Enumeration of Perfect Matchings

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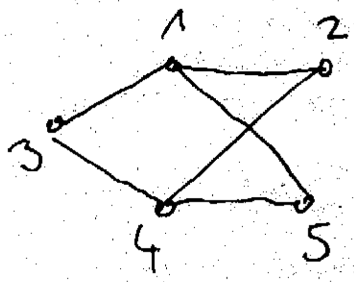
Matchings are good!

beginning of 60's:

Edmonds: Polynomial-time good characterisation $P \neq NP$

Kastelegn, Fisher: combinatorial (2D) statistical physics

$G = (V, E)$ graph
vertices → V, edges → $E \subseteq \binom{V}{2}$



$M \subseteq E$ matching if $e, e' \in M \Rightarrow e \cap e' = \emptyset$

12, 34 is matching
12, 13 not matching

Maximum (weighted) matchings, perfect matchings

The work of Kasteleyn (end of 50's)

$G = (V, E)$ graph, $w : E \longrightarrow \mathbb{Q}$
weights on edges

Partition fction of dimer arrangements:

$$P(G, x) = \sum_{\substack{E' \subseteq E \\ \text{perf. m.}}} x^{w(E')} \quad \text{where } w(E') = \sum_{e \in E'} w(e)$$

$\boxed{\text{$G$ planar then $P(G,x)$ is a determinant-type expression; may be efficiently calculated.}}$

$\boxed{\text{Determinantal Complexity}}$

p: real polynomial with variables $x_1, \ldots, x_n$

formula that calculates p: can use $+, \cdot$; start with variables

$L(p)$: size of the least formula for p.

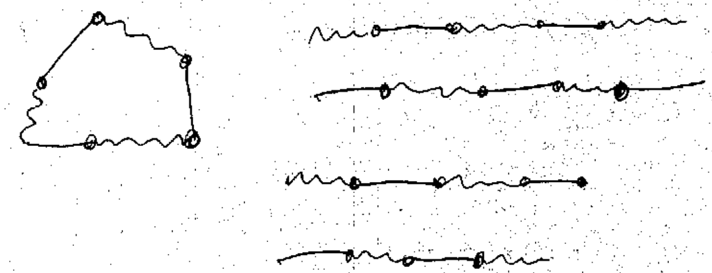
Theorem (Valiant)

If $L(p) = m$ then p is a projection of $\mathrm{Det}_{2m+2}(Y_{ij})$ $\longrightarrow$ $Y_{ij} := x_\ell$ or a const.

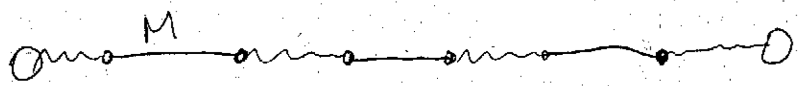
? Does permanent (variables x_{ij}) have exponential determinantal complexity ? Partial results by $\boxed{\text{Mulmuley}}$ methods of algebraic geometry

Efficient algorithm for max matching

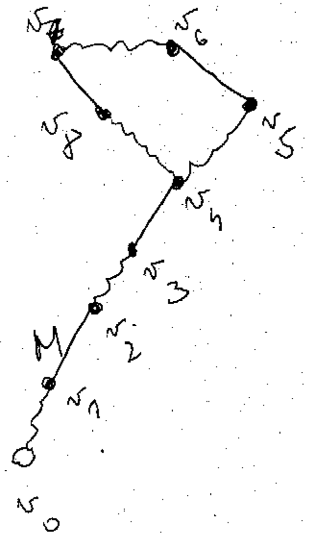
Union of 2 perfect matchings :



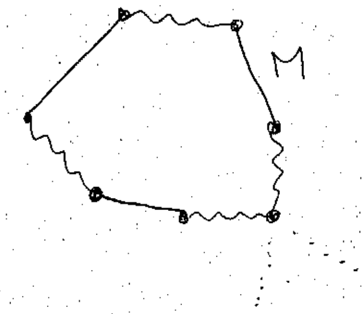
M maximum if there is no augmenting alternating path



o : uncovered by M



flower



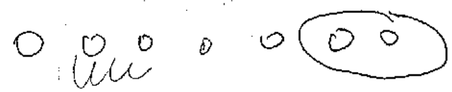
blossom

Key Observation.

B blossom of M. Then M max in G iff $M \setminus B$ max in G/B [contraction]

The algorithm. Given M

1. Grow "flower" from each uncovered vertex
   ⓐ 2 components merge ⇒ augmenting alt. path
   ⓑ constructs a flower ⇒ contract (smaller graph)
   ⓒ otherwise M is maximum matching by Tutte theorem

odd  defect

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The construction of Valiant

① variable x : $\begin{pmatrix} x_1 & \\ 1 & 1 \end{pmatrix}$

In the induction step we assume a particular form of the matrix :

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

ⓐ $f = f_1 \cdot f_2$

$$\begin{array}{|c|c|} \hline f_1 & 0 \\ \hline 0 \;\; \boxed{1} & f_2 \\ \hline \end{array}$$

ⓑ addition : a key observation :

$$d_1 \left[\begin{pmatrix} \alpha_1 & 0 \\ \hline A_1 & \beta_1 \end{pmatrix} \right. \qquad d_2 \left[\begin{pmatrix} \alpha_2 & 0 \\ \hline A_2 & \beta_2 \end{pmatrix} \right.$$

A_i upper $\triangle$ with 1 on D

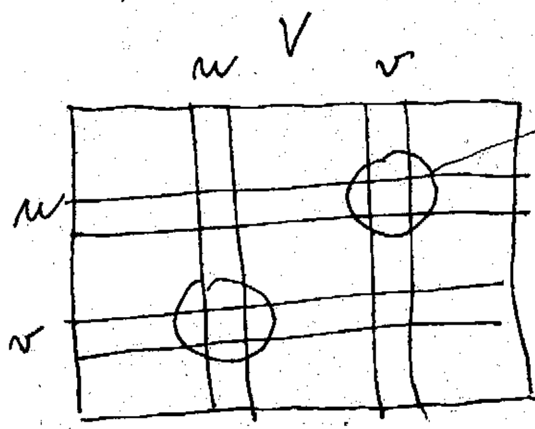
$$M = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \hline A_1 & 0 & \beta_1 \\ 0 & A_2 & \beta_2 \end{pmatrix}$$

$\Rightarrow \det M = (-1)^{d_2} \det(M_1) + (-1)^{d_1} \det(M_2)$

An alternative approach from stat. physics.

$G = (V, E)$ graph $\quad D$ orientation of G

$A(D)$ is $V \times V$ adjacency matrix of D



$$\begin{cases} x^{w(e)} & \text{if } u \xrightarrow{e} v \\ -x^{w(e)} & \text{if } u \xleftarrow{e} v \end{cases} \quad \text{skew-symmetric}$$

$$\mathrm{Pfaf}(A(D)) = \sum_{\substack{M \text{ perf.} \\ \text{m. of } G}} (-1)^{\#(D,M)} \, x^{w(M)}$$

Pfaf is a determinant-type expression
(Gaussian elimination works)

60's: $\boxed{\text{Kasteleyn}}$ conjectured that: if G has genus g

then $P(G,x)$ is linear combination of 4^g terms

of form $\mathrm{Pfaf}(A(D))$, D orientation of G.

$\boxed{\text{Theorem}}$ True for planar graphs $(g=0)$: Pfaf suff.

Theorem (Galluccio, Loebl 99; Tesler 2000)

$$P(G,x) = \sum_{i=1}^{4^g} 2^{-g}\, \mathrm{Pfaf}(A(D_i)) \cdot (-1)^{\#_i}$$

for well-defined orientations D_i and numbers $\#_i$.

Cimasoni, Reshetikhin (recently): topological meaning of $\#_i$

[Additive determinantal complexity]

$\mathcal{X} = (x_e)_{e \in E}$

$P(G, \mathcal{X}) = \sum_{\substack{E' \subseteq E \\ \text{p.m.}}} \prod_{e \in E'} x_e$

Norine (2008, but beginning of 2000's)

Graph G is k-Pfaffian if $P(G, \mathcal{X})$ is linear combination of k terms of form $\mathrm{Pf}(A(D))$, D orientation of G, and k is smallest possible.

Conjecture (Norine): The Pfaffian number of a graph is always a power of 4.

DISPROVED by Miranda, Lucchesi (2009)

L, Maslaum: Conjecture true for Ising partition fction

Ising partition fction

$G=(V,E)$ $\quad s: V \longrightarrow \{1,-1\}$ $\quad$ energy$(s) = -\sum_{e=\{uv\}} w(e)\, s(u)\, s(v)$

$$Z(G,x) = \sum_{s:V\to\{1,-1\}} x^{-\text{energy}(s)}$$

usually: $x = e^{\beta}$

generating fction of edge-cuts

$$\mathcal{C}(G,x) = \sum_{\substack{E'\subseteq E \\ \text{edge-cut}}} x^{w(E')}$$

$$Z(G,x)\; x^{\sum_E w(e)} = \mathcal{C}(G,x^2)$$

$G = (V, E)$; $E' \subseteq E$ even if (V, E') has all degrees even.

$$\mathcal{E}(G, x) = \sum_{\substack{E' \subseteq E \\ \text{even}}} x^{w(E')}$$

<u>Theorem</u> (Van der Waerden, Mc Williams)
$G = (V, E)$ arbitrary. Then $\mathcal{E}(G, x)$ is equivalent to $\mathcal{C}(G, x)$ [and $Z(G, x)$].

"duality of Tutte polynomial" X "geometric duality"

(9A)

$$\sinh(x) = \frac{e^{x}-e^{-x}}{2}, \quad \cosh(x) = \frac{e^{x}+e^{-x}}{2}$$

$$Z(G,\beta) = \sum_{s:V\to\{1,-1\}} e^{\beta \sum_{E} w(uv)\, s(u)\, s(v)}$$

$$= \sum_{s} \prod_{uv\in E} \left[\cosh \beta w(uv) + s(u)s(v) \sinh \beta w(uv)\right]$$

$\prod_{E} \cosh()$

$$= K \sum_{s} \prod_{uv\in E} \left(1 + s(u)s(v) \tanh \beta w(uv)\right) =$$

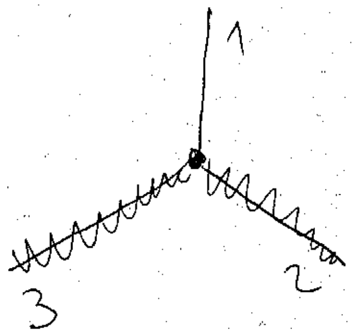
$$K \sum_{s} \sum_{E'\subseteq E} \prod_{uv\in E'} s(u)s(v) \tanh \beta w(uv) = K \sum_{E'\subseteq E} U(E') \prod_{uv\in E'} \tanh \beta w(uv)$$

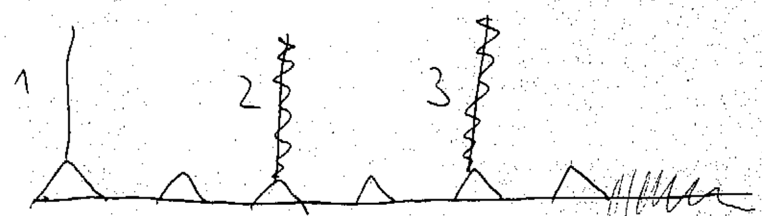
where $U(E') = \sum_{s:V\to\{1,-1\}} \prod_{uv\in E'} s(u)s(v) = \begin{cases} 2^{|V|}, & E' \text{ even} \\ 0 & \text{otherwise} \end{cases}$

Even sets to perfect matchings

$$G \longmapsto G_{\triangle}$$

even sets $\qquad\qquad$ perfect matchings





The gadget chooses a linear order of edges incident with each vertex

<u>Conclusion</u>. Generating fction of edge-cuts, even sets, Ising partition fction is transformed to generating fction of perfect matchings of $G_{\triangle}$.

$\vec{\Delta} = (\vec{\Delta}_v,\ v \in V(G))$ arbitrary choice of orientations of each gadget. Orientation D of G_Δ is $\boxed{\vec{\Delta}\text{-admissible}}$ if D extends $\vec{\Delta}$.

$$\boxed{\begin{array}{ccc} \text{set of } \vec{\Delta}\text{-admissible orientations of } G_\Delta & \equiv & \text{set of orientations of } G \end{array}}$$

$C_{\vec{\Delta}}(G)$: minimum # of $\vec{\Delta}$-admissible orientations of G_Δ so that $\mathcal{E}(G, x)$ is a linear combination of corresponding Pfaffians.

$C_{Ising}(G)$: $\min C_{\vec{\Delta}}(G)$

$$\mathcal{E}(G, x) = \sum_{\substack{E' \subseteq E \\ \text{even}}} \prod_{e \in E'} x_e$$

$$x = (x_e)_{e \in E}$$

Theorem (L, Masbaum 09)

1. $c_{\vec{\Delta}}(G)$ is a power of 4.
2. $c_{\text{Ising}}(G) = 4^g$, g : embedding genus of G.

$\boxed{\text{Some Corollaries ☺}}$

1. $c_{\text{Ising}}(G) = 1$ iff G planar
2. Let G be a graph. The minimal genus of an orientable surface which supports an even drawing of G is equal to the embedding genus of G.

Proof.

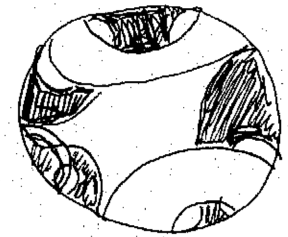
1) Theorem (Tesler)

G drawn in the plane. Then there is $\varepsilon_0 \in \{\pm 1\}$ and crossing orientation D_0 of G so that for each perfect matching M of G, its sign in $\mathrm{Pfaf}(A(G, D_0))$ sat.

$\mathrm{sign}(M, D_0) = \varepsilon_0 (-1)^{K(M)}$

K: # double points of the drawing.

2) g-graph G



bridges $R_1, \dots, R_{2g}$

only edges on bridges

$\Rightarrow$ project bridges to the plane $\rightsquigarrow$ special planar drawing of G.

$q_0(a_i) = q_0(b_i) = 0 \quad \forall i$

Observation. Special embedding. Then

$K(M) = q_0([M])$

[M] hom. class of M.

$H = H_1(\Sigma_g; \mathbb{F}_2)$ first homology group of Σ_g with coefficients in $\mathbb{F}_2$.

$a_1, b_1, \dots a_g, b_g$ basis of H

a_i: corresponds to class of bridge R_{2i-1}

b_i: corr. to bridge R_{2i}

intersection form $\cdot$ on H:

$a_i \cdot a_j = b_i \cdot b_j = 0$

$a_i \cdot b_j = \Delta_i^j$

Quadratic form on $(H, \cdot)$

$q: H \longrightarrow \mathbb{F}_2$

$q(x+y) = q(x) + q(y) + x \cdot y$

determined by the values on the basis $\Rightarrow$ there are 2^{2g} q.f.

Corollary $\forall M$

$$\mathrm{sign}(M, D_0) = \varepsilon_0 (-1)^{q_0([M])}$$

Restatement of Galluccio, Loebl (Cimasoni, Reshetikhin)

There is ε_0 and coll. of (Dq) of 4^g orientations indexed by quadratic forms $q \in Q$ s.t. for each M,

$$\mathrm{sign}(M, D_q) = \varepsilon_0 (-1)^{q([M])}$$

Arf invariant:

$$\sum_{x \in H} (-1)^{q(x)} = (-1)^{\mathrm{Arf}(q)} 2^g$$

$\mathrm{Arf}(q) = 0$ if q takes $2^{g-1}(2^g+1)$ times value 0.

$\mathrm{Arf}(q) = 1$ if q takes $2^{g-1}(2^g-1)$ times value 0.

Lemma

$$\frac{1}{2^g} \sum_{q} (-1)^{\mathrm{Arf}(q)} (-1)^{q(x)} = 1$$

Corollary.

$$\frac{\varepsilon_0}{2^g} \sum_{q \in Q} (-1)^{\mathrm{Arf}(q)} \mathrm{sign}(M, D_q) = 1.$$

How to construct crossing orientation?

G plane graph. D orientation of G is Kasteleyn if every bounded face is clockwise odd.

Observation.

If we remove from a planar drawing of a graph all edges involved in a crossing, then any Kasteley orientation of the rest can be extended to crossing orientation.

We are given $\vec{\Delta}_v, v \in V$.

This defines order of ~~vertices~~ edges at each vertex v, so that the orientation of the gadgets is Kasteleyn.

This defines an embedding of G on a surface Σ.

Let D_0 be <u>crossing orientation</u> extending $\vec{\Delta}_v, v \in V$.

Hence, each admissible orientation D is obtained from D_0 by changing orientation of some edges of G. We have, for E' even:

$$\operatorname{sign}(E', D) = \operatorname{sign}(E', D_0) \cdot (-1)^{|E' \cap S(D)|}$$

$$= \varepsilon_0 (-1)^{q_0([E'])} (-1)^{l_D(E')}$$

where

$l_D(E') = |E' \cap S(D)|$,

$S(D) = \#$ changed edges of D w.r.t. D_0

l_D is a linear form!

Hence:

$$\operatorname{sign}(E', D) = \varepsilon_0 (-1)^{q_D(E')}$$

~~We want too~~

Each homology class is realized by an even set [0 corr. to $\emptyset$]

l_D defines linear form

$l : l(x) = l_D(E_x)$, $x \in H$ [E_x: corr. even set]

$$q_D = q_0 + l$$

We want to characterize all real vectors $(\alpha_q)_q$ s.t. satisfying:

$$\boxed{\text{For each } a \in \mathbb{F}_2^{2g}, \quad 1 = \sum_q (-1)^{q(a)} \alpha(q)}$$

One solution comes from Arf invariant:

$$\boxed{\alpha(\rho_r) = 2^{-g} (-1)^{s(r)}}$$

since for each a,

$$1 = \sum_r (-1)^{\rho_r(a)} 2^{-g} (-1)^{s(r)}$$

Theorem

Let Q denote the set of all the quadratic forms on $\mathbb{F}_2^{2g}$

Then

$$\det\left((-1)^{q(x)} \right)_{(x,q) \in \mathbb{F}_2^{2g} \times Q} \neq 0 .$$