q - chromatic polynomial

$G = (V, E)$ graph

$f : V \longrightarrow \{0, \dots, k-1\}$ is proper coloring if

$e = uv \in E \implies f(u) \neq f(v)$

$M(G, k) = \#$ proper colorings by k colors.

$$M(G, x) = \sum_{E' \subseteq E} (-1)^{|E'|} x^{c(E')}, \quad c(E') : \# \text{ components of } (V, E').$$

Chromatic polynomial

q-chromatic function (M.L., Advances in Math 07) ②

$$M_q(G,k) = \sum_{\substack{f: V \to \{0,\dots,k-1\} \\ \text{proper}}} q^{\sum_{v \in V} f(v)}$$

$$M_q(G,k)\Big|_{q=1} = M(G,k)$$

$$(k)_q = q^{k-1} + \dots + q + 1$$

$$(k)!_q \quad , \quad \binom{n}{k}_q$$

$$\sum_{\substack{0 \le v_i \le k-1 \\ 1 \le i \le n}} q^{\sum v_i} = \left(q^0 + q^1 + \dots + q^{k-1}\right)^n = \left(\frac{q^k - 1}{q - 1}\right)^n$$

$$M_q(K_n, k) = n! \binom{k}{n}_q q^{n(n-1)/2}$$

$$\|$$

$$\sum_{\substack{0 \le v_i \le k-1 \\ \boxed{v_i \neq v_j} \\ 1 \le i \le n}} q^{\sum v_i}$$

$$M_q(G,k) = \sum_{A \subseteq E} (-1)^{|A|} \prod_{W \in C(A)} (k)_{q^{|W|}}$$

$$\sum_{A \subseteq E} (-1)^{|A|} k^{c(A)}$$

<u>Proof</u>. I_e, $e \in E$ finite sets

$$I_A = \bigcap_{e \in A} I_e .$$

$$\left| \bigcup_{e \in E} I_e \right| = \sum_{1 \le a \le |E|} (-1)^{a-1} \sum_{A \subseteq E, |A| = a} |I_A|$$

<u>Principle of inclusion and exclusion</u>

$$M_q(G,k) = \sum_{s : V \to \{0, \dots, k-1\}} q^{\sum_V s(v)} - \sum_{s \in \bigcup_{e \in E} I_e} q^{\sum_V s(v)}$$

where

$$e = uv : I_e = \{ s : V \to \{0, \dots, k-1\} ; s(u) = s(v) \}.$$

$$= \sum_{A \subseteq E} (-1)^{|A|} \sum_{s \in I_A} q^{\sum_V s(v)} = \prod_{W \in C(A)} \sum_{0 \le i \le k-1} q^{i|W|}$$

$\boxtimes$

$$\boxed{\text{Tutte polynomial} \equiv \text{Dichromate}}$$

$$B(G, x, y) = \sum_{A \subseteq E} x^{|A|} y^{c(A)}$$

<u>q-dichromate</u>: $$B_q(G, x, y) = \sum_{A \subseteq E} x^{|A|} \prod_{W \in C(A)} (y)_{q^{|W|}}$$

? Does q-dichromate distinguish non-isomorphic chordal graphs ?

<u>Stanley</u>: Does U-polynomial distinguish all trees ?

$$U_G(z, x_1, x_2, \dots) = \sum_{A \subseteq E(G)} x(\tau_A)\,(z-1)^{|A| - |V| + c(A)}$$

$$\tau_A = (n_1 \geq n_2 \geq \dots \geq n_e) \Rightarrow x(\tau_A) = x_{n_1} \cdot x_{n_2} \cdot \dots \cdot x_{n_e}$$

q-dichromate and Potts partition fction

$E(\mathrm{Potts}^k)(s) \}$

Thm.:

$$\sum_{A \subseteq E} \prod_{W \in C(A)} (k)_{q^{|W|}} \prod_{uv \in A} r_{uv} = \sum_{s : V \to \{0,\dots,k-1\}} q^{\sum_V s(v)} e^{\left(\sum_{uv \in E} J_{uv} \Delta(s(u), s(v))\right)}$$

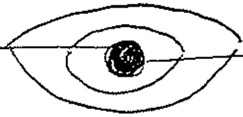
$$\boxed{r_{uv} = e^{J_{uv}} - 1}$$

? Does Potts in variable magnetic field distinguish non-isomorphic chordal graphs ?

?) Is permanent exponentially harder than determinant ?

FACT: permanents can be well aproximated
- for 0,1 matrices (Jerrum, Sinclair, Vigoda 2000)

recent: Leonid Gurvits 2007	⊛ several aproximation algorithms for general matrices (worst case ε-approximation exp. time)



This may help us to (efficiently) solve any decision problem in BQP: computable with bounded error in poly-time using quantum resources (e.g. factoring).

Jones polynomial of a link L is defined by skein relation

$$q^{2} J(\nwarrow\!\!\!\!\nearrow) - q^{-2} J(\nwarrow\!\!\!\!\nearrow) = (q - q^{-1}) J(\uparrow\ \uparrow)$$

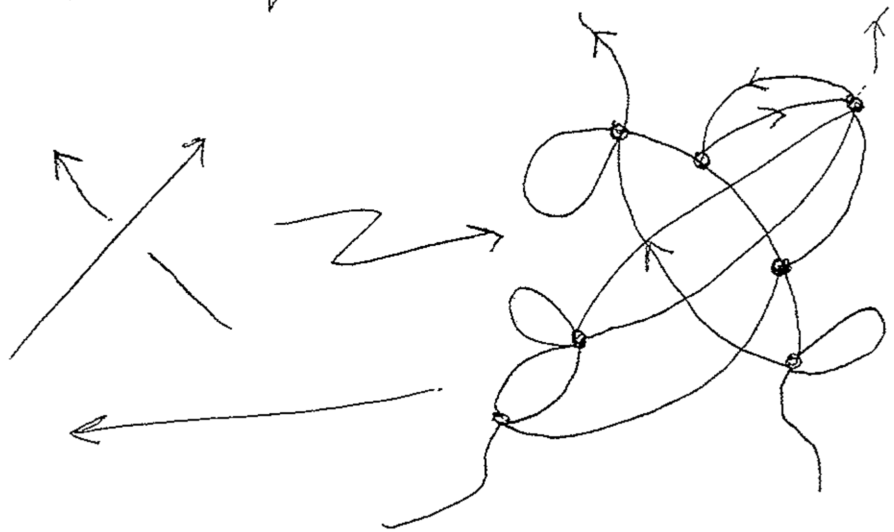
$$J(\text{unknot}) = q + q^{-1}.$$

up to a power of q

<u>Theorem (L, Moffatt 2009)</u>

$$J(L) =_{q} \operatorname{per}(M_L)$$

" and system of equations ass. to the gadget must have a solution "



<u>method</u>: R-matrix state sum for J.

Jones and Quantum Computing

Freedman, Kitaev, Larsen, Wang, Aharonov, Jones ...

<u>Theorem</u> ( )

Suppose we have a BQP language and an input x. Then we can construct a link L, of size polynomial in $|x|$, such that

if x is in the language then $\left| J(L, e^{2\pi i/5}) \right| < [2]_5^{|x|+1} \cdot 0{,}39$

x not — " — $\left| J(L, e^{2\pi i/5}) \right| > [2]_5^{|x|+1} \cdot 0{,}65$

$$\boxed{[2]_5 = 2\cos \pi/5}$$

These are weak, <u>additive</u> approximations of Jones polynomial [Bordewich, Freedman, Lovász, Welsh]

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computer experiments initiated with Moffatt, Plecháč.