon = 1$, the root is chosen to be this arch, oriented from $\varphi(0)$ to $\varphi(s(0))$, and if $\epsilon = -1$, we choose the reverse orientation. See Figure 3 for a summary of the construction.

We are now able to state the key combinatorial result. Let $\mathbf{Q}_n$ be the set of rooted planar quadrangulations with n faces. Let $\mathbf{Q}_n^*$ be the set of pairs $(\mathbf{q}, v_*)$, where $\mathbf{q} \in \mathbf{Q}_n$ and $v_* \in V(\mathbf{q})$ is a distinguished vertex.

Theorem 2.1. *The previous construction yields a bijection between the set* $\mathbf{Q}_n^*$*, and the set* $\mathbb{T}_n \times \{-1, 1\}$*. This construction identifies vertices of* $\mathbf{t}$ *with vertices of* $\mathbf{q}$ *different from* v_**, in such a way that for every* $v \in \mathbf{t}$*,*

$$d_{\mathbf{q}}(v, v_*) = \ell(v) - \ell(v_*) = \ell(v) - \min_{u \in \mathbf{t}} \ell(u) - 1, \qquad (2.2)$$

where $d_{\mathbf{q}}$ *is the graph distance on* $V(\mathbf{q})$*.*

See Figure 3 for an example. The identification of vertices of $\mathbf{q}$ distinct from v_* and the vertices of the labeled tree associated with $\mathbf{q}$ is crucial, and will be systematic in the sequel. Note that the previous theorem admits as a simple corollary the computations of the cardinality of $\mathbf{Q}_n$. First note that for every rooted quadrangulation $\mathbf{q} \in \mathbf{Q}_n$, it holds that $\#V(\mathbf{q}) = n + 2$, by applying the Euler Formula. Now, each choice of one vertex v_* among the $n + 2$ possible yields a

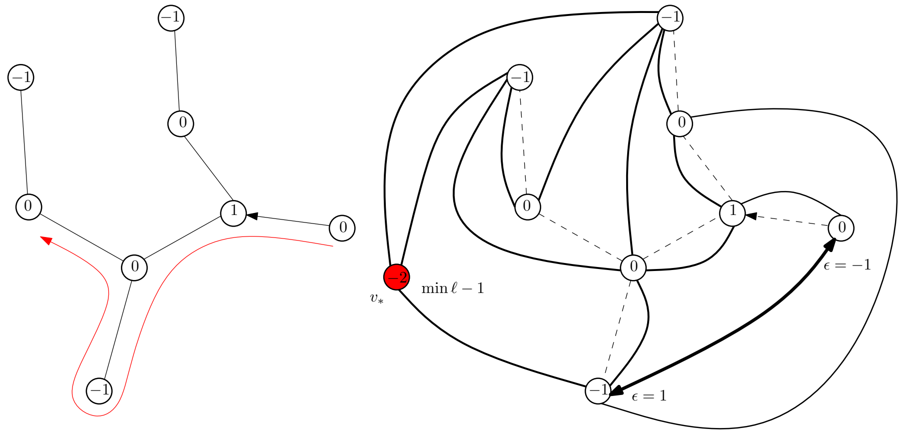

FIGURE 3. A labeled plane tree with the five first steps of the contour exploration and the associated planar quadrangulation with a distinguished vertex v_* (and with the two possible rooting choices)

different element of $\mathbf{Q}_n^*$, so combining with (2.1) yields

$$\#\mathbf{Q}_n = \frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}.$$

Note that the factor 2 appearing in the first term of this formula is due to the choice of ϵ in the construction. This kind of simple enumeration formula is what initially led Cori and Vauquelin [22] on the path of finding bijection between maps and labeled trees.

3. The scaling limit of labeled trees

The Schaeffer bijection leads us to consider the behavior of a uniform element of $\mathbb{T}_n$ as n gets large. We will study this problem with some detail, both because it will be crucial for the sequel, but also because it is a simple example of a derivation of a scaling limit for a random combinatorial structure, which can be seen as a sort of warm-up for the study of maps.

3.1. The Brownian CRT

To begin with, we study random trees without the labels. Let T_n be a random variable with uniform distribution in $\mathbf{T}_n$. It is a well-known fact that the typical distances between vertices of this tree are of order $n^{1/2}$. We want to rescale these distances by this factor and let n go to ∞.

3.1.1. Convergence of the contour process. Introduce the *contour process* $(C_{\mathbf{t}}(i), 0 \leq i \leq 2n)$ of $\mathbf{t} \in \mathbf{T}_n$, defined by

$$C_{\mathbf{t}}(i) = |\varphi(i)|\,,$$

the height of $\varphi(i)$, where as before $\varphi(i), i \geq 0$ is the contour exploration of $\mathbf{t}$ (so that $\varphi(0) = \varphi(2n) = \varnothing$ is the root vertex). The contour process $C_{\mathbf{t}}$ is extended to a continuous function on the segment $[0, 2n]$ by linear interpolation between integer times. This yields a piecewise linear function, also known as *Harris encoding* of the tree $\mathbf{t}$. Conversely, any non-negative walk $(C(i), 0 \leq i \leq 2n)$ of duration $2n$, taking only ± 1 steps, and satisfying $C(0) = C(2n) = 0$, is the contour process of a uniquely defined element of $\mathbf{t}$.

Consequently, when T_n is a random variable uniformly distributed in $\mathbf{T}_n$, its contour process is a simple random walk with duration $2n$, conditioned to remain non-negative and to end at 0. A generalization of the Donsker invariance principle, due to Kaigh [33] shows that

$$\left(\frac{1}{\sqrt{2n}} C_{T_n}(2ns), 0 \leq s \leq 1\right) \xrightarrow[n\to\infty]{(d)} (\mathbb{e}_s, 0 \leq s \leq 1)\,, \tag{3.1}$$

in distribution for the uniform topology on the space $\mathcal{C}([0,1])$ of real-valued continuous functions defined on $[0,1]$, and where the limit is the so-called *normalized Brownian excursion*, which can be understood as an excursion away from 0 of the standard Brownian motion, conditioned to have duration 1. It can be easily defined, by scaling properties of Brownian motion, as a rescaled version of the excursion of a Brownian motion straddling 1 [51]. If $(B_s, s \geq 0)$ is a standard one-dimensional Brownian motion, and

$$g = \sup\{s \leq 1 : B_s = 0\}\,, \qquad d = \inf\{s \geq 1 : B_s = 0\}\,,$$

then the process $\mathbb{e}$ has same distribution as

$$\frac{|B_{(d-g)s+g}|}{\sqrt{d-g}}\,, \quad 0 \leq s \leq 1\,. \tag{3.2}$$

In terms of trees, the interpretation of (3.1) is that the process of heights in the tree, for a particle wrapping around T_n and starting from the root, converges once rescaled properly (the distances being divided by $\sqrt{2n}$) towards the contour process of a limiting structure, called the Brownian CRT [3]. The correct way to view $\mathbb{e}$ as the contour process of a tree structure is to view trees as metric spaces. Let us step back to the discrete contour process once again. Fix $0 \leq i, j \leq 2n$, let $u = \varphi(i), v = \varphi(j)$. It is then a simple exercise to check that the height of the most recent common ancestor $u \wedge v$ is equal to $\min_{i\wedge j \leq k \leq i \vee j} C_{\mathbf{t}}(k)$, moreover, any $k \in [i \wedge j, i \vee j]$ attaining this minimum is such that $\varphi(k) = u \wedge v$. As a consequence, we have a simple formula for the graph distance $d_{\mathbf{t}}$ between vertices of $\mathbf{t}$:

$$\begin{aligned} d_{\mathbf{t}}(u,v) &= |u|+|v|-2|u\wedge v| \\ &= C_{\mathbf{t}}(i)+C_{\mathbf{t}}(j)-2\min_{i\wedge j\leq k\leq i\vee j} C_{\mathbf{t}}(k) \\ &=: d^0_{\mathbf{t}}(i,j)\,. \end{aligned}$$

In the latter formula, the function $d^0_{\mathbf{t}}$ is not a distance, because it is possible to find $i \neq j$ such that $d^0_{\mathbf{t}}(i,j) = 0$. However, it is a *semi-metric*, i.e. it is non-negative, symmetric, null on the diagonal, and satisfies the triangular inequality. The quotient space obtained by identifying points at distance 0 is isometric to $(\mathbf{t}, d_{\mathbf{t}})$.

The advantage of this point of view is that it translates *verbatim* to a continuous setting. In view of (3.1) and the previous discussion, it is natural to define a "distance function" $d^0_{\mathbb{e}}$ on $[0,1]$ by letting

$$d^0_{\mathbb{e}}(s,t) = \mathbb{e}_s + \mathbb{e}_t - 2\inf_{s\wedge t\leq u\leq s\vee t} \mathbb{e}_u\,.$$

This function is the limit of the distance function $d^0_{T_n}$ in the following sense. First, we extend the distance $d^0_{\mathbf{t}}$ to $[0,2n]$ by the formula

$$\begin{aligned} d^0_{\mathbf{t}}(s,t) = {} & (\lceil s\rceil - s)(\lceil t\rceil - t)d^0_{\mathbf{t}}(\lfloor s\rfloor,\lfloor t\rfloor) + (\lceil s\rceil - s)(t-\lfloor t\rfloor)d^0_{\mathbf{t}}(\lfloor s\rfloor,\lceil t\rceil) \\ & +(s-\lfloor s\rfloor)(\lceil t\rceil - t)d^0_{\mathbf{t}}(\lceil s\rceil,\lfloor t\rfloor) + (s-\lfloor s\rfloor)(t-\lfloor t\rfloor)d^0_{\mathbf{t}}(\lceil s\rceil,\lceil t\rceil)\,, \end{aligned} \tag{3.3}$$

where by definition $\lfloor x\rfloor = \sup\{k\in\mathbb{Z}_+ : k\leq x\}$, and $\lceil x\rceil = \lfloor x\rfloor + 1$. It is easy to check that $d^0_{\mathbf{t}}$ defines a semi-metric on $[0,2n]^2$. Moreover, as a consequence of (3.1),

$$\left(\frac{d^0_{T_n}(2ns,2nt)}{\sqrt{2n}}\right)_{0\leq s,t\leq 1} \xrightarrow[n\to\infty]{(d)} (d^0_{\mathbb{e}}(s,t), 0\leq s,t\leq 1)\,, \tag{3.4}$$

in distribution for the uniform topology on $\mathcal{C}([0,1]^2)$. Again, the function $d^0_{\mathbb{e}}$ does not define a distance since, for instance, $d^0_{\mathbb{e}}(0,1)=0$ with this definition. However, it is a semi-metric. Hence, letting $s\sim_{\mathbb{e}} t$ if $d^0_{\mathbb{e}}(s,t)=0$, we can define a quotient metric space $T_{\mathbb{e}} = [0,1]/\sim_{\mathbb{e}}$, endowed with the quotient distance $d_{\mathbb{e}}$. This is a compact space, as it is the image of $[0,1]$ by the canonical projection, and the latter is continuous (even Hölder-continuous as we will see in Sect. 3.1.3).

Definition 3.1. The random metric space $(T_{\mathbb{e}}, \mathrm{d}_{\mathbb{e}})$ is called the Brownian Continuum Random Tree.

From the geometric point of view, this metric space indeed has a tree structure [25]. Namely, a.s. for any $a,b\in T_{\mathbb{e}}$, there is a unique injective continuous path from a to b, so that $T_{\mathbb{e}}$ has 'no loops', moreover, this path is isometric to the real segment $[0,d_{\mathbb{e}}(a,b)]$ (it is a 'geodesic'). Such spaces are usually called $\mathbb{R}$-trees [29].

3.1.2. Convergence in the Gromov-Hausdorff topology. It would be more satisfactory to view the convergence of the distance functions (3.4) as a convergence in a space of trees, rather than using the artifact of encoding spaces as quotients of $[0,1]$. This can be done by reasoning entirely in terms of metric spaces, using a topology developed around the ideas of Gromov starting in the late 1970's [31, 28].

Let $(X,d),(X',d')$ be two compact metric spaces. We let $\mathrm{d}_{\mathrm{GH}}((X,d),(X',d'))$, the Gromov-Hausdorff distance between these spaces, be the infimum of all quantities $\delta_H(\phi(X),\phi'(X'))$, taken over the set of all metric spaces (Z,δ) and isometries $\phi : X \to Z, \phi' : X' \to Z$, where δ_H denotes the Hausdorff distance between closed subsets of Z:

$$\delta_H(A,B) = \sup_{a\in A} \delta(a,B) \vee \sup_{b\in B} \delta(b,A)\,.$$

Of course, two isometric spaces will be at 'distance' 0. In fact, d_{GH} is a class function, where two spaces are identified whenever they are isometric.

Proposition 3.2. *The function* d_{GH} *is a complete, separable distance on the set of isometry classes of compact metric spaces.*

This statement is shown in [29]. We also refer to [17] for important properties of this distance. It is a simple exercise to show that the convergence (3.4) implies the following fact:

Proposition 3.3. *As $n \to \infty$, the isometry class of the random metric space $(T_n,(2n)^{-1/2}d_{T_n})$ converges in distribution to the isometry class of the Brownian CRT $(T_{\mathbb{e}},d_{\mathbb{e}})$, for the topology induced by the Gromov-Hausdorff metric on the set of isometry classes of compact metric spaces.*

3.1.3. Hausdorff dimension. As a warm-up for the later case of scaling limits of random planar maps, let us perform a Hausdorff dimension computation.

Proposition 3.4. *The Hausdorff dimension of the metric space $(T_{\mathbb{e}},d_{\mathbb{e}})$ is 2 a.s.*

This fact is well-known, although complete proofs are relatively recent. Extensions to computations of exact Hausdorff measures for the Brownian CRT and other kinds of continuum trees, appear in [25, 26]. We provide a short, elementary proof, that will be useful in the analogous derivation of the dimension of scaling limits of random maps.

Proof. First of all, we use the fact that the process $\mathbb{e}$ is a.s. Hölder continuous with exponent α, for any $\alpha \in (0,1/2)$. This comes as a consequence of (3.2) and the well-known analog fact for Brownian motion. Therefore, the canonical projection $p : [0,1] \to T_{\mathbb{e}}$ is a.s. Hölder-continuous of exponent $\alpha \in (0,1/2)$. Indeed, for $s,t \in [0,1]$, choose $u \in [s\wedge t, s\vee t]$ such that $\mathbb{e}_u = \inf_{s\wedge t\leq r\leq s\vee t} \mathbb{e}_r$, and note

$$d_{\mathbb{e}}(p(s),p(t)) = \mathbb{e}_s - \mathbb{e}_u + \mathbb{e}_t - \mathbb{e}_u \leq 2\|\mathbb{e}\|_\alpha\,|s-t|^\alpha\,,$$

where

$$\|\mathbb{e}\|_\alpha := \sup_{0\leq s\neq t\leq 1} \frac{|\mathbb{e}_s - \mathbb{e}_t|}{|s-t|^\alpha}$$

is a random, a.s. finite quantity. The upper-bound $\dim_H(T_{\mathbb{e}}, d_{\mathbb{e}}) \leq \alpha^{-1}$, for any $\alpha \in (0, 1/2)$ is a direct and well-known consequence of this last fact, and letting $\alpha \to 1/2$ yields the wanted upper-bound.

Let us prove the lower bound. Let λ be the image measure of the Lebesgue measure on $[0, 1]$ by the projection p. We want to estimate the probability distribution of the distance in $T_{\mathbb{e}}$ between two λ-distributed random points. We will use the well-known fact [27, Proposition 3.4] that if U is a random variable with the uniform distribution in $[0, 1]$, independent of $\mathbb{e}$, then $2\mathbb{e}_U$ has the so-called Rayleigh distribution:

$$P(\mathbb{e}_U \geq r) = \exp(-2r^2)\,, \qquad r \geq 0\,. \tag{3.5}$$

Lemma 3.5. *Let U, V be independent uniform random variables in $[0, 1]$, independent of $\mathbb{e}$. Then $d^0_{\mathbb{e}}(U, V)$ has the same distribution as $\mathbb{e}_U$.*

Proof. This lemma is a special case of a more general invariance of $T_{\mathbb{e}}$ by change of root, since $\mathbb{e}_U = d^0_{\mathbb{e}}(0, U)$ measures the distance to the special point 0, sometimes called the root of $T_{\mathbb{e}}$. The idea of its proof is simple. Let $k, l \in \{0, 1, \dots, 2n - 1\}$. The mapping from $\mathbf{T}_n$ to itself, consisting in re-rooting $\mathbf{t}$ at the edge e_k pointing from the vertex $\varphi(k)$ to $\varphi(k + 1)$, is a bijection. Therefore, it leaves the uniform law on $\mathbf{T}_n$ unchanged. Now,

$$d^0_{\mathbf{t}}(k, l) = d_{\mathbf{t}}(\varphi(k), \varphi(l)) = C_{\mathbf{t}}(k) + C_{\mathbf{t}}(l) - 2 \min_{[k\wedge l, k\vee l]} C_{\mathbf{t}}\,.$$

In the new contour exploration of the tree $\mathbf{t}$ re-rooted at the edge e_k, the vertex $\varphi(l)$ is now visited at step $l - k$ if $k \leq l$, or $2n + l - k$ otherwise. By applying this to the uniform random variable T_n, we thus obtain that $d_{T_n}(\varphi(k), \varphi(l))$ must have the same distribution as $C_{T_n}((l - k) \vee (2n + l - k))$. Letting $k = \lfloor 2ns \rfloor, l = \lfloor 2nt \rfloor$, letting $n \to \infty$ and applying (3.1), this yields

$$d^0_{\mathbb{e}}(s, t) = \mathbb{e}_s + \mathbb{e}_t - 2 \inf_{[s\wedge t, s\vee t]} \mathbb{e} \overset{(d)}{=} \mathbb{e}((t - s) \vee (1 + t - s))\,,$$

for every fixed s, t. By independence, we may apply this to $s = U, t = V$, and using the fact that $(V - U) \vee (1 + V - U)$ has the same law as U, we get the result. $\square$

We are now ready to end the proof of Proposition 3.4. First of all, with the above notations, we have, for $r \geq 0$,

$$P(d^0_{\mathbb{e}}(U, V) \leq r) = P(\mathbb{e}_U \leq r) = 1 - \exp(-2r^2) \leq 2r^2\,,$$

by using Lemma 3.5 and (3.5). On the other hand, the left-hand side in the above displayed expression also equals

$$E\left[\int_{T_{\mathbb{e}}} \lambda(\mathrm{d}a) \int_{T_{\mathbb{e}}} \lambda(\mathrm{d}b)\, \mathbb{1}_{\{d_{\mathbb{e}}(a,b)\leq r\}}\right] = E\left[\int_{T_{\mathbb{e}}} \lambda(\mathrm{d}a)\lambda(B_r(a))\right]\,,$$

where $B_r(a)$ denotes the ball with radius r centered at a in $(T_{\mathbb{e}}, \mathrm{d}_{\mathbb{e}})$. This yields, for $\varepsilon > 0$,

$$E\left[\int_{T_{\mathbb{e}}} \lambda(\mathrm{d}a)\mathbb{1}_{\{\lambda(B_r(a))\geq r^{2-\varepsilon}\}}\right] \leq 2r^2/r^{2-\varepsilon} = 2r^{\varepsilon}\,.$$

Applying this to $r = 2^{-k}$, we obtain that the above quantities have a finite sum as k varies in $\mathbb{N}$. The Borel-Cantelli Lemma shows that P-a.s., for λ-almost every $a \in T_{\mathbb{e}}$, there exists a (random) K such that $\lambda(B_{2^{-k}}(a)) < 2^{-(2-\varepsilon)k}$ for $k \geq K$, so that P-a.s.,

$$\limsup_{k\to\infty} \frac{\lambda(B_{2^{-k}}(a))}{2^{-(2-\varepsilon)k}} \leq 1\,, \qquad \lambda(\mathrm{d}a) - \text{a.e.}$$

We conclude that $\dim_H(T_{\mathbb{e}}, d_{\mathbb{e}}) \geq 2-\varepsilon$, by standard density theorems for Hausdorff measures [44, Theorem 6.9]. □

3.2. Scaling limit of the tree with labels

Let us now turn to the limit of a uniform random element (T_n, L_n) in the set $\mathbb{T}_n$. Such a random variable is obtained by assigning uniformly at random one of the 3^n possible label functions to a uniform random variable in $\mathbf{T}_n$, so the notation is consistent and T_n has the same distribution as in the previous section.

To understand how the labels behave, let us condition on $T_n = \mathbf{t}$ and choose $u = \varphi(i), v = \varphi(j) \in \mathbf{t}$. Let $u(0), \ldots, u(|u|)$, resp. $v(0), \ldots, v(|v|)$ be the two paths of vertices starting from the root of $\mathbf{t}$ and respectively ending at u, v, going upwards in the tree. These two sequences are equal up to the step $|u \wedge v| = \min_{[i\wedge j, i\vee j]} C_{\mathbf{t}}$ when the most recent common ancestor to u, v is reached. Now, the two sequences $(\ell(u(k)), 0 \leq k \leq |u|)$ and $(\ell(v(k)), 0 \leq k \leq |v|)$ both start from 0, share common values up to step $|u \wedge v|$, and then evolve independently from $\ell(u_{|u\wedge v|})$, moreover, their individual distributions are those of a random walk with uniform step distribution in $\{-1, 0, 1\}$, which has variance 2/3. By the central limit theorem, it is to be expected that $\ell(u(k))$ approximates a standard Gaussian distribution in the scale $\sqrt{2k/3}$. More precisely, we expect $(\ell(u(l)), 0 \leq l \leq k)$ to approximate a Brownian motion in this scale. Since by (3.1) the height $|u|$ is typically of order $\sqrt{2n}\mathbb{e}_{i/2n}$, we expect the labels to be Gaussian with variance $\mathbb{e}_{i/2n}$ in the scale $(8n/9)^{1/4}$.

It is now easy to be convinced that the following statement, taken from Chassaing and Schaeffer [21], holds. By abuse of notation we let $L_n(i) := L_n(\varphi(i))$, for every $i \geq 0$. As for contour processes, we extend L_n to a continuous function on $[0, 2n]$ by linear interpolation between integers.

Proposition 3.6. *We have the joint convergence in distribution for the uniform topology on* $\mathcal{C}([0,1])^2$*:*

$$\left(\left(\frac{1}{\sqrt{2n}} C_{T_n}(2ns)\right)_{0\leq s\leq 1}, \left(\left(\frac{9}{8n}\right)^{1/4} L_n(2ns)\right)_{0\leq s\leq 1}\right) \xrightarrow[n\to\infty]{(d)} (\mathbb{e}, Z)\,, \qquad (3.6)$$

where conditionally on $\mathbb{e}$*, the process* $(Z_s, 0 \leq s \leq 1)$ *is a centered Gaussian process with covariance* $\mathrm{Cov}\,(Z_s, Z_t) = \inf_{[s\wedge t, s\vee t]} \mathbb{e}$*.*

The process $(\mathbb{e}, Z)$ is sometimes referred to as the *head of the Brownian snake*. The Brownian snake [36] is a Markov process from which $(\mathbb{e}, Z)$ (which is by no means a Markov process) is obtained as a simple functional. It has an important role in the resolution of certain non-linear PDEs, a fact which we are going to need later. For now, we state two elementary, useful lemmas.

Lemma 3.7. *The process Z is a.s. Hölder continuous with any exponent $\alpha \in (0, 1/4)$.*

Lemma 3.8. *Let U be a uniform random variable in $[0,1]$, independent of $(\mathbb{e}, Z)$. Then $(Z_{U+s} - Z_U, 0 \le s \le 1)$ has the same distribution as Z, where Z_{U+s} should be understood as Z_{U+s-1} whenever $U + s > 1$.*

The proof of the first lemma is an easy application of the Kolmogorov criterion, checking that for $p > 0$,

$$E[|Z_s - Z_t|^p \,|\, \mathbb{e}] = C_p \left(\mathbb{e}_s + \mathbb{e}_t - 2 \inf_{[s\wedge t, s\vee t]} \mathbb{e}\right)^{p/2} \le 2^{p/2} C_p \|\mathbb{e}\|_{2\alpha} |s-t|^{\alpha p} \,,$$

where C_p is the p-th moment of a standard Gaussian random variable, and using the fact that $\|\mathbb{e}\|_{2\alpha} < \infty$ a.s. for $\alpha \in (0, 1/4)$. The second lemma is a re-rooting result whose proof is analogous to that of Lemma 3.5. Details are left to the interested reader.

4. Scaling limits of random planar quadrangulations

Let us now draw consequences of Sections 2 and 3 in the context of random maps.

4.1. Limit laws for the radius and the profile

Let $\mathbf{q} \in \mathbf{Q}_n$ be a rooted planar quadrangulation, and v be a vertex of $\mathbf{q}$. As before, let $d_{\mathbf{q}}$ denote the graph distance on the set of vertices of $\mathbf{q}$. We define the *radius* of $\mathbf{q}$ seen from v as

$$\mathcal{R}(\mathbf{q}, v) = \max_{u \in V(\mathbf{q})} d_{\mathbf{q}}(u, v) \,,$$

and the *profile* of $\mathbf{q}$ seen from v as the sequence

$$I_{\mathbf{q},v}(k) = \mathrm{Card}\,\{u \in V(\mathbf{q}) : d_{\mathbf{q}}(u,v) = k\} \,, \qquad k \ge 0$$

which measures the 'volumes' of the spheres centered at v in the graph metric. The latter can be seen as a measure on $\mathbb{Z}_+$ with total volume $n + 2$.

Theorem 4.1. *Let Q_n be a random variable with uniform distribution in $\mathbf{Q}_n$, and conditionally on Q_n, let v_* be uniformly chosen among the $n+2$ vertices of Q_n. Let also $(\mathbb{e}, Z)$ denote the head of the Brownian snake, as in the previous section.*

(i) We have

$$\left(\frac{9}{8n}\right)^{1/4} \mathcal{R}(Q_n, v_*) \xrightarrow[n\to\infty]{(d)} \sup Z - \inf Z \,.$$

(ii) If v_{**} is another uniform vertex of Q_n chosen independently of v_*,

$$\left(\frac{9}{8n}\right)^{1/4} d_{Q_n}(v_*, v_{**}) \xrightarrow[n\to\infty]{(d)} \sup Z \,.$$

(iii) Finally, the following convergence in distribution holds for the weak topology on probability measures on $\mathbb{R}_+$*:*

$$\frac{I_{Q_n,v_*}((8n/9)^{1/4}\cdot)}{n+2} \xrightarrow[n\to\infty]{(d)} \mathcal{I}\,,$$

where $\mathcal{I}$ *is the occupation measure of* Z *above its infimum, defined as follows: for every non-negative, measurable* $g : \mathbb{R}_+ \to \mathbb{R}_+$*,*

$$\langle \mathcal{I}, g\rangle = \int_0^1 \mathrm{d}s\, g(Z_s - \inf Z)\,.$$

The points (i) and (iii) are due to Chassaing and Schaeffer [21], and (ii) is due to Le Gall [37], although these references state these properties in a slightly different context, namely, in the case where v_* is the root vertex rather than a uniformly chosen vertex. This indicates that as $n \to \infty$, the root vertex plays no particular role.

Proof. We give the proof of (i) and (ii), as they are going to be the most useful. Let (T_n, L_n, ϵ) be the labeled tree associated with (Q_n, v_*) by Schaeffer's bijection (Theorem 2.1), so that (T_n, L_n) is uniform in $\mathbb{T}_n$. By (2.2), the radius of Q_n viewed from v_* then equals as $\max L_n - \min L_n + 1$. The result (i) follows immediately from this and Proposition 3.6. As for (ii), it is clear that we may in fact assume that v_{**} is uniform among the n vertices of Q_n that are distinct from v_* and the root vertex. These are identified with the set $T_n \setminus \{\varnothing\}$. Now, letting U be a uniform random variable in $[0, 1]$, independent of the contour process C_{T_n}, we let $\langle U\rangle_{T_n} = \lceil 2nU\rceil$ if C_{T_n} has slope $+1$ at U, and $\langle U\rangle_{T_n} = \lfloor 2nU\rfloor$ otherwise. One can check as an exercise that $\varphi(\langle U\rangle_{T_n})$ is uniform among the n vertices of T_n distinct from the root vertex, while $|U - \langle U\rangle_{T_n}/2n| \leq 1/2n$. Together with Proposition 3.6, this entails that $(9/8n)^{1/4}(L_n(v_{**}) - \min L_n + 1)$, which equals $d_{Q_n}(v_*, v_{**})$, converges in distribution to $Z_U - \inf Z$. By Lemma 3.8, this has the same distribution as $-\inf Z$, or as $\sup Z$, by an obvious symmetry property. □

4.2. Convergence as a metric space

We would like to be able to understand the full scaling limit picture for random maps, in a similar fashion as it was done for trees, where we showed, relying on the basic result (3.1), that the distances in discrete trees, once rescaled by $\sqrt{2n}$, converge to the distances in the continuum random tree $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}})$. We thus ask if there is an analog of the CRT, that arises as the limit of the properly rescaled metric spaces (Q_n, d_{Q_n}). In view of Theorem 4.1, the correct normalization for the distance should be $n^{1/4}$.

Assume that (T_n, L_n) is uniform in $\mathbb{T}_n$, let ϵ be uniform in $\{-1, 1\}$, independent of (T_n, L_n), and let Q_n be the random uniform quadrangulation with n faces and with a uniformly chosen vertex v_*, obtained from (T_n, L_n, ϵ) by Schaeffer's bijection. Here we follow Le Gall [38][1]. By the usual identification, the set $\{\varphi(i), i \geq$

[1] At this point, it should be noted that [38, 39, 40] consider another version of Schaeffer's bijection, where no distinguished vertex v_* has to be considered. This results in considering pairs (T_n, L_n)

$0\}$ of vertices of T_n explored in contour order, is understood as the set $V(Q_n)\setminus\{v_*\}$. Define a semi-metric on $\{0,\ldots,2n\}$ by letting $d_n(i,j)=d_{Q_n}(\varphi(i),\varphi(j))$. The quotient of this metric space obtained by identifying i,j whenever $d_n(i,j)=0$ is isometric to $(V(Q_n)\setminus\{v_*\},d_{Q_n})$. A major problem is that $d_n(i,j)$ is not a simple functional of (C_{T_n},L_n). Indeed, the distances that we are able to handle in an easy way are distances to v_*, through the formula

$$d_{Q_n}(v_*,v)=L_n(i)-\min L_n+1\,, \tag{4.1}$$

whenever v is a vertex visited at time i in the contour exploration of T_n. A key observation is the following.

Lemma 4.2. *Let*

$$d_n^0(i,j)=L_n(i)+L_n(j)-2\inf_{[i\wedge j,i\vee j]}L_n+2\,.$$

Then it holds that $d_n\leq d_n^0$.

Proof. Assume $i<j$ without loss of generality. It is convenient to extend $L_n(i)=L_n(\varphi(i))$ to all $i\in\mathbb{Z}_+$, by continuing the contour exploration as in Section 2.2. In the construction of the quadrangulation Q_n from (T_n,L_n) via Schaeffer's bijection, successive arches are drawn between $\varphi(i),\varphi(s(i)),\varphi(s^2(i)),\ldots$ until they end at v_*, and similarly for the arches drawn successively from the $\varphi(j)$.

Let $k\in\mathbb{Z}_+\cup\{\infty\}$ be the first step after i, or equivalently after j, such that $L_n(k)=\min_{[i,j]}L_n-1$. Then by construction, the vertex $\varphi(k)$ is the $L_n(i)-L_n(k)$-th successor of $\varphi(i)$, and the $L_n(j)-L_n(k)$-th successor of $\varphi(j)$. The arches between $\varphi(i),\varphi(j)$ and their respective successors, until they arrive at $\varphi(k)$, form a path in Q_n of length $L_n(i)+L_n(j)-2L_n(k)$, which must be larger than the distance $d_n(i,j)$. □

We extend the functions d_n,d_n^0 to $[0,2n]^2$ by adapting the formula (3.3). It is easy to check that d_n thus extended defines a semi-metric on $[0,2n]$ (which is not the case for d_n^0 as it does not satisfy the triangular inequality), and that it still holds that $d_n\leq d_n^0$. We let

$$D_n(s,t)=\left(\frac{9}{8n}\right)^{1/4}d_n(2ns,2nt)\,,\qquad 0\leq s,t\leq 1\,,$$

so that the subspace $(\{i/2n,0\leq i\leq 2n\},D_n)$, quotiented by points at zero D_n-distance, is isometric to $(V(Q_n)\setminus\{v_*\},(9/8n)^{1/4}d_{Q_n})$. We define similarly the functions D_n^0 on $[0,1]^2$. Then, as a consequence of (3.6), it holds that

$$(D_n^0(s,t),0\leq s,t\leq 1)\xrightarrow[n\to\infty]{(d)}(D^0(s,t),0\leq s,t\leq 1)\,, \tag{4.2}$$

---

in which L_n is conditioned to be positive. The scaling limits of such random variables are still tractable, and in fact, are simple functionals of $(\mathbb{e},Z)$, as shown in [41, 37]. So there will be some differences with our exposition, but these turn out to be non-important.

for the uniform topology on $\mathcal{C}([0,1]^2)$, where by definition

$$D^0(s,t) = Z_s + Z_t - 2 \inf_{[s\wedge t, s\vee t]} Z\,.$$

We can now state

Proposition 4.3. *The family of laws of $(D_n(s,t), 0 \leq s,t \leq 1)$, as n varies, is relatively compact for the weak topology on the probability measures on $\mathcal{C}([0,1]^2)$.*

Proof. Let $s,t,s',t' \in [0,1]$. Then by a simple use of the triangular inequality, and Lemma 4.2,

$$|D_n(s,t) - D_n(s',t')| \leq D_n(s,s') + D_n(t,t') \leq D_n^0(s,s') + D_n^0(t,t')\,,$$

which allows to estimate the modulus of continuity at a fixed $\delta > 0$:

$$\sup_{\substack{|s-s'|\leq\delta\\ |t-t'|\leq\delta}} |D_n(s,t) - D_n(s',t')| \leq 2 \sup_{|s-s'|\leq\delta} D_n^0(s,s')\,. \tag{4.3}$$

However, the convergence in distribution (4.2) entails that for every $\varepsilon > 0$,

$$\limsup_{n\to\infty} P\left(\sup_{|s-s'|\leq\delta} D_n^0(s,s') \geq \varepsilon\right) \leq P\left(\sup_{|s-s'|\leq\delta} D^0(s,s') \geq \varepsilon\right)\,,$$

and the latter goes to 0 when $\delta \to 0$, with a fixed ε, by continuity of D^0 and the fact that $D^0(s,s) = 0$. Hence, taking $\eta > 0$ and letting $\varepsilon = \varepsilon_k = 2^{-k}$, we can choose $\delta = \delta_k$ (tacitly depending also on η) such that

$$\sup_{n\geq 1} P\left(\sup_{|s-s'|\leq\delta_k} D_n^0(s,s') \geq 2^{-k}\right) \leq \eta 2^{-k}\,, \qquad k \geq 1,$$

entailing

$$P\left(\bigcap_{k\geq 1}\left\{\sup_{|s-s'|\leq\delta_k} D_n^0(s,s') \leq 2^{-k}\right\}\right) \geq 1-\eta\,,$$

for all $n \geq 1$. Together with (4.3), this shows that with probability at least $1-\eta$, the function D_n is in the set of functions $f : [0,1]^2 \to \mathbb{R}$ such that for every $k \geq 1$,

$$\sup_{\substack{|s-s'|\leq\delta_k\\ |t-t'|\leq\delta_k}} |f(s,t) - f(s',t')| \leq 2^{-k}\,,$$

the latter set being compact by the Arzela-Ascoli Theorem. The conclusion follows from Prokhorov's tightness Theorem [11]. □

At this point, we are allowed to say that the random distance functions D_n admit a limit in distribution, up to taking $n \to \infty$ along a subsequence:

$$(D_n(s,t), 0 \leq s,t \leq 1) \xrightarrow{(d)} (D(s,t), 0 \leq s,t \leq 1) \tag{4.4}$$

for the uniform topology on $\mathcal{C}([0,1]^2)$. In fact, we are going to need a little more than the convergence of D_n. From the relative compactness of its components, we see that the family of laws of $((2n)^{-1}C_{T_n}(2n\cdot), (9/8n)^{1/4}L_n(2n\cdot), D_n), n \geq 1$ is relatively compact in the set of probability measures on $\mathcal{C}([0,1])^2 \times \mathcal{C}([0,1]^2)$. Therefore, it is possible to choose an extraction $(n_k, k \geq 1)$ so that this triple converges in distribution to a limit, which we call $(\mathbb{e}, Z, D)$ with a slight abuse of notation. The joint convergence to the triple $(\mathbb{e}, Z, D)$ gives a coupling of D, D^0 such that $D \leq D^0$, since $D_n \leq D_n^0$ for every n.

Define a random equivalence relation on $[0,1]$ by letting $s \approx t$ if $D(s,t) = 0$. We let $M = [0,1]/\approx$ be the quotient space, endowed with the quotient distance, which we denote by d_M. Let also $s_* \in [0,1]$ be such that $Z_{s_*} = \inf Z$ (such a s_* turns out to be unique [41]). The $\approx$-equivalence class of s_* is denoted by ρ_*, it is intuitively the point of M that corresponds to v_*. The following statement is a relatively elementary corollary of (4.4).

Proposition 4.4. *The isometry class of (M, d_M) is the limit in distribution of the isometry class of $(Q_n, (9/8n)^{1/4}d_{Q_n})$, for the Gromov-Hausdorff topology, along the subsequence $(n_k, k \geq 1)$. Moreover, it holds that a.s. for every $x \in M$ and $s \in x$ a $\approx$-representative*

$$d_M(\rho_*, x) = D(s_*, s) = Z_s - \inf Z\,.$$

The last equation is of course the continuous analog of (2.2) and (4.1), and is proved by combining this with the convergence of L_n. It is tempting to call (M, d_M) the "Brownian map", although the choice of the subsequence poses a problem of uniqueness. As we see in the previous statement, only the distances to ρ_* are *a priori* defined as simple functionals of the process Z. Distances between other points in M seem to be harder to handle, and it is not known whether they are indeed uniquely defined. In the sequel, the words "Brownian map" will refer to any limit in distribution of the form (M, d_M), along some subsequence. Of course, it is natural make the following

Conjecture 4.5. *The isometry class of $(Q_n, n^{-1/4}d_{Q_n})$ converges in distribution for the Gromov-Hausdorff topology.*

Marckert and Mokkadem [43] and Le Gall [38] give a natural candidate for the limit (called the Brownian map in [43]) but at present, it has not been identified as the correct limit. The rest of the section is devoted to some properties that are nevertheless satisfied by *any* limit of the form (M, d_M) as appearing in Proposition 4.4, along some subsequence.

4.3. Hausdorff dimension of the limit space

The goal of this section is to prove the following result, due to Le Gall [38].

Theorem 4.6. *Almost-surely, the Hausdorff dimension of the space (M, d_M) is equal to* 4.

This fact takes its historic roots in the Physics literature [4]. We are going to present a proof that is slightly simpler than that of [38], in the sense that it does not rely on the precise estimates on the behavior of the Brownian snake near its minimum that are developed in [41]. Rather, it relies on more classical properties of the Brownian snake and its connection to PDEs. We are going to need the following formula, of a Laplace transform kind, due to Delmas [23].

Proposition 4.7. *It holds that*

$$\int_0^\infty \frac{\mathrm{d}r}{2\sqrt{2\pi r^3}}\left(1-e^{-\lambda r}P(\sup Z\le r^{-1/4})\right)=\sqrt{\frac{\lambda}{2}}\left(3\coth^2((2\lambda)^{1/4})-2\right).$$

Proving this formula would fall way beyond the scope of the present paper, so we take this for granted. It is interesting to note that this formula for the law of $\sup Z$, which according to Theorem 4.1 (ii) measures the distance between two uniformly chosen points in a large random quadrangulation, is intimately connected to formulas appearing in the Physics literature back in the 1990's, see [5], or [4, Chapter 4.7], under the name of *two-point function.* These were derived using direct counting arguments on maps, without mentioning labeled trees or the random variable $\sup Z$ itself. See also [12] for derivations of this formula using the language of labeled trees, relying on discrete computations and scaling limit arguments.

Corollary 4.8. *There exists a finite constant $K>0$ such that $P(\sup Z\le r)\le Kr^4$ for every $r\ge 0$.*

Proof. In this proof, the numbers K_1, K_2 denote positive, finite, universal constants. By differentiating twice the formula of Proposition 4.7, and by an elementary (but tedious) computation, we find, as $\lambda\to 0$,

$$\int_0^\infty \sqrt{r}e^{-\lambda r}P(\sup Z\le r^{-1/4})\mathrm{d}r=K_1\lambda^{-1/2}+o(\lambda^{-1/2}).$$

Note that the differentiation under the integral sign is licit in this situation. Changing variables $s=r^{-1/4}$ yields

$$\int_0^\infty s^{-7}e^{-\lambda/s^4}P(\sup Z\le s)\mathrm{d}s\le K_2\lambda^{-1/2}, \tag{4.5}$$

for $\lambda>0$. Introducing the function $F(x)=\int_x^\infty u^{-7}e^{-1/u^4}\mathrm{d}u$ for $x\ge 0$, note that this function is positive, decreasing to 0 as $x\to\infty$, so that $\mathbb{1}_{[0,1]}(x)\le F(1)^{-1}F(x)$ for every $x\ge 0$. This yields, using (4.5) in the last step,

$$\begin{aligned} P(\sup Z\le r) &\le F(1)^{-1}E[F(\sup Z/r)] \\ &= F(1)^{-1}\int_0^\infty u^{-7}e^{-1/u^4}P(\sup Z\le ru)\mathrm{d}u \\ &= F(1)^{-1}r^6\int_0^\infty s^{-7}e^{-r^4/s^4}P(\sup Z\le s)\mathrm{d}s \\ &\le F(1)^{-1}K_2r^6(r^4)^{-1/2}=Kr^4, \end{aligned}$$

as wanted. □

Lemma 4.9. *Let U, V be independent uniform random variables on* $[0, 1]$*, independent of D. Then $D(U, V)$ has the same distribution as* $\sup Z$.

Proof. This statement is of course reminiscent of (ii) in Theorem 4.1, and is in some sense a continuum analog. It is similar to Lemma 3.5 as well, and is also proved using a re-rooting argument. Let U, V be as in the statement. Define U_n as follows: with probability $n/(n + 2)$, we let $U_n = \langle U\rangle_{T_n}$, as in the proof of Theorem 4.1, and with equal probability $1/(n+2)$, we let $U_n = *$ or $U_n = 0$. Define V_n similarly. By convention, let $\varphi(*) = v_*$. Then the vertex $\varphi(U_n)$ of Q_n, is uniformly chosen in Q_n. Obviously, $d_{Q_n}(\varphi(U_n), \varphi(V_n))$ has the same distribution as $d_{Q_n}(v_*, \varphi(U_n))$. The first random variable equals $D_n(\langle U\rangle_{T_n}/2n, \langle V\rangle_{T_n}/2n)$ with probability going to 1 as $n \to \infty$, and by (4.4) this converges to $D(U, V)$ in distribution. On the other hand, $d_{Q_n}(v_*, \varphi(U_n))$ converges in distribution to $\sup Z$ by (ii) in Theorem 4.1. □

We are now able to prove Theorem 4.6. The scheme of the proof is very similar to that of Proposition 3.4. First of all, the canonical projection $\pi : [0, 1] \to M = [0, 1]/\approx$ is a.s. Hölder-continuous of index $\alpha \in (0, 1/4)$, since

$$d_M(\pi(s), \pi(t)) \le D^0(s, t) \le 2\|Z\|_\alpha |s - t|^\alpha ,$$

by definition of D^0 and where $\|Z\|_\alpha < \infty$ a.s. by Lemma 3.7. This implies that (M, d_M) has Hausdorff dimension at most 4.

For the lower-bound, we introduce the image measure μ of Lebesgue measure on $[0, 1]$ by π, and note that if $\mathcal{B}_r(a)$ denotes the ball of radius r centered at a in the space (M, d_M), and with the same notation as in Lemma 4.9,

$$E\left[\int_M \mu(\mathrm{d}a)\mu(\mathcal{B}_r(a))\right] = P(D(U, V) \le r) = P(\sup Z \le r) \le Kr^4 ,$$

using Lemma 4.9 and Corollary 4.8, for any $r \ge 0$. From there, the conclusion follows by taking the exact same steps as in the proof of the lower-bound in Proposition 3.4.

4.4. Topology of the limit space

In the previous section, we showed that, even though the scaling limit of uniform random quadrangulations is not yet proved to be uniquely defined, forcing us to consider appropriate extractions, any limit along such an extraction has Hausdorff dimension 4, a.s. Several other features of the limiting map can be studied in a similar way. In particular, Le Gall [38] identifies the topology of (M, d_M):

Theorem 4.10. *The metric d_M a.s. induces the quotient topology of* $[0, 1]/\approx$.

In a subsequent work, Le Gall and Paulin identify the topology of (M, d_M) by establishing the following result [39].

Theorem 4.11. *The space (M, d_M) is a.s. homeomorphic to the 2-dimensional sphere.*

This shows that the limiting space of uniform random quadrangulation is a topological surface, as was expected by physicists. To prove this, one first uses a description of M as a quotient of the CRT $T_{\mathbb{e}}$ rather than $[0,1]$. More precisely, it is easy to see that the function D is a class function of $T_{\mathbb{e}}$, meaning that $D(s,t) = D(s',t')$ for every $s \sim_{\mathbb{e}} s', t \sim_{\mathbb{e}} t'$. Hence, one can see D as a function on $T_{\mathbb{e}}$, and take the alternative definition $M = T_{\mathbb{e}}/\approx$ (instead of $M = [0,1]/\approx$) where, with the obvious abuse of notations, we write $a \approx b$ if and only if $D(a,b) = 0$. The space M is endowed with the image of the semi-metric D under the canonical projection. Theorem 4.10 depends on a careful description of identified points of $T_{\mathbb{e}}$, in a way that mimics, in a continuous framework, the addition of arches to the tree T_n in the Schaeffer bijection. In turn, the tree $T_{\mathbb{e}}$ and the identifications induced by the relation $\approx$ are viewed as a pair of geodesic laminations of the hyperbolic disk. The proof of Theorem 4.11 then rests on a theorem by Moore on quotients of the sphere, and is developed in an entirely "continuum" framework.

In Miermont [48], an alternative proof of Theorem 4.11 is provided, relying on a strengthening of the Gromov-Hausdorff convergence [9] that allows to conserve topological properties of approximating spaces in the limit. It relies on proving the non-existence of small bottlenecks, i.e. of cycles with diameter $o(n^{1/4})$ separating Q_n into two parts that are of diameters $\Omega(n^{1/4})$.

5. Developments

5.1. Universal aspects of the scaling limit

It is a natural question to ask whether the results discussed above are robust, and in particular, to see if similar results hold when quadrangulations are replaced by more general maps. In fact, all the results of [38, 39] are stated and proved in the more general setting of uniform planar 2κ-angulations, meaning that all faces have the same degree equal to 2κ, where κ is an integer larger than or equal to 2.

It has been shown in a series of papers by Marckert, Miermont and Weill [42, 46, 55, 49] with increasing generality, that results similar to Theorem 4.1 are in fact true for much more general models of maps, namely, maps with a so-called *Boltzmann distribution*. This study is allowed by the bijective encodings of general planar maps, that were studied by Bouttier, Di Francesco and Guitter [13], which generalizes the Schaeffer bijection of Sect. 2. This is at the cost of considering labeled trees with a more complicated structure than elements of $\mathbb{T}_n$, but whose enumeration and probabilistic study is still tractable, using some technology on spatial multitype branching processes, developed in [42, 47].

Let us focus on the simplest case [42] of bipartite planar maps, i.e. maps whose faces all have even degrees. We let $\mathcal{M}$ be the set of rooted bipartite planar maps. Let $\mathbf{w} = (w_1, w_2, \ldots)$ be a non-negative sequence of weights, such that $w_i > 0$ for at least one index $i \geq 2$. Then one can define a non-negative measure

$W_{\mathbf{w}}$ on $\mathcal{M}$ by letting

$$W_{\mathbf{w}}(\mathbf{m}) = \prod_{f \in F(\mathbf{m})} w_{\deg(f)/2} \,.$$

To motivate this definition, suppose that $w_i = w \mathbb{1}_{\{i=\kappa\}}$ for some $\kappa \geq 2$ and $w > 0$. In this case, $W_{\mathbf{w}}$ charges only 2κ-angulations, and assigns same weight w^m to all 2κ-angulations with m faces. By the Euler Formula (1.1), these have also $n = (\kappa - 1)m + 2$ vertices. Therefore, the probability distribution

$$W_{\mathbf{w}}^{(n)} := W_{\mathbf{w}}(\cdot | \mathcal{M}_n) = \frac{W_{\mathbf{w}}(\cdot \cap \mathcal{M}_n)}{W_{\mathbf{w}}(\mathcal{M}_n)} \,,$$

where $\mathcal{M}_n \subset \mathcal{M}$ is constituted of those $\mathbf{m}$ that have n vertices (assuming the latter set has positive $W_{\mathbf{w}}$-mass), is the uniform distribution on planar 2κ-angulations with n vertices.

We say that $\mathbf{w}$ is admissible if $W_{\mathbf{w}}(\mathcal{M}) < \infty$, in which case we an define the *Boltzmann probability distribution* $P_{\mathbf{w}} = W_{\mathbf{w}}/W_{\mathbf{w}}(\mathcal{M})$. We have the following simple criterion for admissibility. For $x \geq 0$, let

$$f_{\mathbf{w}}(x) = \sum_{k \geq 0} \binom{2k+1}{k} w_{k+1} x^k \in [0, \infty] \,,$$

hence defining a completely positive power series. Let $R_{\mathbf{w}}$ denote its radius of convergence.

Proposition 5.1. *The sequence* $\mathbf{w}$ *is admissible if and only if the equation*

$$f_{\mathbf{w}}(x) = 1 - 1/x \,, \qquad x > 1 \tag{5.1}$$

admits a solution. We say that $\mathbf{w}$ *is* critical *if furthermore it holds that the solution* $Z_{\mathbf{w}}$ *is unique and*

$$Z_{\mathbf{w}}^2 f'_{\mathbf{w}}(Z_{\mathbf{w}}) = 1 \,,$$

i.e. if the graphs of the functions $f_{\mathbf{w}}$ *and* $x \mapsto 1 - 1/x$ *are tangent at* $Z_{\mathbf{w}}$*. Finally, we say that* $\mathbf{w}$ *is* regular critical *if* $R_{\mathbf{w}} > Z_{\mathbf{w}}$*, in which case we define*

$$C_{\mathbf{w}} = \frac{9}{8 + 4 Z_{\mathbf{w}}^3 f''_{\mathbf{w}}(Z_{\mathbf{w}})} \,. \tag{5.2}$$

Now recall the definitions of the radius and the profile of a quadrangulation seen from a particular vertex as in Section 4.1, and extend them *verbatim* to any planar map. Of course, we let $d_{\mathbf{m}}$ be the graph distance associated with a map $\mathbf{m}$.

Theorem 5.2. *Assume* $\mathbf{w}$ *is a regular critical sequence. Let* M_n *have distribution* $W_{\mathbf{w}}^{(n)}$*, where it is assumed that* n *varies in the set* $\{k \geq 1 : W_{\mathbf{w}}(\mathcal{M}_n) > 0\}$*. Conditionally on* M_n*, let* v_*, v_{**} *be two vertices of* M_n *chosen uniformly at random.*

Then,

$$\begin{aligned}(C_{\mathbf{w}}n)^{-1/4}\mathcal{R}(M_n,v_*) &\xrightarrow[n\to\infty]{(d)} \sup Z-\inf Z\,,\\ (C_{\mathbf{w}}n)^{-1/4}d_{M_n}(v_*,v_{**}) &\xrightarrow[n\to\infty]{(d)} \sup Z\,,\\ \frac{I_{M_n,v_*}((C_{\mathbf{w}}n)^{1/4}\cdot)}{n} &\xrightarrow[n\to\infty]{(d)} \mathcal{I}\,,\end{aligned}$$

where the constant $C_{\mathbf{w}}$ is defined in (5.2).

This result is implicit in [42], although it is not stated in the exact form above. In the previous reference, the Boltzmann measures are defined on the set $\mathcal{M}^*$ of pointed, rooted maps, i.e. of pairs $(\mathbf{m},v_*)$ with v_* a distinguished vertex of $\mathbf{m}$. One then defines the measure

$$W^*_{\mathbf{w}}(\mathbf{m},v_*)=\prod_{f\in F(\mathbf{m})}w_{\deg(f)/2}\,,\qquad (\mathbf{m},v_*)\in\mathcal{M}^*$$

instead of using $W_{\mathbf{w}}$ and choosing a vertex v_* at random. However, it is immediate that

$$W^*_{\mathbf{w}}(\{(\mathbf{m},v_*):\mathbf{m}\in\mathcal{M}_n\})=nW_{\mathbf{w}}(\mathcal{M}_n)\,,$$

so that a random variable with law $W^*_{\mathbf{w}}$ conditioned on $\{(\mathbf{m},v_*):\mathbf{m}\in\mathcal{M}_n\}$ is the same as a random variable with law $W^{(n)}_{\mathbf{w}}$, together with a distinguished vertex chosen uniformly at random among the n possible choices. Another small difference is that in [42], it is assumed that the root edge of $\mathbf{m}$ points from a vertex u to a vertex v such that $d_{\mathbf{m}}(u,v_*)=d_{\mathbf{m}}(v,v_*)-1$. However, this restriction can be lifted by considering the involution of $\mathcal{M}^*$ that consists in inverting the orientation of the root, since the latter has no fixed points.

The idea of the proof is as follows. Using the bijections of [13], and under the hypothesis that $\mathbf{w}$ is critical, one can show that the tree encoding a random map with Boltzmann distribution $P_{\mathbf{w}}$ is the genealogy of a two-type critical branching process with spatial labels. This allows to understand their limiting behavior thanks to invariance principles for spatial multitype branching processes developed in [42]. These results are generalized in Miermont [47] to allow to treat the case of maps without restriction on the degree.

The condition of being regular critical is not easily read directly on the sequence of weights $\mathbf{w}$, so it is not clear to see *a priori* which are the sequences that are covered by Theorem 5.2. So let us discuss some examples.

5.1.1. Uniform 2κ-angulations. Fix $\kappa\geq 2$. As discussed at the beginning of Section 5.1, in the case where $w_i=w\mathbb{1}_{\{i=\kappa\}}$, for any $w>0$, the measure $W^{(n)}_{\mathbf{w}}$ is the uniform measure over uniform 2κ-angulations with n vertices. We have

$$f_{\mathbf{w}}(x)=w\binom{2\kappa-1}{\kappa-1}x^{\kappa-1}\,,$$

and it is easy to see that $\mathbf{w}$ is (regular) critical if and only if

$$w = \frac{(\kappa-1)^{\kappa-1}}{\kappa^\kappa \binom{2\kappa-1}{\kappa-1}},$$

in which case

$$C_{\mathbf{w}} = \frac{9}{4\kappa}.$$

In particular, when $\kappa = 2$, we recover the case of quadrangulations of Theorem 4.1.

5.1.2. A more general example. Let us assume that

- the sequence $\mathbf{w}$ decreases fast enough so that the radius of convergence of $f_{\mathbf{w}}$ is infinite. This includes in particular the case where $\mathbf{w}$ has finite support.
- $w_1 = 0$, so that $W_{\mathbf{w}}$ does not charge maps with faces of degree 2 (note that such faces are non-important from the point of view of the graph distance).
- $w_2 < 1/12$ and there exists $i > 2$ such that $w_i > 0$, so that $W_{\mathbf{w}}$ is not supported only by the set of quadrangulations, this last case having been studied before.

We can freely change $\mathbf{w}$ into the sequence $a \bullet \mathbf{w} := (a^{i-1} w_i, i \geq 1)$, for some $a > 0$, without changing the distribution $W_{\mathbf{w}}^{(n)}$. Indeed, a simple use of the Euler Formula shows that

$$W_{a\bullet\mathbf{w}}(\mathbf{m}) = a^{\sum_{f\in F(\mathbf{m})}(\deg(f)/2-1)} W_{\mathbf{w}}(\mathbf{m}) = a^{\#V(\mathbf{m})-2} W_{\mathbf{w}}(\mathbf{m}),$$

since $\sum_{f\in F(\mathbf{m})} \deg(f)/2$ is the number of edges of $\mathbf{m}$. This has the effect of changing the function $f_{\mathbf{w}}$ to

$$f_{a\bullet\mathbf{w}}(x) = \frac{f_{\mathbf{w}}(ax)}{a} = 3w_2 x + 10 a w_3 x^2 + 35 a^2 w_4 x^3 + \dots,$$

and the latter converges to $3w_2 x < x/4$ as $a \downarrow 0$. Since the graphs of the functions $x \mapsto x/4$ and $x \mapsto 1 - 1/x$ are tangent, it easily follows that there exists a unique value $a_c > 0$ such that $a_c \bullet \mathbf{w}$ is critical, and it is necessarily regular critical since $R_{\mathbf{w}} = R_{a\bullet\mathbf{w}} = \infty$ for every $a > 0$. Therefore, Theorem 5.2 applies to the conditioned measure $W_{\mathbf{w}}^{(n)} = W_{a_c\bullet\mathbf{w}}^{(n)}$, with the scaling constant $C_{a_c\bullet\mathbf{w}}$.

5.2. Beyond the radius and the profile

It is of course tempting, using Theorem 5.2 as a basis, to try and generalize the convergence theorems obtained from Section 4.2 onwards. This turns out to be possible for most of them, without much more effort.

It is indeed an easy exercise to check, along the lines explained above, that Propositions 4.3 and 4.4, and Theorem 4.6 remain true in the more general setting of Theorem 5.2, i.e. that maps with distribution $W_{\mathbf{w}}^{(n)}$ with regular critical $\mathbf{w}$, and with graph distances rescaled by $n^{1/4}$, admit scaling limits for the Gromov-Hausdorff topology, the latter having Hausdorff dimension 4. It is also to be expected that the result identifying the topology (Theorem 4.10) holds in this setting as well.

On a more speculative basis, the natural conjecture is of course that the limiting space is "always the same", i.e. does not depend on $\mathbf{w}$ up to scaling constants.

5.3. Geodesics

There has been also a recent interest in the study of the geodesic paths in discrete maps and in the Brownian map, and the bijective methods are again good enough to give a lot of information on these aspects. In Bouttier and Guitter [14], the authors discuss the existence of "truly distinct" geodesics between two typical vertices in a large random quadrangulation, by extending the Schaeffer bijection to a family of quadrangulations with a distinguished geodesic path. In a different direction, Miermont [45] shows the uniqueness of the geodesic between two typical points chosen in the scaling limit of a critical Boltzmann-distributed quadrangulation, i.e. with distribution $P_{\mathbf{w}}$ where $w_i = 12^{-1}\mathbb{1}_{\{i=2\}}$ with the notations of Sect. 5.1. The latter uses a new family of bijections inspired by the Schaeffer bijection, called k-pointed bijections, in which an arbitrary number k of vertices of the quadrangulation are distinguished instead of only one (the vertex that we called v_*). This bijection allows to study certain geometric loci of multi-pointed maps, which are variants of the Voronoi tessellation with sources at the distinguished vertices.

A recent, deep result of Le Gall [40] shows that it is in fact possible to identify *all* the geodesics from the point ρ_* in the Brownian map[2], as defined around Proposition 4.4. Among other results, it shows that any point in M is linked to ρ_* by $1, 2$ or 3 distinct geodesics, and identifies the *cut-locus* of ρ_*, i.e. the set of points linked to ρ_* by more than one geodesic. More precisely, recall that the Brownian map is a quotient of the Brownian CRT, $M = T_{\mathbb{e}}/\approx$ as mentioned in Sect. 4.4. Let $\pi : T_{\mathbb{e}} \to T_{\mathbb{e}}/\approx$ be the canonical projection. We let $\mathrm{Sk}(T_{\mathbb{e}})$ be the set of points that disconnect $T_{\mathbb{e}}$. Then the cut-locus of ρ_* is exactly the set $\pi(\mathrm{Sk}(T_{\mathbb{e}}))$, and moreover the restriction of π to $\mathrm{Sk}(T_{\mathbb{e}})$ is a homeomorphic embedding. This nicely identifies $T_{\mathbb{e}}$ not only as a convenient tool to build the Brownian map, but also as a natural geometric object associated with it. It also entails a confluence property of the geodesics, namely, any two geodesic paths emanating from ρ_* will share a common initial segment. This shows that there is essentially a unique way to leave the point ρ_* along a geodesic, suggesting that the space (M, d_M) is very rough from a metric point of view, and very far from being a smooth surface.

5.4. Multi-point functions

Theorems 4.1 and 5.2 identify the so-called *two-point* function in the Brownian map, i.e. the distribution of the distance between two uniformly chosen points. It is natural to wonder about the distribution of mutual distances

$$(D(U_i, U_j), 1 \leq i, j \leq k)$$

---

[2] Again, there is a difference with [40] as the latter reference considers rooted, non-pointed maps, see however Remark (i) after Theorem 1.4 therein.

between k independent uniformly chosen points $U_1, \ldots, U_k$ in $[0,1]$, independent of the distance function D of (4.4). It turns out that knowing these distributions (in fact, just knowing that these distributions are uniquely defined and do not depend on the choice of the subsequence of Section 4.2) would be sufficient in addressing the uniqueness problem of the Brownian map, and getting rid of subsequences in Proposition 4.4.

In a recent paper [15], Bouttier and Guitter made an important step in this direction, by identifying the $k = 3$-point function. One of the ingredients is a careful use of the 3-pointed bijection of [45]. Unfortunately, the method does not seem to generalize to more points.

5.5. Higher genera

It is also natural to consider maps on an orientable, compact surface with genus g, i.e. a cellular embedding of a graph in the torus with g handles. The (asymptotic) enumeration of such maps has been studied, starting from work of Bender and Canfield [10], along the lines of Tutte's enumeration methods. It is also covered by the matrix integral methods we alluded to in Section 1.

It turns out that the Schaeffer bijection generalizes nicely to this setting as well, replacing labeled trees with labeled maps with one face (of same genus), as shown by Chapuy, Marcus and Schaeffer [19]. This naturally paves the way to the probabilistic exploration of these classes of maps. In particular, the essential uniqueness of geodesics can be also obtained in this setting [45], while Chapuy [18] provides a very nice closed representation for the 2-point function and the profile of these more general maps.

Acknowledgment. Thanks to the referee for carefully pointing several typos and inaccuracies in the first version of this work.

References

[1] D. J. Aldous. The continuum random tree. I. *Ann. Probab.*, 19(1):1–28, 1991.

[2] D. J. Aldous. The continuum random tree. II. An overview. In *Stochastic analysis (Durham, 1990)*, volume 167 of *London Math. Soc. Lecture Note Ser.*, pages 23–70. Cambridge Univ. Press, Cambridge, 1991.

[3] D. J. Aldous. The continuum random tree. III. *Ann. Probab.*, 21(1):248–289, 1993.

[4] J. Ambjørn, B. Durhuus, and T. Jonsson. *Quantum geometry. A statistical field theory approach*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1997.

[5] J. Ambjørn and Y. Watabiki. Scaling in quantum gravity. *Nuclear Phys. B*, 445(1):129–142, 1995.

[6] O. Angel. Growth and percolation on the uniform infinite planar triangulation. *Geom. Funct. Anal.*, 13(5):935–974, 2003.

[7] O. Angel and O. Schramm. Uniform infinite planar triangulations. *Comm. Math. Phys.*, 241(2-3):191–213, 2003.

[8] D. Arquès. Les hypercartes planaires sont des arbres très bien étiquetés. *Discrete Math.*, 58(1):11–24, 1986.

[9] E. G. Begle. Regular convergence. *Duke Math. J.*, 11:441–450, 1944.

[10] E. A. Bender and E. R. Canfield. The asymptotic number of rooted maps on a surface. *J. Combin. Theory Ser. A*, 43(2):244–257, 1986.

[11] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[12] J. Bouttier, P. Di Francesco, and E. Guitter. Geodesic distance in planar graphs. *Nuclear Phys. B*, 663(3):535–567, 2003.

[13] J. Bouttier, P. Di Francesco, and E. Guitter. Planar maps as labeled mobiles. *Electron. J. Combin.*, 11:Research Paper 69, 27 pp. (electronic), 2004.

[14] J. Bouttier and E. Guitter. Statistics in geodesics in large quadrangulations. *J. Phys. A*, 41(14):145001, 30, 2008.

[15] J. Bouttier and E. Guitter. The three-point function of planar quadrangulations. *J. Stat. Mech. Theory Exp.*, (7):P07020, 39, 2008.

[16] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber. Planar diagrams. *Comm. Math. Phys.*, 59(1):35–51, 1978.

[17] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[18] G. Chapuy. The structure of dominant unicellular maps, and a connection between maps of positive genus and planar labelled trees. *Probab. Theory Relat. Fields*, 2008. To appear.

[19] G. Chapuy, M. Marcus, and G. Schaeffer. A bijection for rooted maps on orientable surfaces. arXiv:0712.3649.

[20] P. Chassaing and B. Durhuus. Local limit of labeled trees and expected volume growth in a random quadrangulation. *Ann. Probab.*, 34(3):879–917, 2006.

[21] P. Chassaing and G. Schaeffer. Random planar lattices and integrated superBrownian excursion. *Probab. Theory Related Fields*, 128(2):161–212, 2004.

[22] R. Cori and B. Vauquelin. Planar maps are well labeled trees. *Canad. J. Math.*, 33(5):1023–1042, 1981.

[23] J.-F. Delmas. Computation of moments for the length of the one dimensional ISE support. *Electron. J. Probab.*, 8:no. 17, 15 pp. (electronic), 2003.

[24] B. Duplantier and S. Sheffield. Liouville Quantum Gravity and KPZ. arXiv:08081560.

[25] T. Duquesne and J.-F. Le Gall. Probabilistic and fractal aspects of Lévy trees. *Probab. Theory Related Fields*, 131(4):553–603, 2005.

[26] T. Duquesne and J.-F. Le Gall. The Hausdorff measure of stable trees. *ALEA Lat. Am. J. Probab. Math. Stat.*, 1:393–415 (electronic), 2006.

[27] R. T. Durrett and D. L. Iglehart. Functionals of Brownian meander and Brownian excursion. *Ann. Probability*, 5(1):130–135, 1977.

[28] S. N. Evans. Snakes and spiders: Brownian motion on **R**-trees. *Probab. Theory Related Fields*, 117(3):361–386, 2000.

[29] S. N. Evans, J. Pitman, and A. Winter. Rayleigh processes, real trees, and root growth with re-grafting. *Probab. Theory Related Fields*, 134(1):81–126, 2006.

[30] I. P. Goulden and D. M. Jackson. The KP hierarchy, branched covers, and triangulations. *Adv. Math.*, 219(3):932–951, 2008.

[31] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999.

[32] G. 't Hooft. A planar diagram theory for strong interactions. *Nucl. Phys. B*, 72:461–473, 1974.

[33] W. D. Kaigh. An invariance principle for random walk conditioned by a late return to zero. *Ann. Probability*, 4(1):115–121, 1976.

[34] M. A. Krikun. A uniformly distributed infinite planar triangulation and a related branching process. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 307(Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 10):141–174, 282–283, 2004.

[35] S. K. Lando and A. K. Zvonkin. *Graphs on surfaces and their applications*, volume 141 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004.

[36] J.-F. Le Gall. *Spatial branching processes, random snakes and partial differential equations*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1999.

[37] J.-F. Le Gall. A conditional limit theorem for tree-indexed random walk. *Stochastic Process. Appl.*, 116(4):539–567, 2006.

[38] J.-F. Le Gall. The topological structure of scaling limits of large planar maps. *Invent. Math.*, 169(3):621–670, 2007.

[39] J.-F. Le Gall and F. Paulin. Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. *Geom. Funct. Anal.*, 18(3):893–918, 2008.

[40] J.-F. Le Gall. Geodesics in large planar maps and in the Brownian map. arXiv:0804.3012.

[41] J.-F. Le Gall and M. Weill. Conditioned Brownian trees. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(4):455–489, 2006.

[42] J.-F. Marckert and G. Miermont. Invariance principles for random bipartite planar maps. *Ann. Probab.*, 35(5):1642–1705, 2007.

[43] J.-F. Marckert and A. Mokkadem. Limit of normalized random quadrangulations: the Brownian map. *Ann. Probab.*, 34(6):2144–2202, 2006.

[44] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[45] G. Miermont. Tessellations of random maps of arbitrary genus. arXiv:0712.3688.

[46] G. Miermont. An invariance principle for random planar maps. In *Fourth Colloquium on Mathematics and Computer Sciences CMCS'06*, Discrete Math. Theor. Comput. Sci. Proc., AG, pages 39–58 (electronic). Nancy, 2006.

[47] G. Miermont. Invariance principles for spatial multitype Galton-Watson trees. *Ann. Inst. H. Poincaré Probab. Statist.*, 44(6):1128–1161, 2008.

[48] G. Miermont. On the sphericity of scaling limits of random planar quadrangulations. *Electron. Commun. Probab.*, 13:248–257, 2008.

[49] G. Miermont and M. Weill. Radius and profile of random planar maps with faces of arbitrary degrees. *Electron. J. Probab.*, 13:no. 4, 79–106, 2008.

[50] B. Mohar and C. Thomassen. *Graphs on surfaces*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2001.

[51] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, third edition, 1999.

[52] G. Schaeffer. *Conjugaison d'arbres et cartes combinatoires aléatoires*. PhD thesis, Université Bordeaux I, 1998.

[53] O. Schramm. Conformally invariant scaling limits: an overview and a collection of problems. In *International Congress of Mathematicians. Vol. I*, pages 513–543. Eur. Math. Soc., Zürich, 2007.

[54] W. T. Tutte. A census of planar maps. *Canad. J. Math.*, 15:249–271, 1963.

[55] M. Weill. Asymptotics for rooted planar maps and scaling limits of two-type spatial trees. *Electron. J. Probab.*, 12:Paper no. 31, 862–925 (electronic), 2007.

Grégory Miermont
CNRS & DMA, École Normale Supérieure
45, rue d'Ulm
F-75230 Paris Cedex 05
e-mail: `gregory.miermont@ens.fr`