

Some wonderful conjectures
(but almost no theorems)
at the boundary between
analysis, combinatorics and probability:

The entire function $F(x, y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} y^{n(n-1)/2}$,

the polynomials $P_N(x, w) = \sum_{n=0}^N \binom{N}{n} x^n w^{n(N-n)}$,

and the generating polynomials of connected graphs

Alan Sokal
New York University / University College London

in collaboration with

Alex Eremenko (Purdue)
Alex Scott (Oxford)

Talk at Institut Henri Poincaré, 9 November 2009

The entire function $F(x, y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} y^{n(n-1)/2}$

- Defined for complex x and y satisfying $|y| \leq 1$
- Analytic in $\mathbb{C} \times \mathbb{D}$, continuous in $\mathbb{C} \times \overline{\mathbb{D}}$
- $F(\cdot, y)$ is entire for each $y \in \overline{\mathbb{D}}$
- Valiron (1938): “from a certain viewpoint the simplest entire function after the exponential function”

Applications:

- Statistical mechanics: Partition function of one-site lattice gas
- Combinatorics: Generating function for Tutte polynomials on K_n
(also acyclic digraphs, inversions of trees, ...)
- Functional-differential equation: $F'(x) = F(yx)$ where $' = \partial/\partial x$
- Complex analysis: Whittaker and Goncharov constants

Application to Tutte polynomials of complete graphs

- Finite graph $G = (V, E)$
- Multivariate Tutte polynomial $Z_G(q, \mathbf{v}) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} v_e$
where $k(A) = \#$ connected components in (V, A)
- Connected-spanning-subgraph polynomial $C_G(\mathbf{v}) = \lim_{q \rightarrow 0} q^{-1} Z_G(q, \mathbf{v})$
- Write $Z_G(q, v)$ and $C_G(v)$ if $v_e = v$ for all edges e
[standard Tutte polynomial is $Z_G(q, v)$ in different variables]

Specialization to complete graphs K_n :

$$Z_n(q, v) = \sum_{m, k} a_{n, m, k} v^m q^k$$

$$C_n(v) = \sum_m c_{n, m} v^m$$

Exponential generating functions:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} Z_n(q, v) = F(x, 1 + v)^q$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} C_n(v) = \log F(x, 1 + v)$$

[see Tutte (1967) and Scott–A.D.S., arXiv:0803.1477]

- Usually considered as formal power series
- But series are *convergent* if $|1 + v| \leq 1$
[see also Flajolet–Salvy–Schaeffer (2004)]

Elementary analytic properties of $F(x, y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} y^{n(n-1)/2}$

- $\mathbf{y} = \mathbf{0}$: $F(x, 0) = 1 + x$

- $\mathbf{0} < |\mathbf{y}| < \mathbf{1}$: $F(\cdot, y)$ is a nonpolynomial entire function of order 0:

$$F(x, y) = \prod_{k=0}^{\infty} \left(1 - \frac{x}{x_k(y)} \right)$$

where $\sum |x_k(y)|^{-\alpha} < \infty$ for every $\alpha > 0$

- $\mathbf{y} = \mathbf{1}$: $F(x, 1) = e^x$

- $|\mathbf{y}| = \mathbf{1}$ with $\mathbf{y} \neq \mathbf{1}$: $F(\cdot, y)$ is an entire function of order 1 and type 1:

$$F(x, y) = e^x \prod_{k=0}^{\infty} \left(1 - \frac{x}{x_k(y)} \right) e^{x/x_k(y)} .$$

where $\sum |x_k(y)|^{-\alpha} < \infty$ for every $\alpha > 1$

[see also Ålander (1914) for y a root of unity; Valiron (1938) and Eremenko–Ostrovskii (2007) for y not a root of unity]

- $|\mathbf{y}| > \mathbf{1}$: The series $F(\cdot, y)$ has radius of convergence 0

Consequences for $C_n(v)$

- Make change of variables $y = 1 + v$:

$$\bar{C}_n(y) = C_n(y - 1)$$

- Then for $|y| < 1$ we have

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} \bar{C}_n(y) = \log F(x, y) = \sum_k \log\left(1 - \frac{x}{x_k(y)}\right)$$

and hence

$$\bar{C}_n(y) = -(n-1)! \sum_k x_k(y)^{-n} \quad \text{for all } n \geq 1$$

(also holds for $n \geq 2$ when $|y| = 1$)

- This is a *convergent* expansion for $\bar{C}_n(y)$
- In particular, gives large- n asymptotic behavior

$$\bar{C}_n(y) = -(n-1)! x_0(y)^{-n} [1 + O(e^{-\epsilon n})]$$

whenever $F(\cdot, y)$ has a unique root $x_0(y)$ of minimum modulus

Question: What can we say about the roots $x_k(y)$?

Small- y expansion of roots $x_k(y)$

- For small $|y|$, we have $F(x, y) = 1 + x + O(y)$, so we expect a convergent expansion

$$x_0(y) = -1 - \sum_{n=1}^{\infty} a_n y^n$$

(easy proof using Rouché: valid for $|y| \lesssim 0.441755$)

- More generally, for each integer $k \geq 0$, write $x = \xi y^{-k}$ and study

$$F_k(\xi, y) = y^{k(k+1)/2} F(\xi y^{-k}, y) = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} y^{(n-k)(n-k-1)/2}$$

Sum is dominated by terms $n = k$ and $n = k + 1$; gives root

$$x_k(y) = -(k+1)y^{-k} \left[1 + \sum_{n=1}^{\infty} a_n^{(k)} y^n \right]$$

Rouché argument valid for $|y| \lesssim 0.207875$ uniformly in k :
all roots are simple and given by convergent expansion $x_k(y)$

- Can also use theta function in Rouché (Eremenko)

Might these series converge for all $|y| < 1$?

Two ways that $x_k(y)$ could fail to be analytic for $|y| < 1$:

1. Collision of roots (\rightarrow branch point)
2. Root escaping to infinity

Theorem (Eremenko): No root can escape to infinity for y in the open unit disc \mathbb{D} .

In fact, for any compact subset $K \subset \mathbb{D}$ and any $\epsilon > 0$, there exists an integer k_0 such that for all $y \in K \setminus \{0\}$ we have:

- (a) The function $F(\cdot, y)$ has exactly k_0 zeros (counting multiplicity) in the disc $|x| < k_0|y|^{-(k_0-\frac{1}{2})}$, and
- (b) In the region $|x| \geq k_0|y|^{-(k_0-\frac{1}{2})}$, the function $F(\cdot, y)$ has a simple zero within a factor $1 + \epsilon$ of $-(k+1)y^{-k}$ for each $k \geq k_0$, and no other zeros.

- Proof is based on comparison with a theta function (whose roots are known by virtue of Jacobi's product formula)
- *Conjecture* that roots cannot escape to infinity even in the *closed* unit disc except at $y = 1$

Big Conjecture #1. All roots of $F(\cdot, y)$ are simple for $|y| < 1$.
[and also for $|y| = 1$, I suspect]

Consequence of Big Conjecture #1. Each root $x_k(y)$ is analytic in $|y| < 1$.

But I conjecture more ...

Big Conjecture #2. The roots of $F(\cdot, y)$ are non-crossing *in modulus* for $|y| < 1$:

$$|x_0(y)| < |x_1(y)| < |x_2(y)| < \dots$$

[and also for $|y| = 1$, I suspect]

Consequence of Big Conjecture #2. The roots are actually separated in modulus by a factor at least $|y|$, i.e.

$$|x_k(y)| < |y| |x_{k+1}(y)| \quad \text{for all } k \geq 0$$

PROOF. Apply the Schwarz lemma to $x_k(y)/x_{k+1}(y)$.

Consequence for the zeros of $\overline{C}_n(y)$

Recall

$$\overline{C}_n(y) = -(n-1)! \sum_k x_k(y)^{-n}$$

and use a variant of the Beraha–Kahane–Weiss theorem [A.D.S., arXiv:cond-mat/0012369, Theorem 3.2] \implies the limit points of zeros of \overline{C}_n are the values y for which the zero of minimum modulus of $F(\cdot, y)$ is *nonunique*.

So if $F(\cdot, y)$ has a *unique* zero of minimum modulus for all $y \in \mathbb{D}$ (a weakened form of Big Conjecture #2), then the zeros of \overline{C}_n do not accumulate anywhere in the open unit disc.

I actually conjecture more (based on computations up to $n \approx 80$):

Big Conjecture #3. For each n , $\overline{C}_n(y)$ has no zeros with $|y| < 1$. [and, I suspect, no zeros with $|y| = 1$ except the point $y = 1$]

What is the evidence for these conjectures?

Evidence #1: Behavior at real y .

Theorem (Laguerre): For $0 \leq y < 1$, all the roots of $F(\cdot, y)$ are simple and negative real.

Corollary: Each root $x_k(y)$ is analytic in a complex neighborhood of the interval $[0, 1)$.

[Real-variables methods give further information about the roots $x_k(y)$ for $0 \leq y < 1$: see Langley (2000).]

Now combine this with

Evidence #2: From numerical computation of the series $x_k(y) \dots$

Three methods for computing the series $x_k(\mathbf{y})$

1. Insert $x_k(\mathbf{y}) = -(k+1)y^{-k} \left[1 + \sum_{n=1}^{\infty} a_n^{(k)} y^n \right]$ and solve term-by-term

2. Use “explicit implicit function theorem” (generalization of Lagrange inversion formula) given in arXiv:0902.0069:

solve $z = G(z, w)$ with $G(0, 0) = 0$ and $\left| \frac{\partial G}{\partial z}(0, 0) \right| < 1$ by

$$\varphi(w) = \sum_{m=1}^{\infty} \frac{1}{m} [\zeta^{m-1}] G(\zeta, w)^m$$

and more generally

$$H(\varphi(w), w) = H(0, w) + \sum_{m=1}^{\infty} \frac{1}{m} [\zeta^{m-1}] \frac{\partial H(\zeta, w)}{\partial \zeta} G(\zeta, w)^m$$

Methods 1 and 2 work *symbolically* in k .

3. Use

$$\overline{C}_n(\mathbf{y}) = -(n-1)! \sum_k x_k(\mathbf{y})^{-n}$$

together with recursion

$$\overline{C}_n(\mathbf{y}) = y^{n(n-1)/2} - \sum_{j=1}^{n-1} \binom{n-1}{j-1} \overline{C}_j(\mathbf{y}) y^{(n-j)(n-j-1)/2}$$

[cf. Leroux (1988) and Scott–A.D.S., arXiv:0803.1477]

— can go to very high n , at least for small k

And let MATHEMATICA run for a weekend ...

$$\begin{aligned} -x_0(y) = & 1 + \frac{1}{2}y + \frac{1}{2}y^2 + \frac{11}{24}y^3 + \frac{11}{24}y^4 + \frac{7}{16}y^5 + \frac{7}{16}y^6 \\ & + \frac{493}{1152}y^7 + \frac{163}{384}y^8 + \frac{323}{768}y^9 + \frac{1603}{3840}y^{10} + \frac{57283}{138240}y^{11} \\ & + \frac{170921}{414720}y^{12} + \frac{340171}{829440}y^{13} + \frac{22565}{55296}y^{14} \\ & + \dots + \text{terms through order } y^{775} \end{aligned}$$

and all the coefficients (so far) are nonnegative!

Big Conjecture #4. For each k , the series $-x_k(y)$ has all nonnegative coefficients.

Combine this with the known analyticity for $0 \leq y < 1$, and Vivanti–Pringsheim gives:

Consequence of Big Conjecture #4. Each root $x_k(y)$ is analytic in the open unit disc.

NEED TO DO: Extended computations for $k = 1, 2, \dots$ and for symbolic k .

But more is true . . .

Look at the *reciprocal* of $x_0(y)$:

$$\begin{aligned} -\frac{1}{x_0(y)} = & 1 - \frac{1}{2}y - \frac{1}{4}y^2 - \frac{1}{12}y^3 - \frac{1}{16}y^4 - \frac{1}{48}y^5 - \frac{7}{288}y^6 \\ & - \frac{1}{96}y^7 - \frac{7}{768}y^8 - \frac{49}{6912}y^9 - \frac{113}{23040}y^{10} - \frac{17}{4608}y^{11} \\ & - \frac{293}{92160}y^{12} - \frac{737}{276480}y^{13} - \frac{3107}{1658880}y^{14} \\ & - \dots - \text{terms through order } y^{775} \end{aligned}$$

and all the coefficients (so far) beyond the constant term are *nonpositive*!

Big Conjecture #5. For each k , the series $-(k+1)y^{-k}/x_k(y)$ has all *nonpositive* coefficients after the constant term 1.

[This implies the preceding conjecture, but is stronger.]

- Relative simplicity of the coefficients of $-1/x_0(y)$ compared to those of $-x_0(y)$ \longrightarrow simpler combinatorial interpretation?
- Note that $x_k(y) \rightarrow -\infty$ as $y \uparrow 1$ (this is fairly easy to prove). So $1/x_k(y) \rightarrow 0$. Therefore:

Consequence of Big Conjecture #5. For each k , the coefficients (after the constant term) in the series $-(k+1)y^{-k}/x_k(y)$ are the *probabilities* for a positive-integer-valued random variable.

What might such a random variable be???

Could this approach be used to *prove* Big Conjecture #5?

AGAIN NEED TO DO: Extended computations for $k = 1, 2, \dots$ and for symbolic k .

But I conjecture that even more is true ...

Define $D_n(y) = \frac{\overline{C}_n(y)}{(-1)^{n-1}(n-1)!}$ and recall that $x_0(y) = \lim_{n \rightarrow \infty} D_n(y)^{-1/n}$

Big Conjecture #6. For each n ,

(a) the series $D_n(y)^{-1/n}$ has all nonnegative coefficients,

and even more strongly,

(b) the series $D_n(y)^{1/n}$ has all nonpositive coefficients after the constant term 1.

Since $D_n(y) > 0$ for $0 \leq y < 1$, Vivanti–Pringsheim shows that Big Conjecture #6a implies Big Conjecture #3:

For each n , $\overline{C}_n(y)$ has no zeros with $|y| < 1$.

Moreover, Big Conjecture #6b \implies for each n , the coefficients (after the constant term) in the series $D_n(y)^{1/n}$ are the *probabilities* for a positive-integer-valued random variable.

Such a random variable would generalize the one for $-1/x_0(y)$ in roughly the same way that the binomial generalizes the Poisson.

Roots $x_k(y)$ computed *symbolically* in k

$$x_k(y) = -(k+1)y^{-k} \left[1 + \sum_{n=1}^{\infty} \frac{P_n(k)}{Q_n(k)} y^n \right]$$

where I have computed up to $n = 15$:

$$P_1(y) = 1$$

$$P_2(y) = 2 + 6k + 3k^2$$

$$P_3(y) = 11 + 29k + 63k^2 + 65k^3 + 28k^4 + 4k^5$$

$$P_4(y) = 22 + 146k + 273k^2 + 359k^3 + 355k^4 + 211k^5 + 63k^6 + 7k^7$$

⋮

$$Q_1(y) = (k+1)(k+2)$$

$$Q_2(y) = (k+1)^2(k+2)^2$$

$$Q_3(y) = (k+1)^3(k+2)^3(k+3)$$

$$Q_4(y) = (k+1)^4(k+2)^4(k+3)$$

$$Q_5(y) = (k+1)^5(k+2)^5(k+3)$$

$$Q_6(y) = (k+1)^6(k+2)^6(k+3)^2(k+4)$$

⋮

- $P_n(k)$ has nonnegative coefficients for $n \leq 9$ but not for $n = 10, 15$
- $P_n(k) \geq 0$ for all *real* $k \geq 0$ for $n \leq 14$ but not for $n = 15$
- But ... $P_n(k) \geq 0$ for all *integer* $k \geq 0$ at least for $n \leq 15$

which gives evidence that Big Conjecture #4 holds for all k :

For each k , the series $-x_k(y)$ has all nonnegative coefficients.

Reciprocals of roots $x_k(y)$ computed *symbolically* in k

$$\frac{-(k+1)y^{-k}}{x_k(y)} = \left[1 - \sum_{n=1}^{\infty} \frac{\widehat{P}_n(k)}{Q_n(k)} y^n \right]$$

where I have computed up to $n = 15$:

$$\begin{aligned} \widehat{P}_1(y) &= 1 \\ \widehat{P}_2(y) &= 1 + 6k + 3k^2 \\ \widehat{P}_3(y) &= 2 - 10k + 33k^2 + 59k^3 + 28k^4 + 4k^5 \\ \widehat{P}_4(y) &= 3 + 71k + 24k^2 + 82k^3 + 236k^4 + 194k^5 + 63k^6 + 7k^7 \\ &\vdots \end{aligned}$$

and $Q_n(y)$ are the same as before

- $\widehat{P}_n(k)$ does not have nonnegative coefficients (except for $n = 1, 2, 4$)
- $\widehat{P}_n(k) \geq 0$ for all *real* $k \geq 0$ for $n = 1, 2, 3, 4, 5, 7, 8$ but not in general
- But ... $\widehat{P}_n(k) \geq 0$ for all *integer* $k \geq 0$ at least for $n \leq 15$

which gives evidence that Big Conjecture #5 holds for all k :

For each k , the series $-(k+1)y^{-k}/x_k(y)$ has all *nonpositive* coefficients after the constant term 1.

Ratios of roots $x_k(y)/x_{k+1}(y)$

The series

$$\frac{x_0(y)}{x_1(y)} = \frac{1}{2}y + \frac{1}{6}y^2 + \frac{5}{72}y^3 + \frac{11}{216}y^4 + \frac{29}{1296}y^5 + \dots$$

has nonnegative coefficients at least up to order y^{136} .

(But its reciprocal does not have any fixed signs.)

Big Conjecture #7. The series $x_0(y)/x_1(y)$ has all nonnegative coefficients.

Consequence of Big Conjecture #7. Since $\lim_{y \uparrow 1} x_0(y)/x_1(y) = 1$, Big Conjecture #7 implies that $|x_0(y)| < |x_1(y)|$ for all $y \in \mathbb{D}$ (a special case of Big Conjecture #2 on the separation in modulus of roots).

- But unfortunately ... the series

$$\frac{x_1(y)}{x_2(y)} = \frac{2}{3}y + \frac{1}{18}y^2 + \frac{17}{216}y^3 + \frac{23}{810}y^4 + \frac{343}{17280}y^5 + \dots$$

has a negative coefficient at order y^{13} . This doesn't contradict the conjecture that $|x_1(y)/x_2(y)| < 1$ in the unit disc, but it does rule out the simplest method of proof.

- Symbolic computation of $x_k(y)/x_{k+1}(y)$ shows that, up to order y^{16} , the *only* cases of a negative coefficient for *integer* $k \geq 0$ are the coefficient of y^{13} for $k = 1, 2, 3$.

Asymptotics of roots as $y \rightarrow 1$

Write $y = e^{-\gamma}$ with $\text{Re } \gamma > 0$.

Want to study $\gamma \rightarrow 0$ (non-tangentially in the right half-plane).

I *believe* I will be able to prove that

$$-x_k(e^{-\gamma}) \approx \frac{1}{e} \gamma^{-1} + c_k \gamma^{-1/3} + \dots$$

for suitable constants $c_0 < c_1 < c_2 < \dots$. But I have not yet worked out all the details.

Overview of method:

1. Develop an asymptotic expansion for $F(x, e^{-\gamma})$ when $\gamma \rightarrow 0$ and x is taken to be of order γ^{-1} , because this is the regime where the zeros will be found.
2. Use this expansion for $F(x, e^{-\gamma})$ to deduce an expansion for $x_k(e^{-\gamma})$.

Sketch of step #1: Insert Gaussian integral representation for $e^{-\frac{\gamma}{2}n^2}$ to obtain

$$F(x, e^{-\gamma}) = (2\pi\gamma)^{-1/2} \int_{-\infty}^{\infty} \exp[g(t)] dt$$

with

$$g(t) = -\frac{t^2}{2\gamma} + xe^{\gamma/2} e^{it}$$

Saddle-point equation $g'(t) = 0$ is $-ite^{-it} = \gamma e^{\gamma/2}x$, so it makes sense to make the change of variables

$$x = \gamma^{-1}e^{-\gamma/2}we^w,$$

which puts the saddle point at $t_0 = iw$. (Note that this brings in the Lambert W function, i.e. the inverse function to $w \mapsto we^w$.) We then have

$$F(\gamma^{-1}e^{-\gamma/2}we^w, e^{-\gamma}) = (2\pi\gamma)^{-1/2} \int_{-\infty}^{\infty} dt \exp\left[-\frac{t^2}{2\gamma} + \frac{we^w}{\gamma}e^{it}\right]$$

Now shift the contour to go through the saddle point (parallel to the real axis) and make the change of variables $t = s + iw$: we have

$$F(\gamma^{-1}e^{-\gamma/2}we^w, e^{-\gamma}) = (2\pi\gamma)^{-1/2} \exp\left[\frac{w^2}{2\gamma} + \frac{w}{\gamma}\right] \int_{-\infty}^{\infty} ds \exp[h(s)]$$

where

$$h(s) = -\frac{(1+w)}{2\gamma}s^2 + \frac{w}{\gamma}\left(e^{is} - 1 - is + \frac{s^2}{2}\right)$$

and the integration goes along the real s axis.

These formulae should allow computation of asymptotics

- (a) $\gamma \rightarrow 0$ (in a suitable way) for (suitable values of) fixed w ; and
- (b) $w \rightarrow \infty$ (in a suitable direction) for (suitable values of) fixed γ .

Focus for now on (a).

Recall that

$$h(s) = -\frac{(1+w)}{2\gamma}s^2 + \frac{w}{\gamma}\left(e^{is} - 1 - is + \frac{s^2}{2}\right)$$

Consider for simplicity γ and x real. There seem to be three regimes:

- **“High temperature”**: $w > -1$ (i.e. $we^w > -1/e$).

Easiest case: $s = 0$ saddle point is Gaussian, and can compute the asymptotics to all orders in terms of 3-associated Stirling subset numbers $\{^n_m\}_{\geq 3}$. [Still need to justify this formal calculation by showing that only the $s = 0$ saddle point contributes.]

- **“Low temperature”**: $w = -\eta \cot \eta + \eta i$ with $-\pi < \eta < \pi$ (i.e. $we^w < -1/e$).

Saddle points at $s = 0$ and $s = 2\eta$ contribute; I *think* this is all.

- **“Critical regime”**: $w = -(1 + \xi\gamma^{1/3})$ with ξ fixed, which corresponds to

$$x = -\frac{1}{e\gamma} \left[1 - \frac{\xi^2}{2}\gamma^{2/3} + O(\gamma) \right]$$

– At the “critical point” $\xi = 0$: Dominant behavior at $s = 0$ saddle point is no longer Gaussian (it vanishes) but rather the cubic term $is^3/(6\gamma)$. Can compute the asymptotics to all orders in terms of 4-associated Stirling subset numbers $\{^n_m\}_{\geq 4}$ (at least formally).

– In the critical regime (ξ arbitrary): Expect to have Airy asymptotics as in Flajolet–Salvy–Schaeffer (2004). This is where the roots will lie.

I would appreciate help with the details!!!

The polynomials $P_N(x, w) = \sum_{n=0}^N \binom{N}{n} x^n w^{n(N-n)}$

- Partition function of Ising model on complete graph K_N , with $x = e^{2h}$ and $w = e^{-2J}$
- Related to binomial $(1+x)^N$ in same way as our $F(x, y)$ is related to exponential e^x [but we have written $w^{n(N-n)}$ instead of $y^{n(n-1)/2}$]
- $\lim_{N \rightarrow \infty} P_N\left(\frac{xw^{1-N}}{N}, w\right) = F(x, w^{-2})$ when $|w| > 1$
- So results about zeros of P_N generalize those about F (just as results about the binomial generalize those about the exponential function)
- Lee–Yang theorem: In ferromagnetic case ($0 \leq w \leq 1$), all zeros are on the unit circle $|x| = 1$
- Laguerre: In antiferromagnetic case ($w \geq 1$), all zeros are real and negative
- What about “complex antiferromagnetic” case $|w| > 1$??

Big Conjecture #8. For $|w| > 1$, all zeros of $P_N(\cdot, w)$ are separated in modulus (by at least a factor $|w|^2$).

Taking $N \rightarrow \infty$, this implies Big Conjecture #2 about the separation in modulus of the zeros of $F(\cdot, y)$.

Differential-equation approach to $P_N(x, w) = \sum_{n=0}^N \binom{N}{n} x^n w^{n(N-n)}$

On the space of polynomials $Q_N(x) = \sum_{n=0}^N a_n x^n$ of degree N with $a_0 \neq 0$, define the semigroup

$$(\mathcal{A}_t Q_N)(x) \equiv \sum_{n=0}^N a_n x^n e^{tn(N-n)}$$

Roots of $\mathcal{A}_t Q_N$ evolve according to an *autonomous* differential equation, which is best expressed in terms of *logarithms* of roots $\zeta_i = \log x_i$:

$$\frac{d\zeta_i}{dt} = \sum_{j \neq i} F(\zeta_i - \zeta_j)$$

where

$$F(z) = \coth(z/2)$$

These are first-order (“Aristotelian”) equations of motion for a system of n “particles” (in \mathbb{R} or \mathbb{C}) with a translation-invariant “force” F

For polynomials Q_N with *real* roots and *real* $t > 0$, this approach gives interesting results on separation of zeros. (In particular, it gives a new proof of Laguerre’s theorem.)

Is this approach useful for *complex* t with $\operatorname{Re} t > 0$???

Can it be used to prove Big Conjecture #8?