ASEPs, PASEPs and Paths

Des Johnston Richard Blythe
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The Grand-Canonical Asymmetric Exclusion Process and the One-Transit Walk, Dyck Paths, Motzkin Paths and Traffic Jams, J. Stat. Mech. (2004)
Continued Fractions and the Partially Asymmetric Exclusion Process, J. Phys. A (2009)

IHP, Sept, 2009

Some philosophical waffle

Big picture understood for equilibrium thermodynamics

Universality, critical exponents....

Non-equilibrium less well understood

Driven, diffusive systems, ASEP

Integrable system, XXZ, Bethe ansatz

Plan of Talk

Definition of the model

Solution of the model (matrix product ansatz)

Partition function zeros

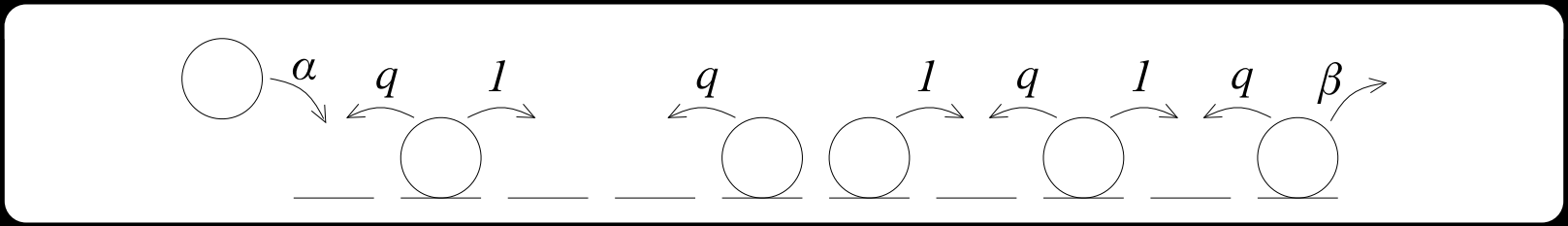
Paths

Definition of the model

One dimensional lattice, N sites

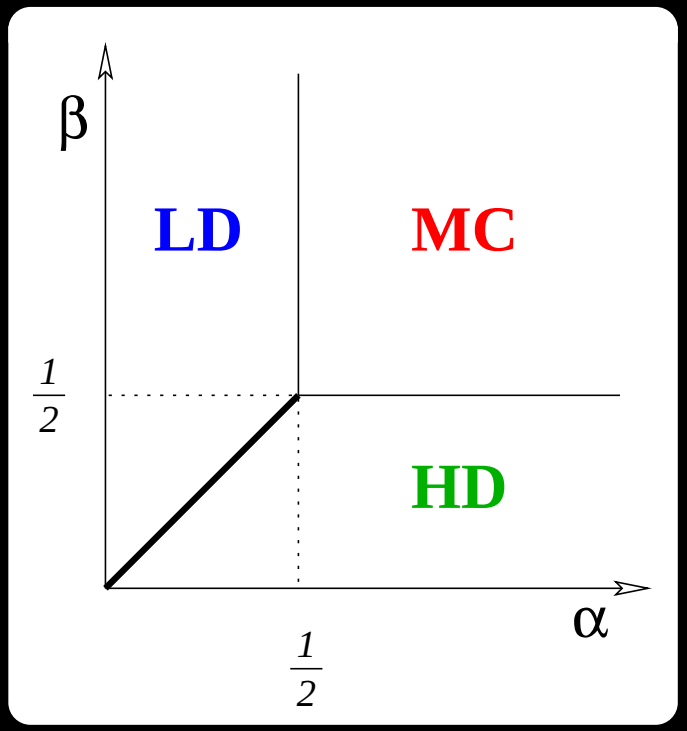
"Hard" particles

Forcing



The Phase Diagram ($q = 0$)

Give away the punchline



Roll of honour (TASEP $q = 0$): Derrida,Evans, Hakim Pasquier
Roll of honour (PASEP $q \neq 0$): Blythe, Evans, Colaiori, Essler

Setup for solution

Configuration

$$\mathcal{C}$$

Statistical weights for configuration

$$f(\mathcal{C})$$

Normalized probability

$$P(\mathcal{C}) = f(\mathcal{C})/Z$$

Normalization

$$Z = \sum_{\mathcal{C}} f(\mathcal{C})$$

Master Equation

$$\frac{\partial P(\mathcal{C},t)}{\partial t} = \sum_{\mathcal{C}' \neq \mathcal{C}} \left[P(\mathcal{C}',t) W(\mathcal{C}' \to \mathcal{C}) - P(\mathcal{C},t) W(\mathcal{C} \to \mathcal{C}') \right]$$

Setup for (Matrix Product) solution

Represent ball with

$$X_i = D$$

Represent space with

$$X_i = E$$

Represent $P(\mathcal{C})$ as

$$P(\mathcal{C}) = \frac{\langle W|X_1 X_2 \ldots X_N|V\rangle}{Z_N}$$

Make sure behaviour of D, E is compatible with dynamics:

$$\begin{aligned} DE - qED &= D + E \\ \alpha\langle W|E &= \langle W| \\ \beta D|V\rangle &= |V\rangle \end{aligned}$$

Now what?

Various ways to proceed

Purely algebraic - "normal order"

Get a representation of D, E - not unique (and infinite)

Evaluate $Z_N = \langle W|(D+E)^N|V\rangle = \langle W|C^N|V\rangle$ having done this

Matrices for the ASEP

$$D = \begin{pmatrix} 1+\frac{1}{\beta} & \sqrt{\kappa} & 0 & \cdots \\ 0 & 1 & 1 & \\ 0 & 0 & 1 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix},$$

$$E = \begin{pmatrix} 1+\frac{1}{\alpha} & 0 & 0 & \cdots \\ \sqrt{\kappa} & 1 & 0 & \\ 0 & 1 & 1 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}.$$

$$\langle W| = (1, 0, 0, \cdots) \qquad |V\rangle = (1, 0, 0, \cdots)^T$$

$$\kappa = \frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\alpha\beta}$$

The solution

DEHP matrix algebra

$$D + E = DE$$

Normalization is the quantity to calculate

$$\langle W|(D+E)^N|V\rangle$$

Given, for random sequential dynamics by

$$Z_N = \sum_{p=1}^{N} \frac{p(2N-1-p)!}{N!(N-p)!} \frac{(1/\beta)^{p+1} - (1/\alpha)^{p+1}}{(1/\beta) - (1/\alpha)}$$

Current is then given by

$$J_N = \frac{Z_{N-1}}{Z_N}$$

Zeroes For *equilibrium* models

Partition function can be written as a polynomial in the fugacity $y = \exp(-2h)$

$$Z_L = \sum_r D_r y^r \ ,$$

completely expressed in terms of its zeros, y_r

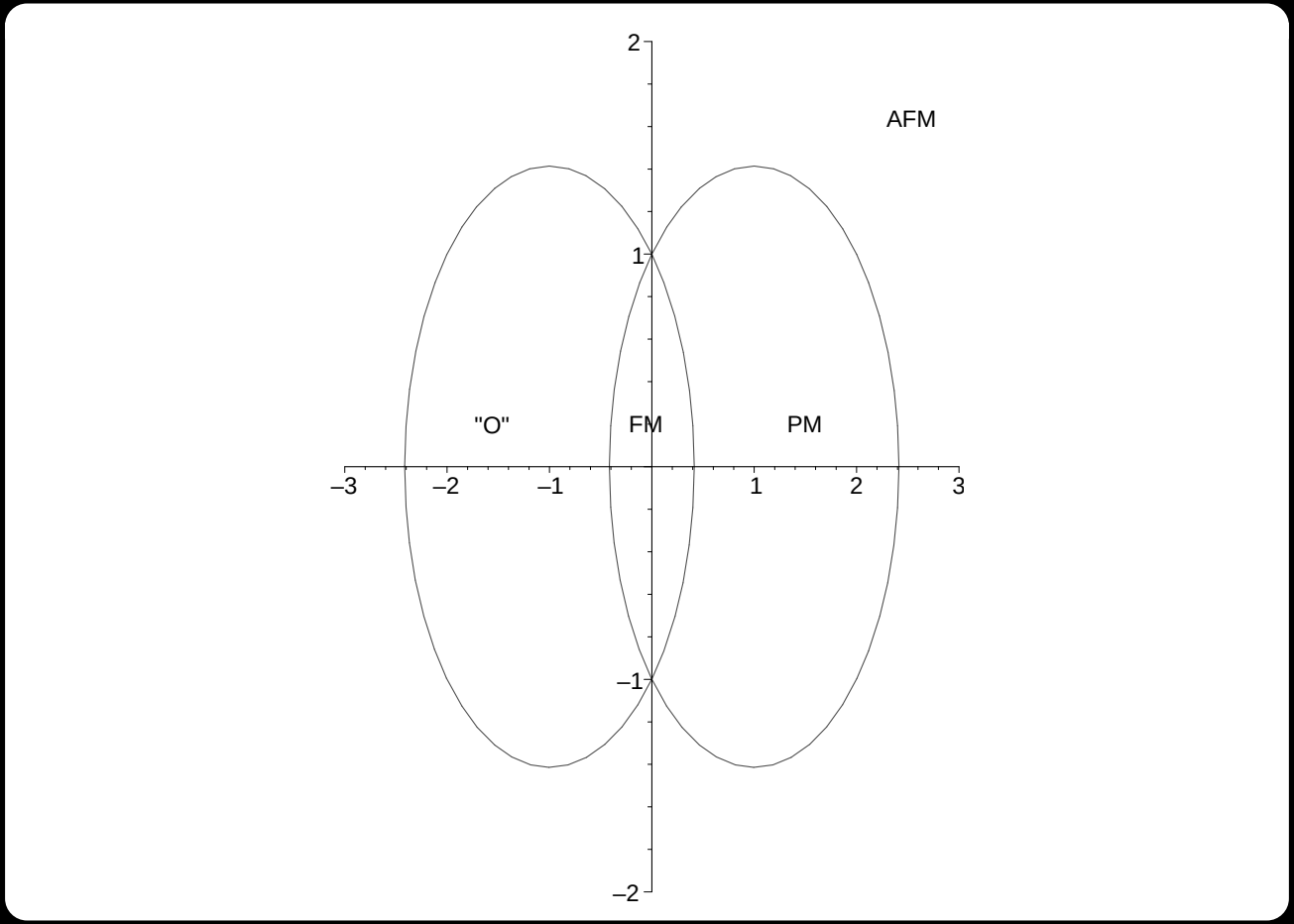
so too may the (reduced) finite-size free energy:

$$f_L(h) \sim -\frac{1}{L} \ln \prod_r (y - y_r(h))$$

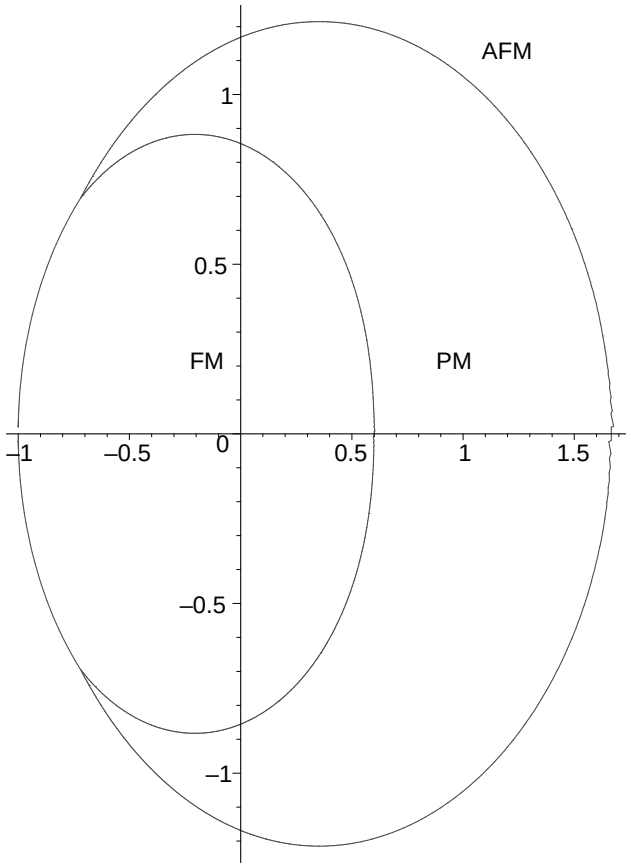
In the thermodynamic limit

$$f(h) \sim -\int_C dy \rho(y) \ln(y - y(C))$$

Works nicely in practice

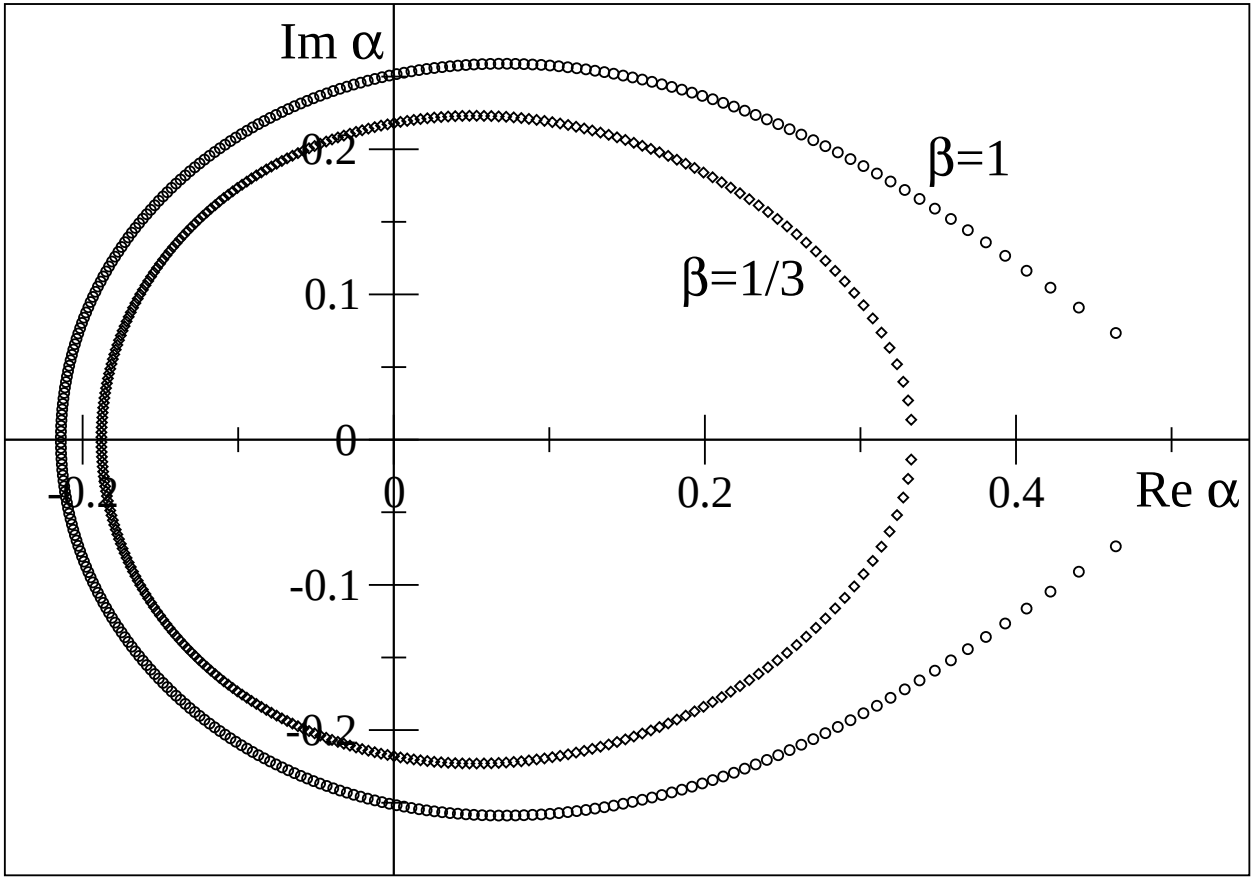


Works nicely in practice II



ASEP zeroes

Blythe and Evans simply did this for ASEP normalization



Exactly as in equilibrium case - why?

Grand Canonical

Look at *Grand*-canonical normalization

$$\mathcal{Z}(z) = \sum_{N=0}^{\infty} Z_N z^N \, .$$

Carrying out sum with the help of

$$\left(\frac{1-\sqrt{1-4z}}{2z}\right)^p = \sum_{N=0}^{\infty} \frac{p(2N+p-1)!}{p!(N+p)!} z^N \, ,$$

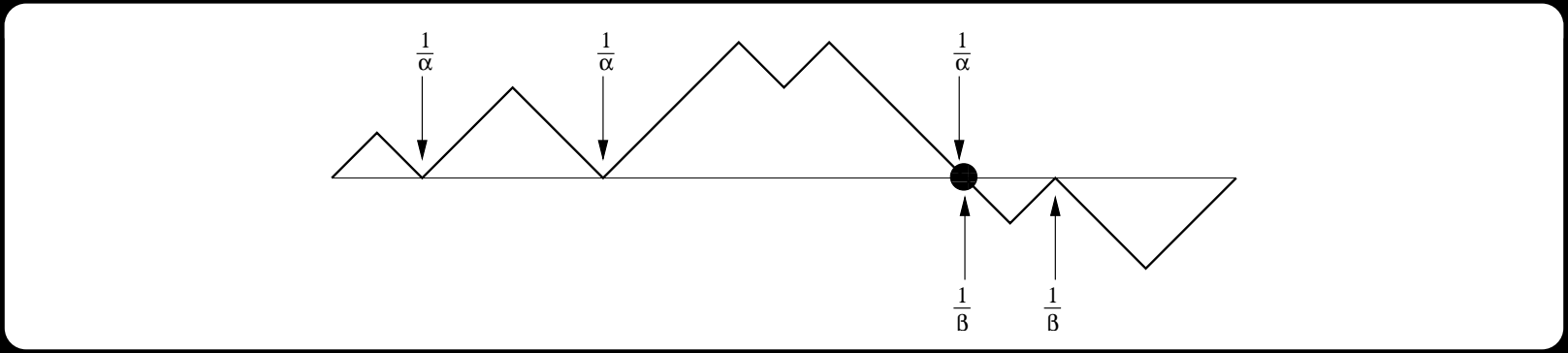
Gives

$$\mathcal{Z}(z) = \frac{\alpha\beta}{(x(z)-\alpha)(x(z)-\beta)}$$

$$x(z) = \frac{1}{2}\left(1-\sqrt{1-4z}\right)$$

Generating Function of What?

$\mathcal{Z}(z)$ is recognizable as the generating function for a lattice path problem.



Get generating function by iterating

$$G_E(z) = z\left(1 + G_E(z) + [G_E(z)]^2 + \cdots\right) = \frac{z}{1 - G_E(z)}$$

Then iterate again with contact weights $1/\alpha$

$$\mathcal{Z}(z) = \alpha\beta\, G_D(1/\alpha, z)\, G_D(1/\beta, z)$$

Using Grand Canonical

Singularities give Z_N

In maximal current phase with $\alpha < \beta$ the $\sqrt{1-4z}$ gives rise to $4^N/N^{3/2}$ so

$$Z_N \sim \frac{4^N}{\pi^{1/2} N^{3/2}} \left[\frac{1}{(2\alpha-1)^2} - \frac{1}{(2\beta-1)^2} \right]$$

In regions (i) and (ii) poles dominate

$$Z_N \sim \frac{\alpha(1-2\beta)}{(\alpha-\beta)(1-\beta)} \frac{1}{(\beta(1-\beta))^N}$$

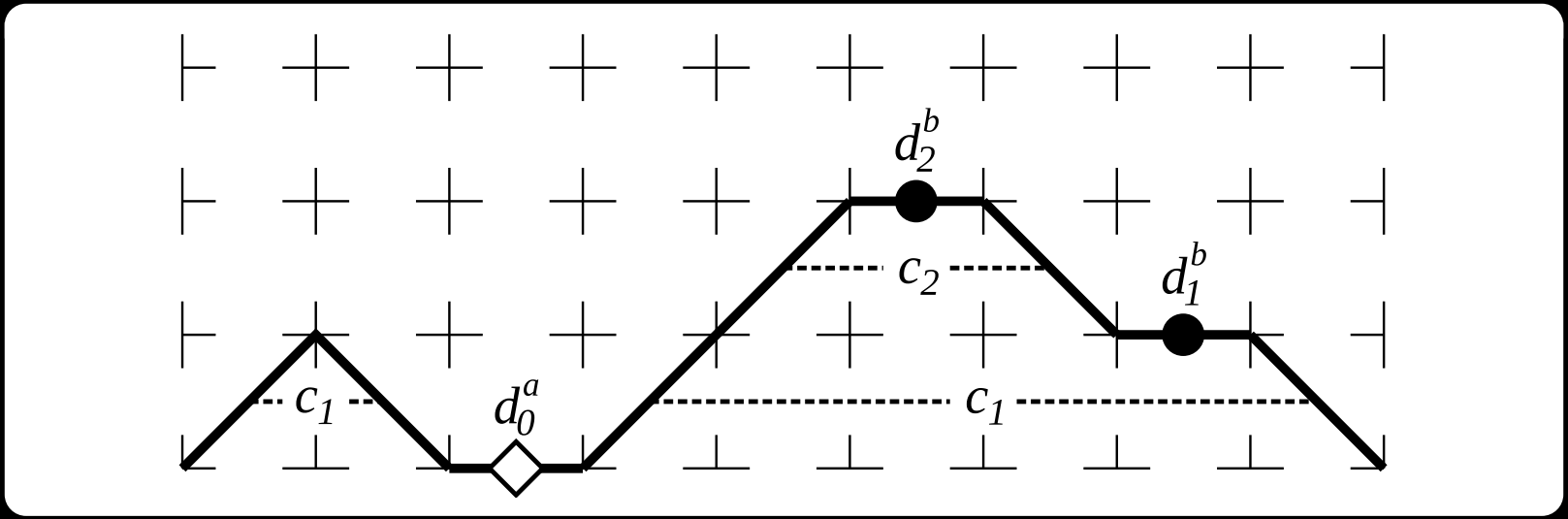
And the PASEP?

$$D_q = \frac{1}{1-q}\begin{pmatrix} 1+\tilde{\beta} & \sqrt{c_1} & 0 & \cdots \\ 0 & 1+\tilde{\beta}q & \sqrt{c_2} & \\ 0 & 0 & 1+\tilde{\beta}q^2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix},$$

$$E_q = \frac{1}{1-q}\begin{pmatrix} 1+\tilde{\alpha} & 0 & 0 & \cdots \\ \sqrt{c_1} & 1+\tilde{\alpha}q & 0 & \\ 0 & \sqrt{c_2} & 1+\tilde{\alpha}q^2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}.$$

$$\langle W| = h_0^{1/2}(1,0,0,\cdots) \qquad |V\rangle = h_0^{1/2}(1,0,0,\cdots)^T$$

(Bicoloured) Motzkin Paths



$$\tilde{d}_n = \frac{2 + (\tilde{\alpha} + \tilde{\beta})\, q^n}{1 - q}$$

$$\tilde{c}_n = \frac{(1 - q^n)(1 - \tilde{\alpha}\tilde{\beta}\, q^{n-1})}{(1 - q)^2}$$

Generating Function as a Continued Fraction

$$\mathcal{Z}(z) = \sum_N Z_N z^N$$

$$\mathcal{Z}(\tilde{\alpha}, \tilde{\beta}, q, z) = \cfrac{1}{1 - \tilde{d}_0 z - \cfrac{\tilde{c}_1 z^2}{1 - \tilde{d}_1 z - \cfrac{\tilde{c}_2 z^2}{1 - \tilde{d}_2 z - \cfrac{\tilde{c}_3 z^2}{\cdots}}}}$$

Singularities of Continued Fractions

generic, poles
Worpitzsky: A continued fraction of the form

$$\cfrac{1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cfrac{a_4}{\dots}}}}$$

converges if the partial numerators a_p satisfy

$$|a_p| < 1/4,\ p = 2, 3, 4, \dots$$

Hunting Poles

Generic, poles
Look at n^{th} *convergent* of the continued fraction

$$\begin{aligned} K_0 &= \frac{1}{1-\tilde{d}_0 z}\,, \\ K_1 &= \frac{1}{1-\tilde{d}_0 z - \frac{\tilde{c}_1 z^2}{1-\tilde{d}_1 z}}\,, \end{aligned}$$

Continued fraction is given *exactly* by the convergent K_n if $\tilde{c}_{n+1} = 0$.

$$\tilde{\alpha}\tilde{\beta} = q^{-n}\,.$$

THE END :)