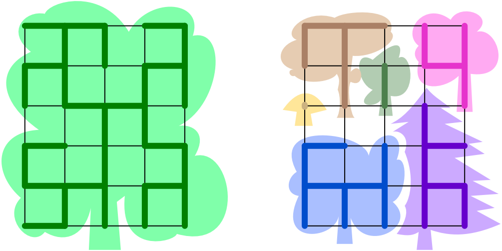
Trees and forests⋆ in 2-D⋆ Statistical Mechanics

Andrea Sportiello

September 8^{th} 2009, at Centre Émile Borel, Institut Henri Poincaré
Trimester on Statistical physics, combinatorics and probability



⋆ but not only!

Two words on 2-D Statistical Mechanics

Minimal intro to Critical Phenomena

A paradigm: Lattice models of Statistical Mechanics...

➊ work at finite volume; ➋ introduce an ensemble and a Gibbs measure; ➌ consider (connected) k-point correlation functions; ➍ large volume asymptotics and definition of correlation length

$\longrightarrow$ notion of criticality

At criticality, there is no typical length scale

$\longrightarrow$ in D dimensions: scale invariance

further (RG) reasonings (better within QFT) lead to universality: critical exponents are only determined by the symmetry property of the 'physical' and 'target' spaces.

This is why we study prototypal 'Ising-like' models!

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Specialties at $D = 2$

scale invariance ➨ conformal invariance ➨ CFT
➥ Schramm-Loewner evolution (SLE)

S-matrix in "1+1" ➨ Yang-Baxter eqs. ➨ Integrability
➥ for lattice loop models: Temperley-Lieb Algebra

Certain properties are shared by "all and only" the planar graphs

ex1 ➨ many uses of planar duality (e.g. Peierls contours);
ex2 ➨ Kasteleyn orientability for Dimer models (and Ising);
ex3 ➨ canonical basis for the cyclomatic vector space;
ex3bis ➥ canonical leg-ordering for Bernardi partitionability;
ex4 ➨ restrictions on how to draw bunches of non-crossing paths;
ex4bis ➥ Lindström-Gessel-Viennot–type formulas;

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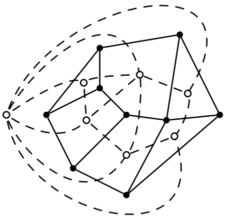
Planar duality

Let the connected planar $G = (V; E; F)$ *(vertices, edges, faces)* a planar dual graph $\widehat{G} \simeq (F, E, V)$ is defined,



Duality induces a natural bijection among subgraphs $H \subseteq G$ and $K \subseteq \widehat{G}$: $E(\widehat{H}) = \widehat{E(H)}^{c}$. One gets $L(\widehat{H}) = K(H) - 1$, so that:

- Spanning Forests and Connected Subgraphs are dual;
- Trees are self-dual, and the intersection of the two.

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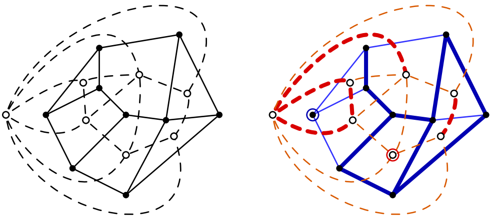

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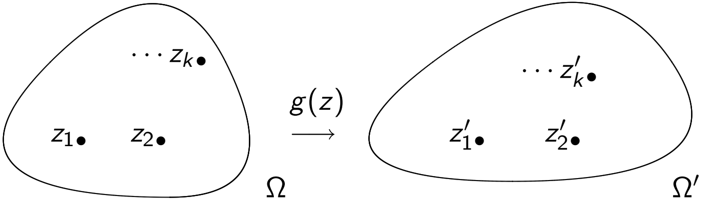
CFT and covariance of k-point functions



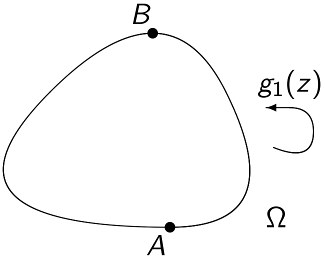
For primary fields, k-point fns. have covariance property

$$\langle \phi_1(z_1) \cdots \phi_k(z_k) \rangle_{\Omega}^{\mathrm{conn}} = \prod_{i=1}^{k} \left| \frac{\partial z'}{\partial z} \right|_{z=z_i}^{\Delta_i/d} \langle \phi_1(z'_1) \cdots \phi_k(z'_k) \rangle_{\Omega'}^{\mathrm{conn}}$$

P. Ginsparg, *Applied Conformal Field Theory*

A picture of SLE

Riemann thm.: $\forall\ \Omega, \Omega' \simeq \mathbb{D}$
$\{A, B\} \in \partial\Omega$, $\{A', B'\} \in \partial\Omega'$,
$\exists\, g(z) : \Omega \to \Omega'$ holomorphic
$A, B \xrightarrow{g} A', B'$; $g'(B) = 1$.



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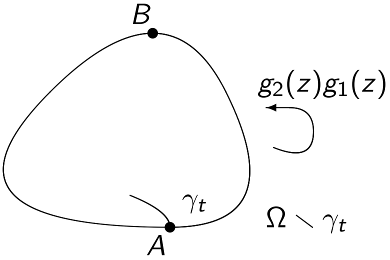

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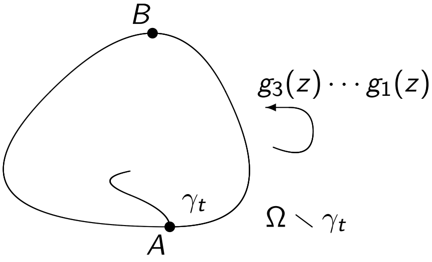

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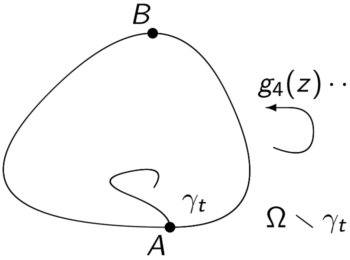

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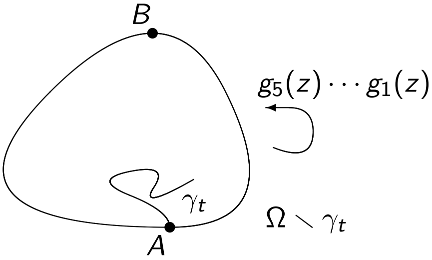

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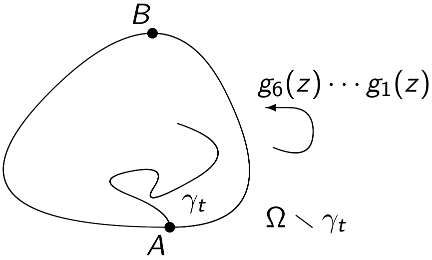

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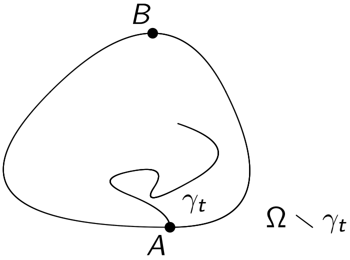

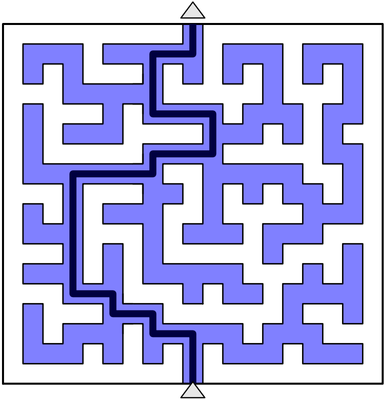

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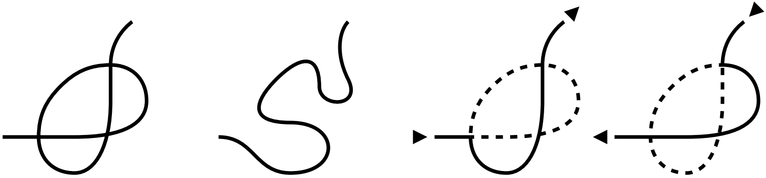
Random Walks, Self-Avoiding RW, Loop-Erased RW



self-crossing $\kappa = 8/3$ $\kappa = 2$

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Schramm's first paper on SLE!

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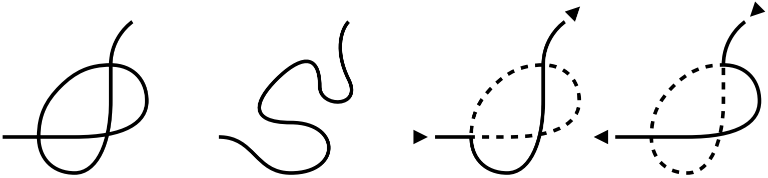

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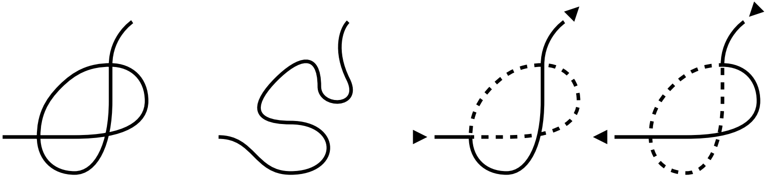

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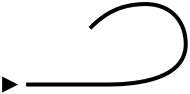

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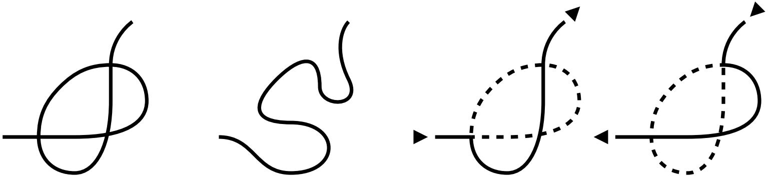

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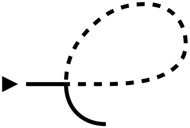

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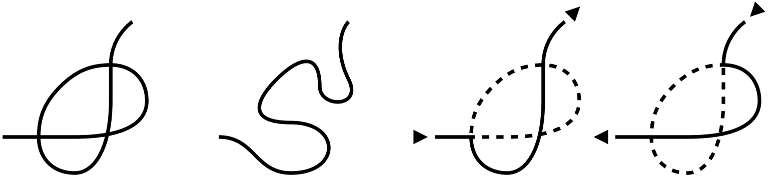

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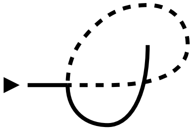

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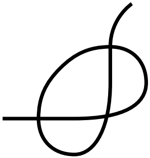

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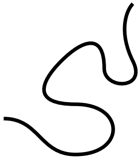

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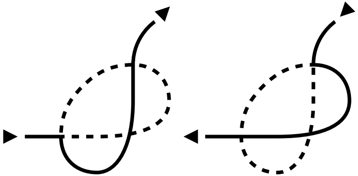

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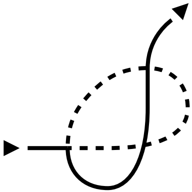

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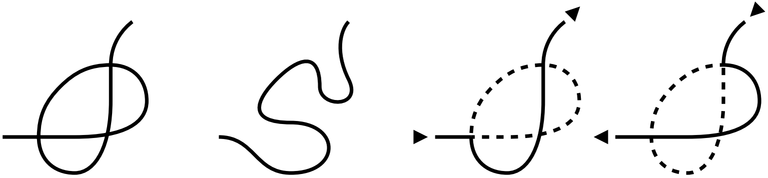

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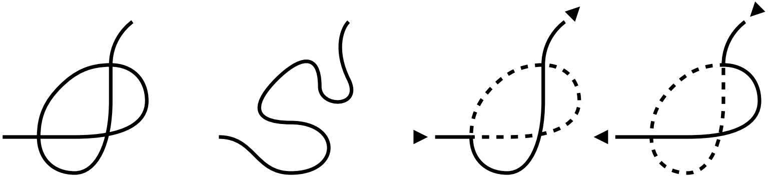

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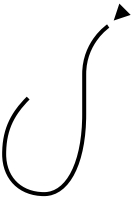

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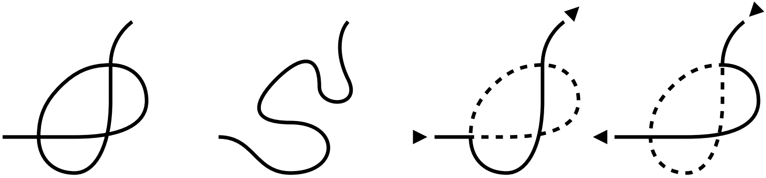

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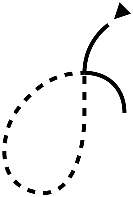

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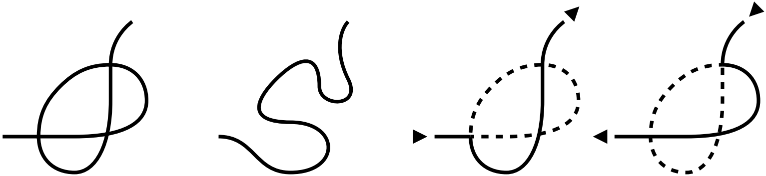

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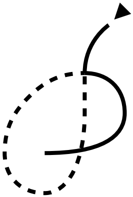

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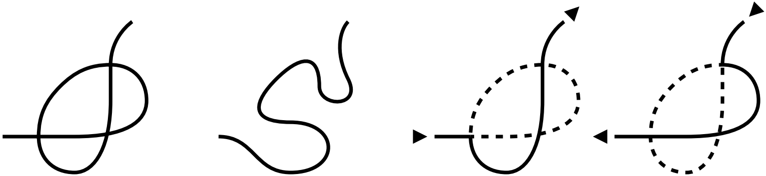

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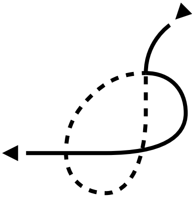
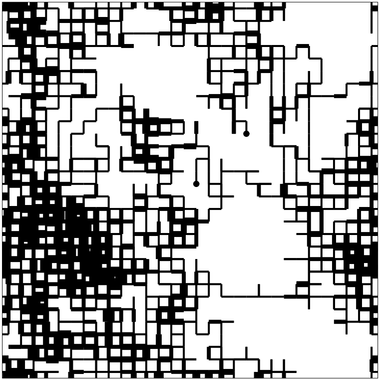
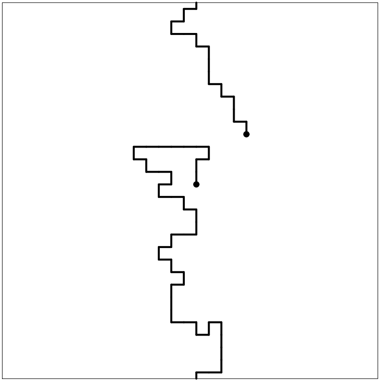
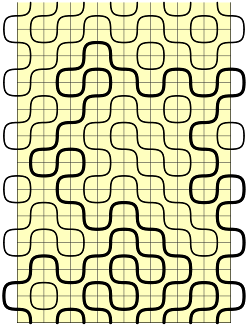


Not much is saved after loop-erasure... the LERW is a fractal with Haussdorff dimension 'only' $5/4 = 1 + \kappa/8$ (at $\kappa = 2$)

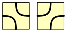




$O(n)$ Loop model on a strip and Temperley-Lieb algebra



The rules:

❶ fill the square lattice with 

❷ give weight n to each cycle.

This model of dense loops has special algebraic properties ➽ TL Algebra

$$e_i^2 = n\, e_i \qquad e_i e_{i\pm1} e_i = e_i$$
$$[e_i, e_j] = 0 \quad \text{if } |i-j| > 1.$$

YBE ➽ comm. of transfer matrices ➽ results from integrability

The ensemble of Random Planar Graphs

Planar vs. non-planar: topological genus $\mathfrak{h}$
'Natural' ensemble of random graphs weighted with their genus, and thus a 'natural' ensemble of random planar graphs (RPG)

For statistical models on RPG, the solution often comes from Random Matrix techniques (a collection of sophisticated tools emerging from Wick theorem for tensor fields), whose main thm. is

$$\sum_{\substack{G: \\ V_k \text{vert. deg. } k}} \frac{N^{-2\mathfrak{h}}}{|\mathrm{Aut}(G)|} = \frac{1}{N^2} \ln \int_{N \times N} dM \; e^{-\frac{N}{2} \operatorname{tr} M^2} \prod_k \frac{(N \operatorname{tr} M^k)^{V_k}}{V_k! \, k^{V_k}}$$

Random Matrices in one slide

❶ Choose your 'combinatorial' Feynman rules, get the action $\mathcal{S}(M^{(\alpha)})$, as a trace of a matrix-valued polynomial.

❷ As we have $\exp[N\mathcal{S}(M^{(\alpha)})]$, and $N \to \infty$ for RPG, it looks like we can use a saddle-point technique. Not still! we have $\sim N^2$ d.o.f.

❸ Exploit properties of the trace, to factor out the $\mathcal{O}(N^2)$ 'angular' d.o.f. from the N 'eigenvalue' d.o.f. Now we can use saddle point.

❹ The Jacobian gives a squared Vandermonde determinant, acting as a 'log' coulomb repulsion (on $\mathbb{R}$) among the eigenvalues.

❺ Fine-tuning the (polynomial) potential, can get the (KPZ image of) the $(m, m+1)$ conformal hierarchy.

P. Di Francesco, *Matrix Model Combinatorics: Applications to Folding and Coloring*

More than one matrix

Theory of characters for unitary groups $U(N)$ and $SU(N)$:
IZHC formula for AB-interaction, and also $ABAB$-interaction

✍ P. Zinn-Justin, J.-B. Zuber, *On some integrals over the $U(N)$ unitary group and their large N limit*

✍ V.A. Kazakov, P. Zinn-Justin, *Two-Matrix model with ABAB interaction*

Also feasible if $\mathcal{S}$ is overall quadratic in all but one matrix: reduce to 1-matrix via Gaussian integration, but get further prefactors besides Vandermonde

This is what happens in Kostov solution of the $O(n)$-loop model on RPG...

✍ I.K. Kostov, M. Staudacher, *Multicritical Phases of the $O(n)$ Model on a Random Lattice*

The KPZ correspondence

Stat. Mech. Lattice model defn. on any (planar, degree-k) graph

On $2D$ periodic lattice:
at $\beta = \beta_c$ and $L \to \infty$
a CFT of central charge c

$\Leftrightarrow$

On RPG's:
at $\tilde{\beta} = \tilde{\beta}_c$ and $g = g^\star(\tilde{\beta}_c)$,
non-trivial exponents, e.g.
the string susceptibility γ.

Related exponents! E.g.

$$\gamma = 2 - \frac{1}{12}\left(25 - c + \sqrt{(1-c)(25-c)}\right)$$

✍ B. Duplantier, *Conformal Random Geometry*

Spanning trees for all seasons

Fortuin-Kasteleyn expansion for Potts Model

$$
\begin{aligned}
Z_{\text{Potts}} &= \sum_{\sigma} e^{-\sum_{\langle ij \rangle} J_{ij} \delta(\sigma_i, \sigma_j)} \\
&= \sum_{\sigma} \prod_{(ij)} \big(1 + v_{ij}\, \delta(\sigma_i, \sigma_j)\big) \qquad \Big[v_{ij} := e^{J_{ij}} - 1 \Big] \\
&= \sum_{H \subseteq G} \prod_{(ij) \in E(H)} v_{ij} \Bigg(\sum_{\sigma} \prod_{(ij) \in E(H)} \delta(\sigma_i, \sigma_j) \Bigg) \\
&= \sum_{H \subseteq G} q^{K(H)} \prod_{(ij) \in E(H)} v_{ij} \,. \qquad \left[K(H) = \# \left\{ \begin{matrix} \text{comp.} \\ \text{in } H \end{matrix} \right\} \right]
\end{aligned}
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You recognize the multivariate Tutte Polynomial of G,
(slightly reparametrized and rescaled) ... wait until next slide!

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Recall:

- $L(H)$, the *cyclomatic number*, is the number of linearly-independent cycles in H.
- Euler formula states that $V - K = E - L$.

$$Z_{\mathrm{RC}}(G; \vec{w}; \lambda, \rho) = \sum_{H \subseteq G} \lambda^{K(H)-K(G)} \, \rho^{L(H)} \prod_{(ij) \in E(H)} w_{ij} \qquad \begin{bmatrix} \lambda\rho = q \\ w_{ij} = v_{ij}/\rho \end{bmatrix}$$

Tutte: $w = 1$; $x := Z[\bullet\!\!-\!\!-\!\!\bullet] = 1 + \lambda$ and $y := Z[\ \ ] = 1 + \rho$.

$$\begin{tikzcd}
& \boxed{\text{Max.-connected subgraphs}} \arrow[r] \arrow[d, leftrightarrow, "\text{(dual if } G \text{ planar)}"'] & \\
\boxed{\text{Random Cluster Model}} \arrow[ur, "\lambda \to 0"] \arrow[r, "\rho \to 0"'] & \boxed{\text{Spanning Forests}} \arrow[r, "\lambda, \rho \to 0"] & \boxed{\text{Gen. funct. Spanning Trees}}
\end{tikzcd}$$

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 & \lambda \to 0 & \boxed{\text{Max.-connected subgraphs}} & & \\
\boxed{\text{Random Cluster Model}} & \longrightarrow & \text{(dual if } G \text{ planar)} \Updownarrow & \longrightarrow \; \lambda, \rho \to 0 \; \longrightarrow & \boxed{\text{Gen. funct. Spanning Trees}} \\
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Random Cluster Model $\xrightarrow{\lambda \to 0}$ Max.-connected subgraphs $\longrightarrow$ Gen. funct. Spanning Trees

(dual if G planar) $\Updownarrow$ $\qquad \lambda, \rho \to 0$

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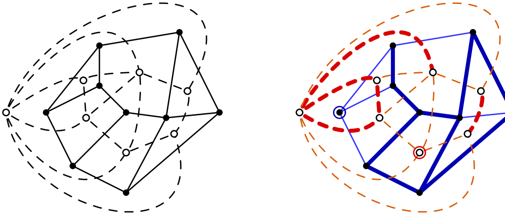
The Random Cluster Model on planar graphs

Recall from "Planar duality" slide: if G is connected and planar

- $E(\widehat{H}) = \widehat{E(H)}^{c}$, and $L(\widehat{H}) = K(H) - 1$
- Spanning Forests and Connected Subgraphs are dual;
- Trees are self-dual, and the intersection of the two.

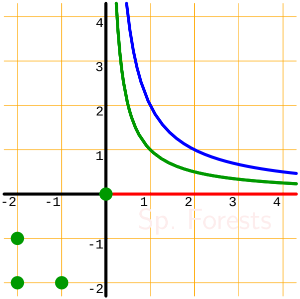
So duality acts as $\lambda \leftrightarrow \rho$ and $w_{ij} \leftrightarrow 1/w_{ij}$.



Temperley-Lieb Algebra with parameter $\sqrt{\lambda\rho}$ plays a role.

Comput. complexity of Random-Cluster Partition Function

$Z_{\rm RC}(G;\vec{w};\lambda,\rho)$ is 'hard' to calculate (#P) in general, except for some special loci in the (λ,ρ) plane: [Jaeger, Welsh, 90's]



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- Computable in poly-time as a Pfaffian if $\lambda\rho = 2$ (Ising) and G is planar [Kasteleyn; Kač, Ward; 60's]
- Computable in poly-time at exceptional special points $(\lambda,\rho) = (-2,-2)$, $(-2,-1)$, $(-1,-2)$ and $(0,0)$.

$(0,0)$: Spanning Trees, counted by a determinant through Matrix-Tree Theorem [Kirchhoff, 1848 (!)]

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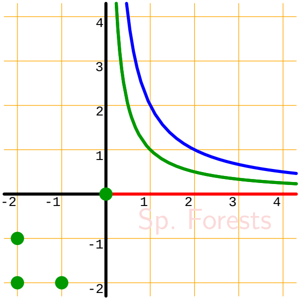

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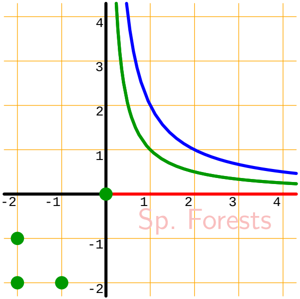

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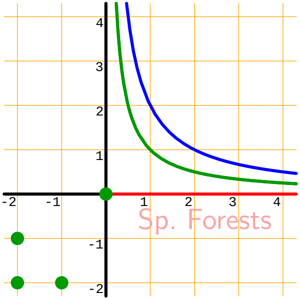

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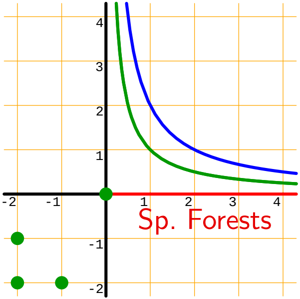

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The Matrix-Tree Theorem

$$Z_{\text{Tree}}(G; \vec{w}) = \sum_{\substack{T \subseteq G \\ \text{trees}}} \prod_{(ij) \in E(T)} w_{ij} = \det L(i_0)$$

where i_0 is any vertex of G (the 'root'), $L(i_0)$ is the minor of L with row and col. i_0 removed, and L is the graph Laplacian matrix:

$$L_{ij} = \begin{cases} -w_{ij} & (ij) \in E(G) \\ 0 & (ij) \notin E(G) \\ \sum_{k \sim i} w_{ik} & i = j \end{cases} \qquad\qquad L \sim -\nabla^2$$

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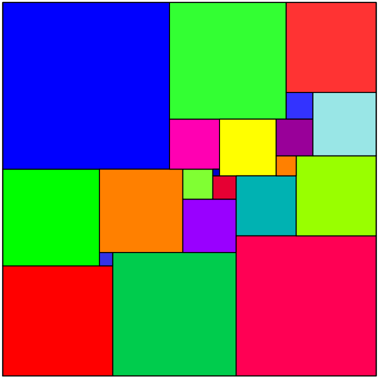


Also famous the application:

R.L. Brooks, C.A.B. Smith, A.H. Stone and W.T. Tutte, *The Dissection of Rectangles into Squares,* Duke Math. J. 7 (1940)

A primer in Grassmann Algebra

Introduce the formal symbols θ_i, with $\theta_i \theta_j \overset{*}{=} -\theta_j \theta_i$,
and symbols $(\int \mathrm{d}\theta_i)$ with the rule $\int \mathrm{d}\theta_i \theta_i = 1$ and $\int \mathrm{d}\theta_i 1 = 0$.
As $\theta_i^2 = 0$, the most general monomial $\prod_i \theta_i^{n_i}$ has $n_i = 0, 1$ *
* ➡ θ is a 'real fermion' of spin zero (no spin indices)!
Remark

$$\int \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1 \prod_i \theta_i^{n_i} = \begin{cases} 1 & n_i = 1 \quad \forall i \\ 0 & \text{otherwise} \end{cases}$$

Special application, for $n \times n$ antisymmetric matrix A,

$$\int \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1 \exp\left(\tfrac{1}{2}\theta A \theta\right) = \mathrm{pf} A = (\det A)^{\frac{1}{2}} \,.$$

Going to "complex" is good and natural...

Now take $2n$ symbols $\bar{\psi}_1, \ldots, \bar{\psi}_n$ and $\psi_1, \ldots, \psi_n$,
(➡ *charge:* $= \deg \bar{\psi} - \deg \psi$) and $\mathcal{D}(\psi, \bar{\psi}) := \mathrm{d}\psi_n \mathrm{d}\bar{\psi}_n \cdots \mathrm{d}\psi_1 \mathrm{d}\bar{\psi}_1$.
Then, for any matrix A

$$\int \mathcal{D}(\psi, \bar{\psi})\, f(\bar{\psi}, A\psi) = \det A \int \mathcal{D}(\psi, \bar{\psi})\, f(\bar{\psi}, \psi)\,;$$

$$\int \mathcal{D}(\psi, \bar{\psi})\ \exp(\bar{\psi} A \psi) = \det A\,;$$

$$\int \mathcal{D}(\psi, \bar{\psi})\, \bar{\psi}_{i_1} \psi_{j_1} \cdots \bar{\psi}_{i_k} \psi_{j_k}\ \exp(\bar{\psi} A \psi) = \epsilon(I, J) \det A_{I^c, J^c}\,.$$

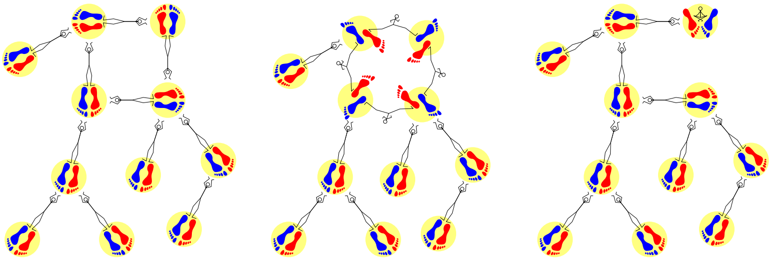
Fermionic counterparts of Jacobian rule for change of variables, Gaussian Integral and Wick Theorem

Fermionic formulation of the Matrix-Tree Theorem

From Gaussian Integral formula in complex Grassmann Algebra:

$$\exp(\bar{\psi} L \psi) = \prod_{i,j} (1 + w_{ij}\, \bar{\psi}_i \psi_i - w_{ij}\, \bar{\psi}_i \psi_j)$$

$$Z_{\mathrm{Tree}}(G; \vec{w}) = \int \mathcal{D}_{V(G)}(\psi, \bar{\psi})\, \bar{\psi}_{i_0} \psi_{i_0} \exp(\bar{\psi} L \psi)$$



Determinantal processes

Lattice versions of point processes:

Potts		$\sim$	Process	$\sim$	Statistics
$q = 0$	(trees)		Determinantal		Fermi
$q = 1$	(percol.)		Poisson/Bernoulli		Classical
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In particular, Spanning Trees are a realization of a lattice Determinantal Process

$$\mathrm{prob}(e_1, \ldots, e_k \in T) = \det \left(\mathcal{K}(e_i, e_j) \right)_{i,j=1,\ldots,k}$$

k-point functions fully encoded by 1- and 2-point functions!

B.J. Hough, M. Krishnapur, Y. Peres and B. Virag,
Zeros of G.A.F.s and Determinantal Point Processes

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Negative Association

For spanning trees, and $w_e \in \mathbb{R}^+$:

$$\mathrm{prob}(e_1, e_2 \in T) \leq \mathrm{prob}(e_1 \in T)\mathrm{prob}(e_2 \in T)$$

Highly non-trivial! (Feder-Mihail "Balanced Matroids", 1992)

For comparison, proving that for Random Cluster $q < 1$ and $w_e \in \mathbb{R}^+$ the *converse* holds

$$\mathrm{prob}(e_1, e_2 \in H) \geq \mathrm{prob}(e_1 \in H)\mathrm{prob}(e_2 \in T)$$

is fairly standard (Ginibre, 1970; FKG, 1971)

The state-of-the-art understanding of all this is in:

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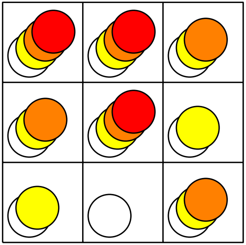
The Abelian Sandpile Model

The ASM is a non-equilibrium model:

Rules: ❶ Graph G. Height vars $z_i \in \mathbb{N}$ at vertices (the sand). A "border". ❷ If $z_i >$ number of neighs of i, donates a grain to each neighbour. Sand possibly falls out of the border. ❸ Any reasonable Markov dynamics for sand addition, then at each time perform the relaxation above (➽ well-defined because of abelianity!)



D. Dhar, *Studying Self-Organized Criticality with Exactly Solved Models*

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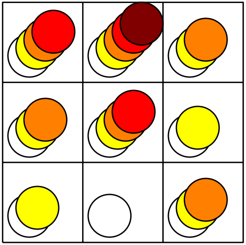

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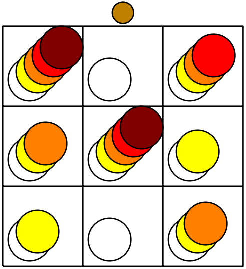

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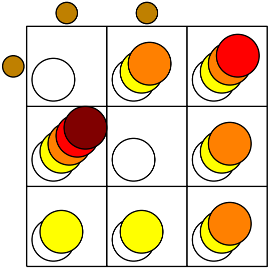
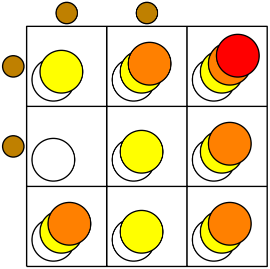


D. Dhar, *Studying Self-Organized Criticality with Exactly Solved Models*

The Abelian Sandpile Model

The ASM is a non-equilibrium model:

Rules: ❶ Graph G. Height vars $z_i \in \mathbb{N}$ at vertices (the sand). A "border". ❷ If $z_i >$ number of neighs of i, donates a grain to each neighbour. Sand possibly falls out of the border. ❸ Any reasonable Markov dynamics for sand addition, then at each time perform the relaxation above (➽ well-defined because of abelianity!)



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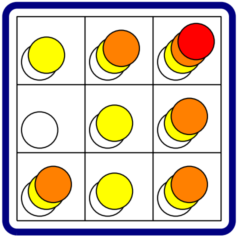
The Markov Chain has an uniform-measure core of recurrent configurations, and an arborescence of transient configs.

Recurrent configs. are characterized by the burning test. This graphical construction has as outcome a bijection between recurrent configs. and spanning trees (with the border counting as a single root vertex)



Natural combinatorial quantities in the ASM recognized as having the appropriate logarithmic-CFT properties

P. Ruelle et al., arXiv:`cond-mat/0609284`, `0707.3766`, `0710.3051`

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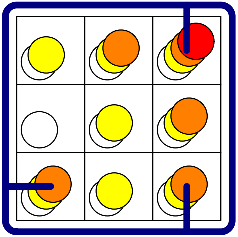

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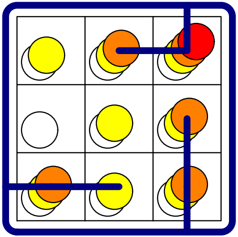

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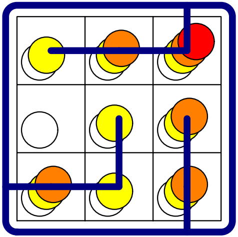

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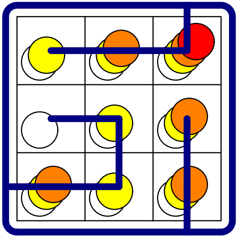

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The Propp and Wilson algorithm

Exact sampling in CS $\Leftrightarrow$ Exact solution (for Z) in SM

Exact sampling of uniform spanning trees:

J.G. Propp and D.B. Wilson, *How to get a perfectly random sample from a generic Markov chain and generate a random spanning tree of a directed graph*, J. Alg. 27 (1998)

The algorithm:

❶ Choose any ordering $v_1, \ldots, v_n$ of the vertices of G;

❷ $T_1 = \{v_1\}$;

❸ For $(k = 2, \ldots, n)$: $\{T_k = T_{k-1} \cup \mathrm{LERW}(v_k \to T_{k-1})\}$;

❹ Return T_n.

If v_1, v_2 are the boundary points in SLE protocol ➽ SLE duality among the $\kappa = 8$ Peano-like profile of the spanning tree, and the $\kappa = 2$ LERW curve.

Spanning Trees on RPG

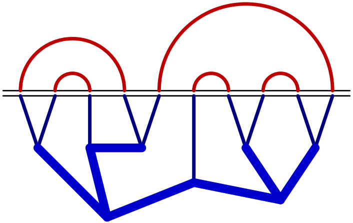
The easiest ever model on RPG: can be reduced to "one-vertex" expectations $\langle \mathrm{tr}\, M^{2k} \rangle = C_k$ (Catalan numbers)

➽ link-pattern and cubic-tree d.o.f. factorize:

$$Z^{RPG}_{\mathrm{Trees}}(g) = \sum_k g^{2k} \frac{C_{2k} C_{k+1}}{2k+2} = \sum_k g^{2k} \frac{(4k)!}{(k+1)!(k+2)!(2k)!}$$

$$\sim \mathrm{const.} \sum_k g^{2k} k^{-4}$$

$$\Rightarrow \ \gamma = -1$$



Spanning Trees on planar graphs counted with genus

One can even do higher genus! (with no need of loop equations)
The 'hard' part (counting unicellular maps) is a classic result of

T. Walsh and A. Lehman, *Counting rooted maps by genus,* J. Combin. Theory Ser.B 13 (1975)

A. Goupil and G. Schaeffer, *Factoring N-cycles and counting maps of given genus,* Eur. J. Comb. 19 (1998)

➽ Also γ' is derived easily and rigorously.

Towards a comprehension of forests

Even/Odd Temperley-Lieb algebra

In the Dense Loop Model formulation of Random Cluster Model on planar graphs, we had a TL algebra with the rules

$$
\begin{aligned}
e_{2i}^2 &= \lambda e_{2i} & \qquad e_i e_{i\pm1} e_i &= e_i \\
e_{2i+1}^2 &= \rho e_{2i+1} & \qquad [e_i, e_j] &= 0 \quad |i-j| > 1
\end{aligned}
$$

Invariant under $e_{2i} \to \alpha e_{2i}$, $e_{2i+1} \to e_{2i+1}/\alpha$, $\lambda \to \lambda/\alpha$, $\rho \to \alpha\rho$.

This is why mostly studied is $\lambda = \rho \quad (= \sqrt{q})$
Only missing case: $\lambda \neq 0$ and $\rho = 0$, i.e. forests (+dual)

ASM: Merino Theorem

If we sum over recurrent configs. in the ASM, with weight $\prod_i (1+t)^{z_i}$, we get the generating function of spanning forests, counted with $t^{K(F)-1}$, up to a simple overall factor $(1+t)^{\mathrm{const.}}$.

Equivalently, we could make a (possibly non-equilibrium) Monte-Carlo dynamics on the ASM, with rates compatible with the weight above.

First proven by Merino López

C. Merino López, *Chip firing and Tutte polynomial,* Ann. Comb. (1997)

Also bijective proof in

R. Cori and Y. Le Borgne, *The sandpile model and Tutte polynomials,* Adv. Appl. Math. 30 (2003)

Negative association for forests

Conjectured to hold (actually, for the whole $0 \leq q < 1$ and $w_e \in \mathbb{R}^+$ Random Cluster model), but no proof so far!

cfr.:

R. Pemantle, *Toward a theory of negative dependence,* J. Math. Phys. 41 (2000)

G.R. Grimmett and S.N. Winkler, *Negative association in uniform forests and connected graphs,* RSA 24 (2004)

An extension of the Matrix-Tree Theorem

From Kirchhoff Matrix-Tree Theorem we had

$$Z_{\mathrm{Tree}}(G; \vec{w}) = \lim_{\lambda,\rho\to 0} \frac{1}{\lambda} Z_{\mathrm{RC}}(G; \vec{w}; \lambda, \rho) = \int \mathcal{D}(\psi, \bar{\psi}) \exp(\bar{\psi} L \psi) \bar{\psi}_i \psi_i$$

This "free-fermion" expression can be extended to forests:

$$Z_{\mathrm{Forest}}(G; \vec{w}; \lambda) = Z_{\mathrm{RC}}(G; \vec{w}; \lambda, \rho = 0) = \int \mathcal{D}(\psi, \bar{\psi}) \exp(\bar{\psi} L \psi) \times \exp\left[\lambda\left(\sum_i \bar{\psi}_i \psi_i + \sum_{(ij)} w_{ij}\, \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j\right)\right]$$

Non-Gaussian integral, as expected from intrinsic hardness of the counting problem. However consequences can be drawn from such an expression.

$\mathrm{O}(n)$ and $\mathrm{OSP}(n|2m)$ non-linear σ-models

Generalize models with $\mathrm{O}(n)$ symmetry to $\mathrm{OSP}(n|2m)$:

$$\vec{\sigma} = (\phi^a)_{a=1,\dots,n} \qquad |\vec{\sigma}|^2 = \sum_{a=1}^{n} (\phi^a)^2$$

$$\Downarrow$$

$$\vec{\sigma} = (\underbrace{\phi^a}_{B}; \underbrace{\bar{\psi}^b, \psi^b}_{F})_{\substack{a=1,\dots,n \\ b=1,\dots,m}} \qquad |\vec{\sigma}|^2 = \sum_{a=1}^{n} (\phi^a)^2 + 2\lambda \sum_{a=1}^{m} \bar{\psi}^b \psi^b$$

"Non-linear σ-model": if we have $\mu(\sigma) \propto \prod_i \delta(|\vec{\sigma}_i|^2 - 1)$

For $n \in \mathbb{N}^+$ and $m \in \mathbb{N}$, analytic continuation should depend on $n - 2m$ only. [Parisi, Sourlas, 1979; Cardy, 1983]

Simplest non-trivial choice: $\mathrm{OSP}(1|2)$, i.e. $\vec{\sigma} = (\phi; \bar{\psi}, \psi)$.

$\mathrm{OSP}(1|2)$ – Spanning-Forest correspondence

Thm: the $\mathrm{OSP}(1|2)$ non-linear σ-model partition function is related to the Random Cluster partition function at $\rho = 0$

$$Z_{\mathrm{OSP}(1|2)}(G; -\vec{w}/\lambda) = Z_{\mathrm{RC}}(G; \vec{w}; \lambda, \rho = 0)$$

at a perturbative level. For the $\mathrm{RP}^{0|2}$ model, the relation is non-perturbative.

Main points:

1. $\int d\phi_i \delta(\phi_i^2 + 2\lambda\bar{\psi}_i\psi_i - 1)$ ➜ $\vec{\sigma}_i = \epsilon_i(1 - \lambda\bar{\psi}_i\psi_i; \bar{\psi}_i, \psi_i)$
2. Forget about ϵ's (say, all $+1$). [this why 'perturbative'...]
3. $e^{\lambda \sum_i \bar{\psi}_i\psi_i}$ comes as a Jacobian in the resolution of the δ's.

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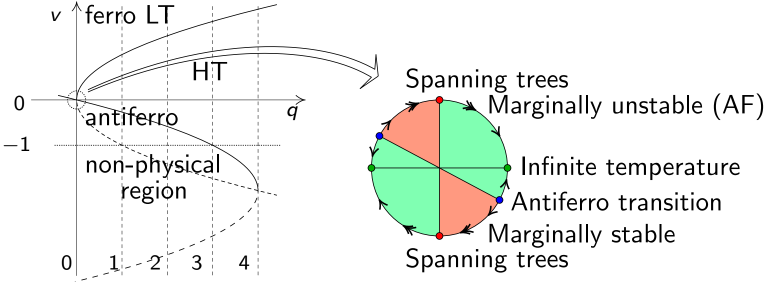
Critical behaviours of spanning forests

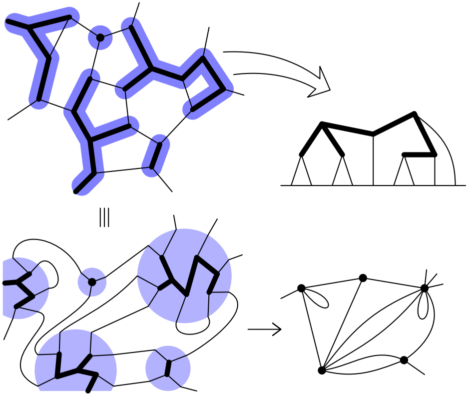
Numerics for $D > 2$ and results for K_n give a percolation transition at $t_{\rm perc} > 0$, besides criticality at $t = 0$



One guesses that $t_{\rm perc} \to 0^+$ for $D \to 2^+$, and this causes "asymptotic freedom" (i.e., a double zero in the beta-fn. $\beta(t)$)
However, we miss a rigorous non-perturbative proof that nothing else happens in the ferromagnetic regime $t > 0$, at $D = 2$

A Random Matrix formulation of the problem

Recall that trees gave a “one-vertex” model. Similarly, forests with k components, of sizes V_i, may be related to RPG's with k vertices, the i-th having $V_i + 2$ ‘legs’.



some combinatorics
+ change of variables
➽ Kostov $O(n)$ model at $n = -2$.

Critical points:
$t = 0$, $g = \frac{1}{8}$
(spanning trees);
$t = -1$, $g = \frac{\pi}{8\sqrt{6}}$
(antiferro transition?)
(Tutte partitionability?)