CELLULAR AUTOMATA & BRANCHING BALLISTIC ANNIHILATION

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USP - Universidade de São Paulo

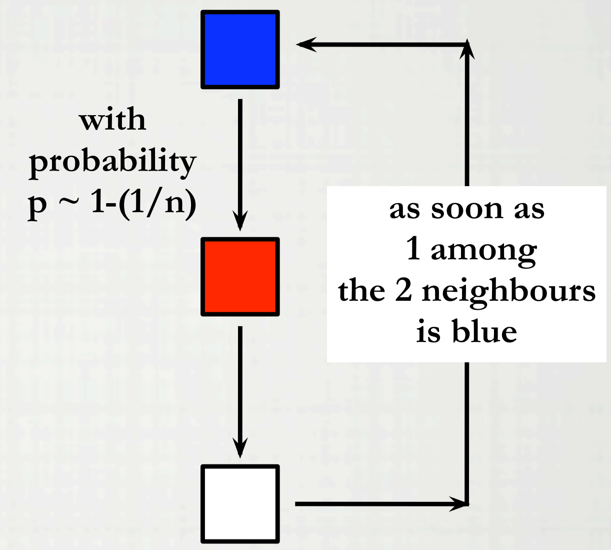
StatComb, IHP, 2009

SIMPLE CELLULAR AUTOMATA #2

Let us see the set Z of integers as a set of cells with 3 possible colors

- blue (= infected),
- red (= healing),
- white (= healthy),

evolving according to the following synchronous rule :



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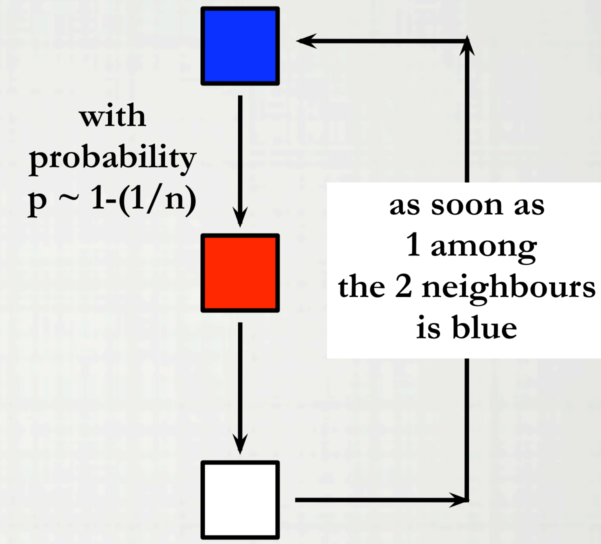

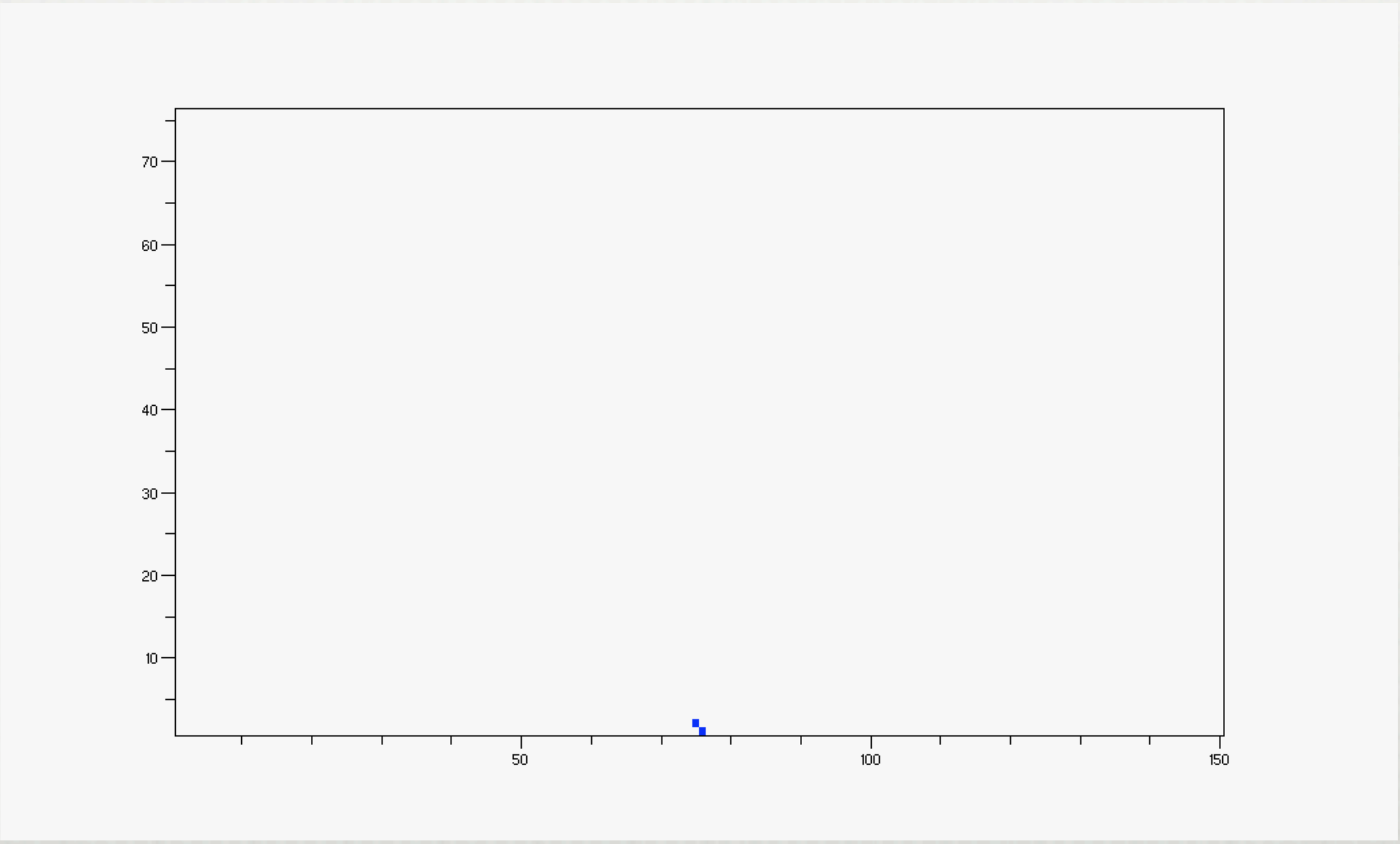

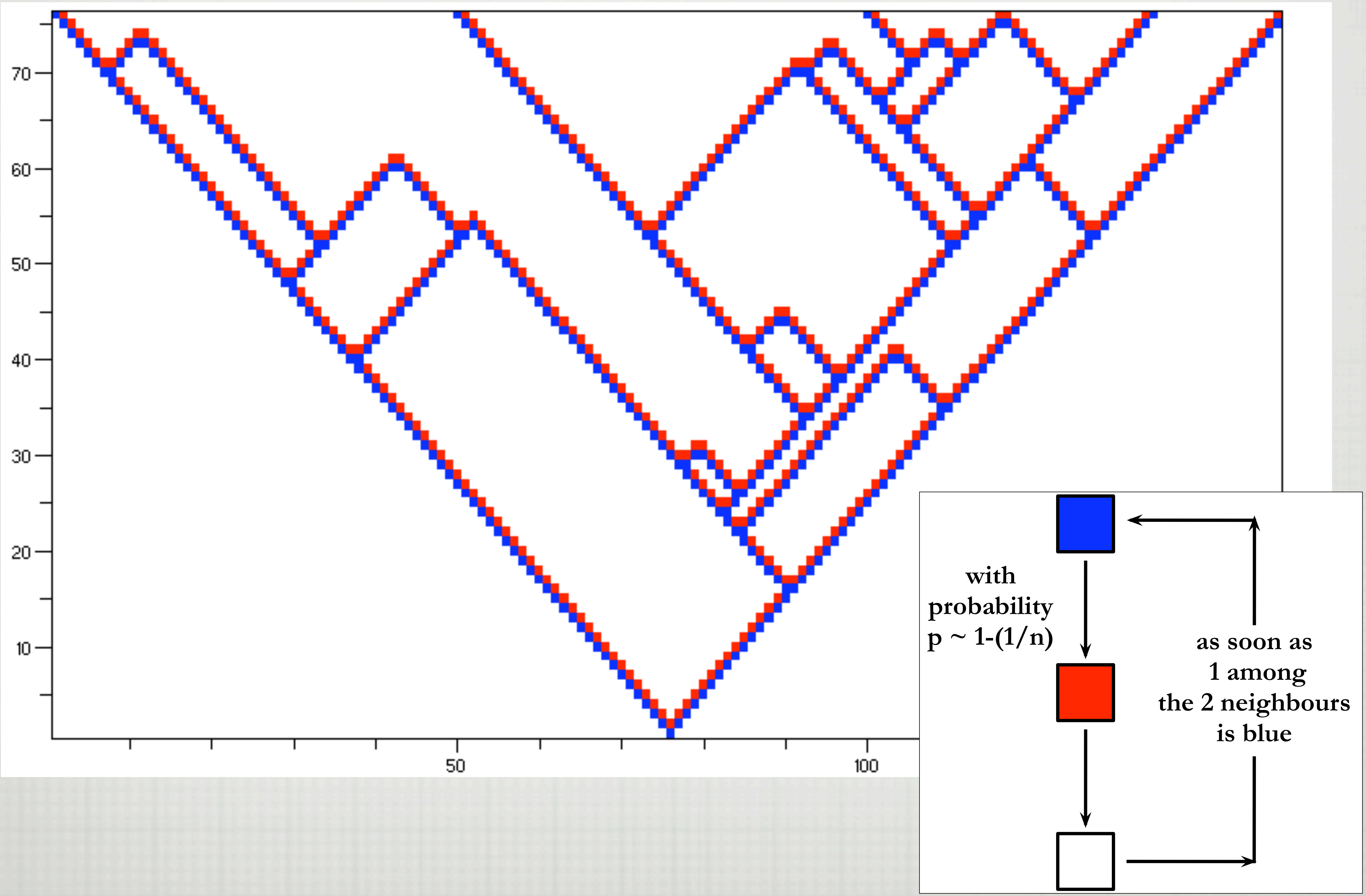

70
60
50
40
30
20
10
50
100
with probability p ~ 1-(1/n)
as soon as 1 among the 2 neighbours is blue

MOTIVATIONS

CELLULAR AUTOMATA IN GENERAL

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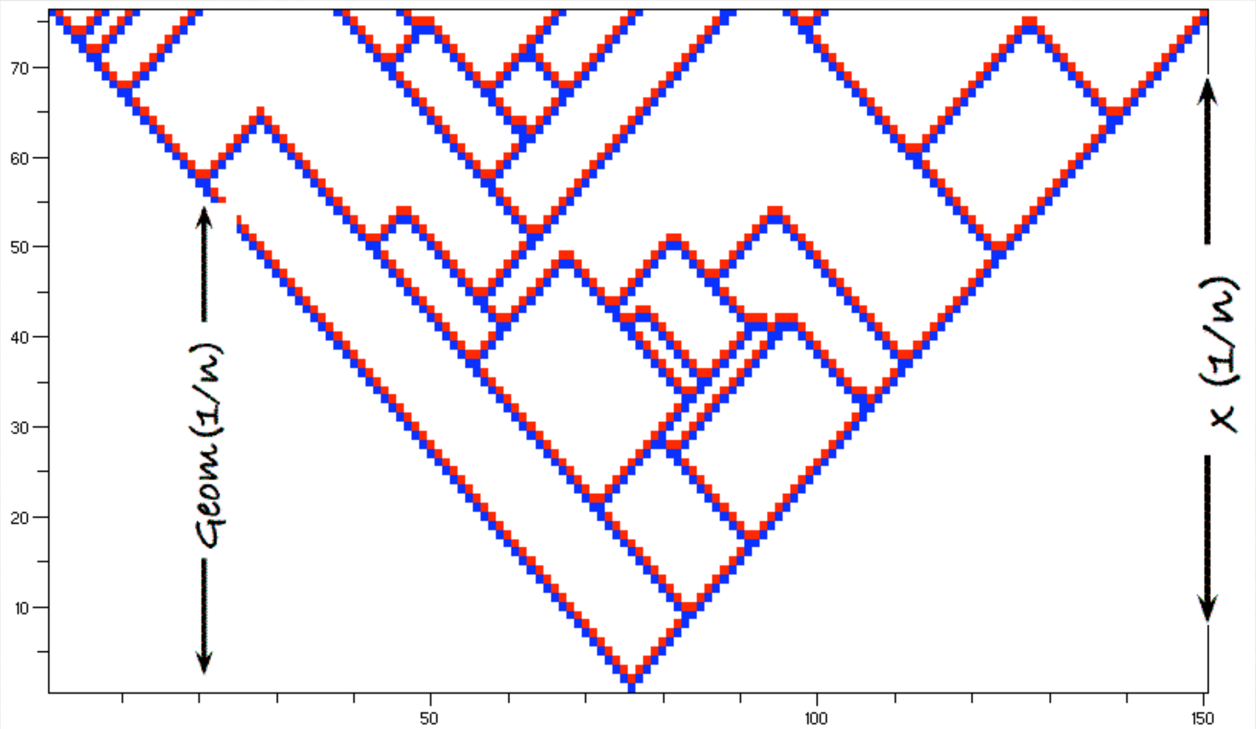
THIS MODEL

- Ferrari, Belitsky, Blythe, Cafri, Evans, Cardy, ... use it for the modelisation of some chemical reactions, of highway traffic, etc ...

THE ASYMPTOTIC MODEL



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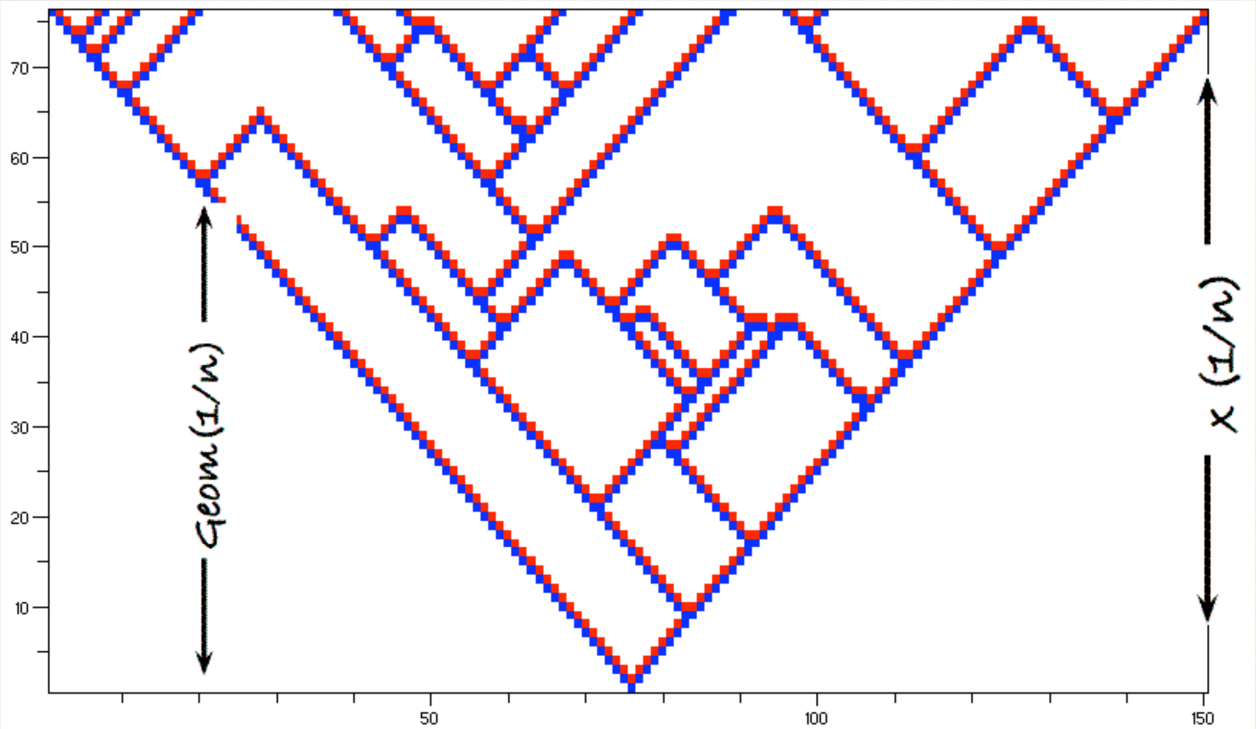

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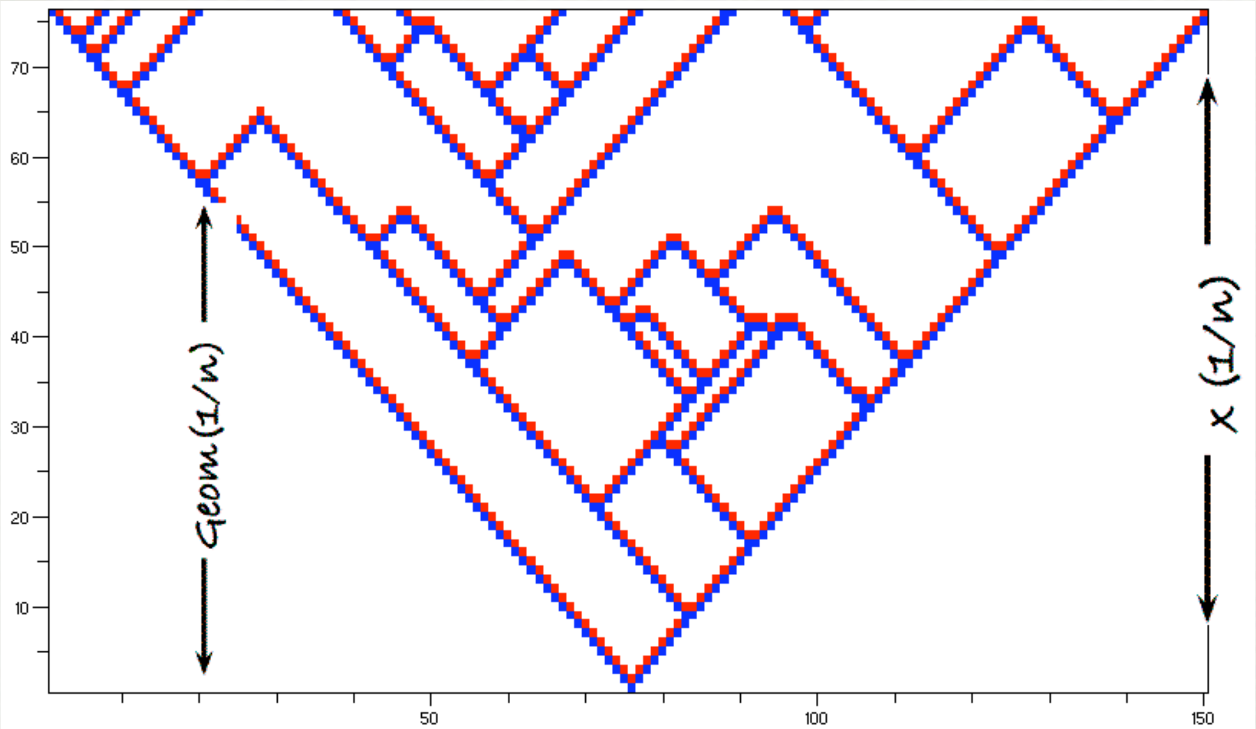

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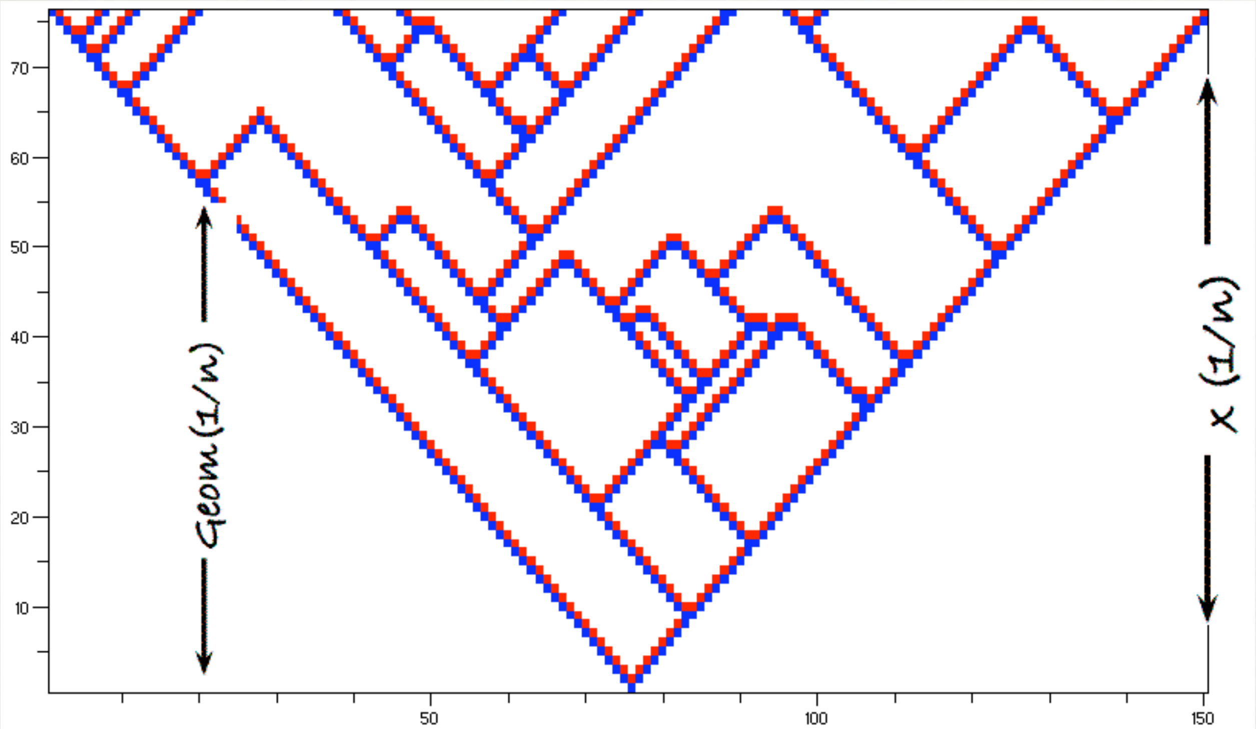

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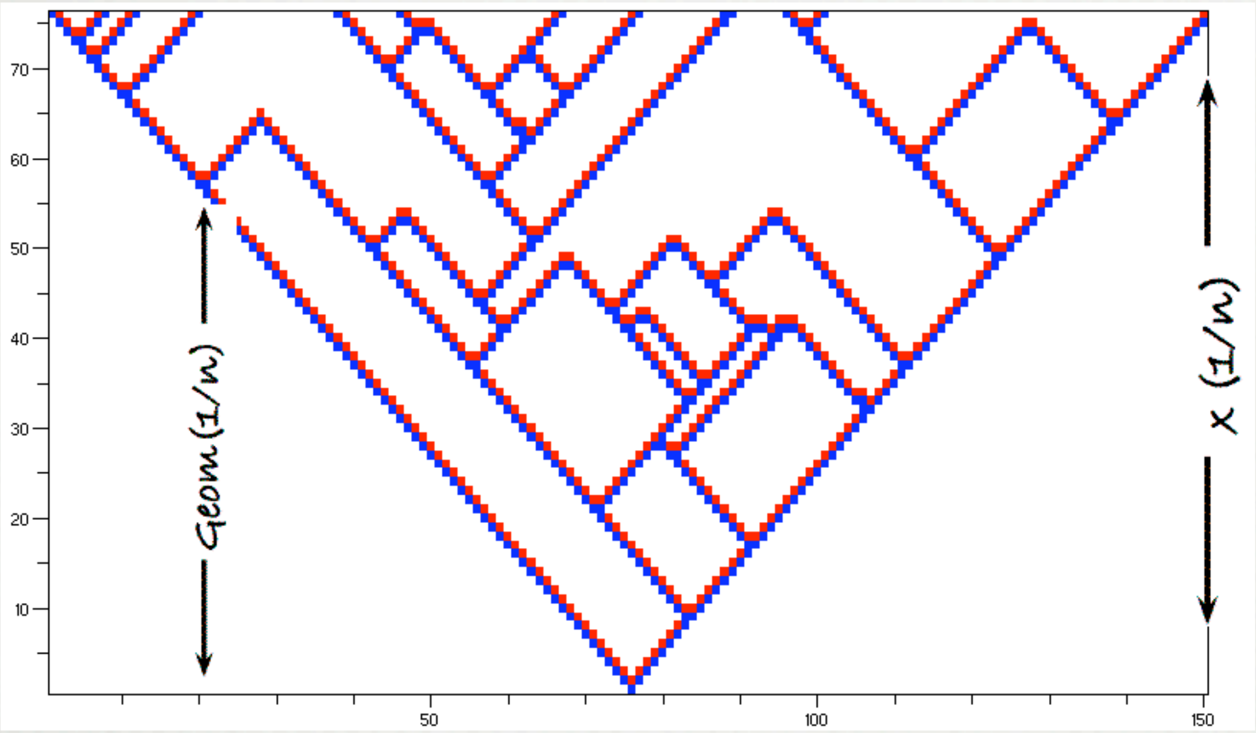

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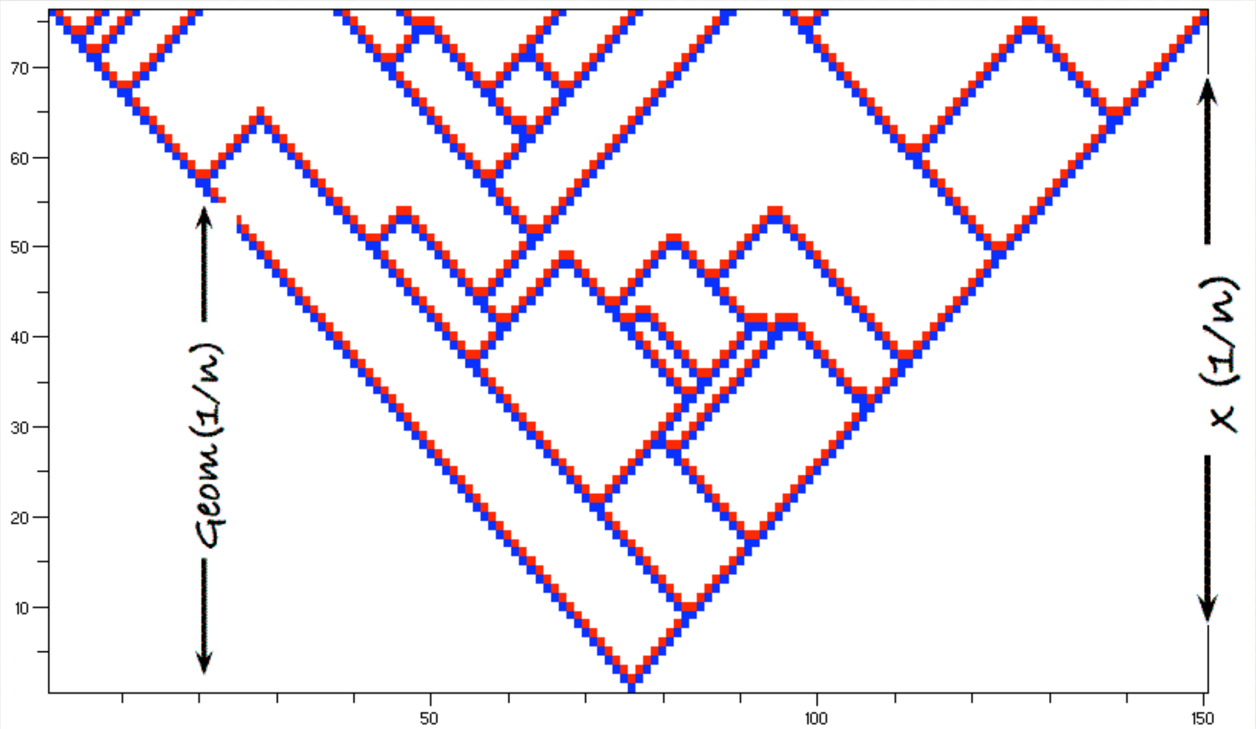

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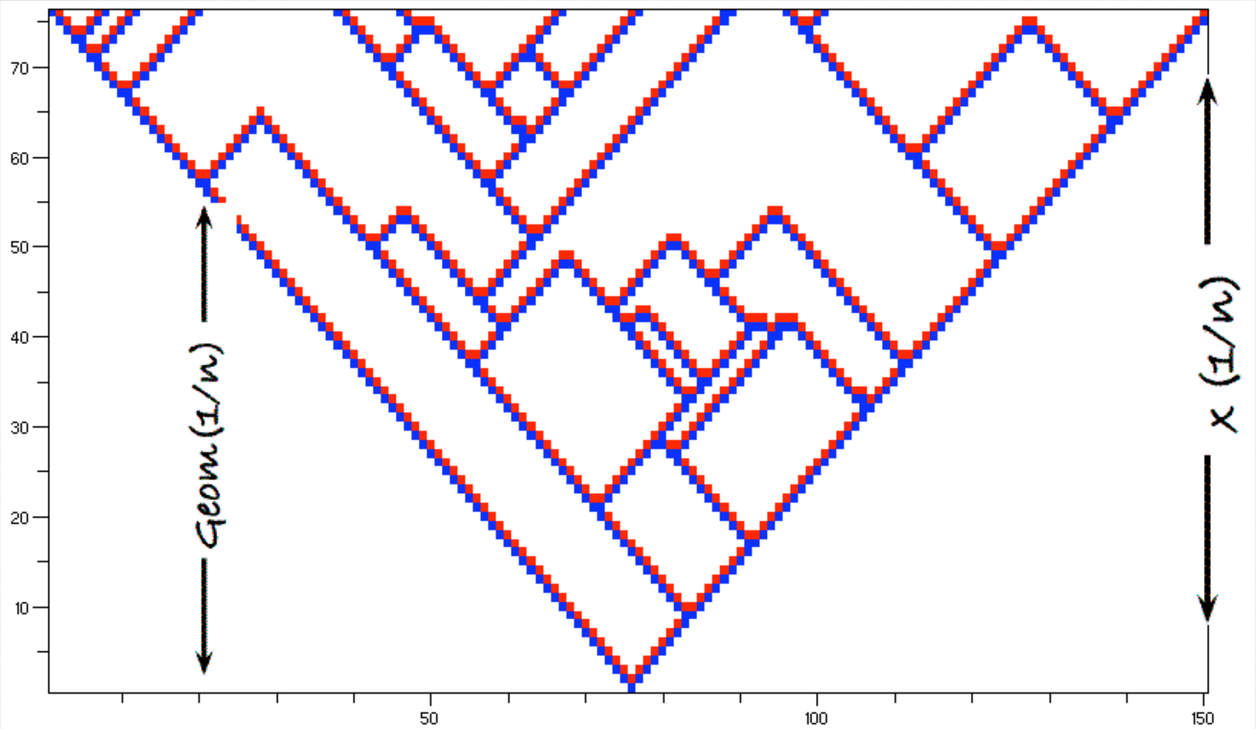

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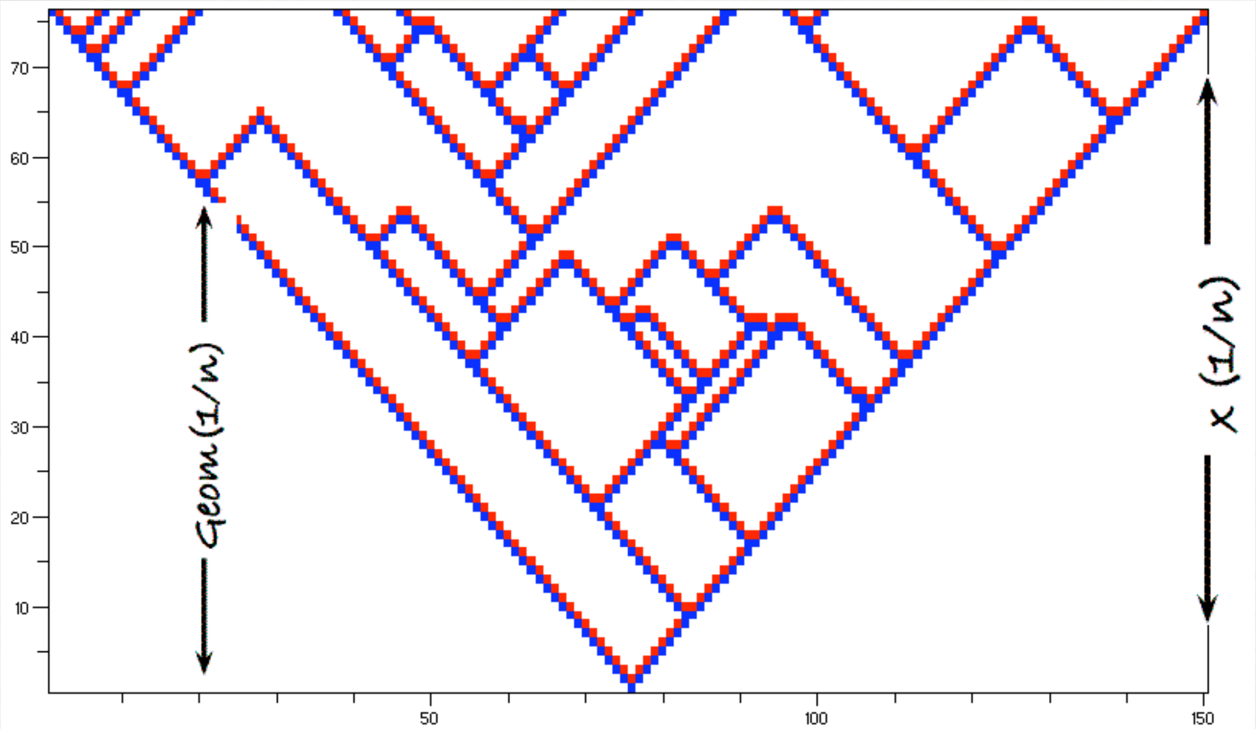

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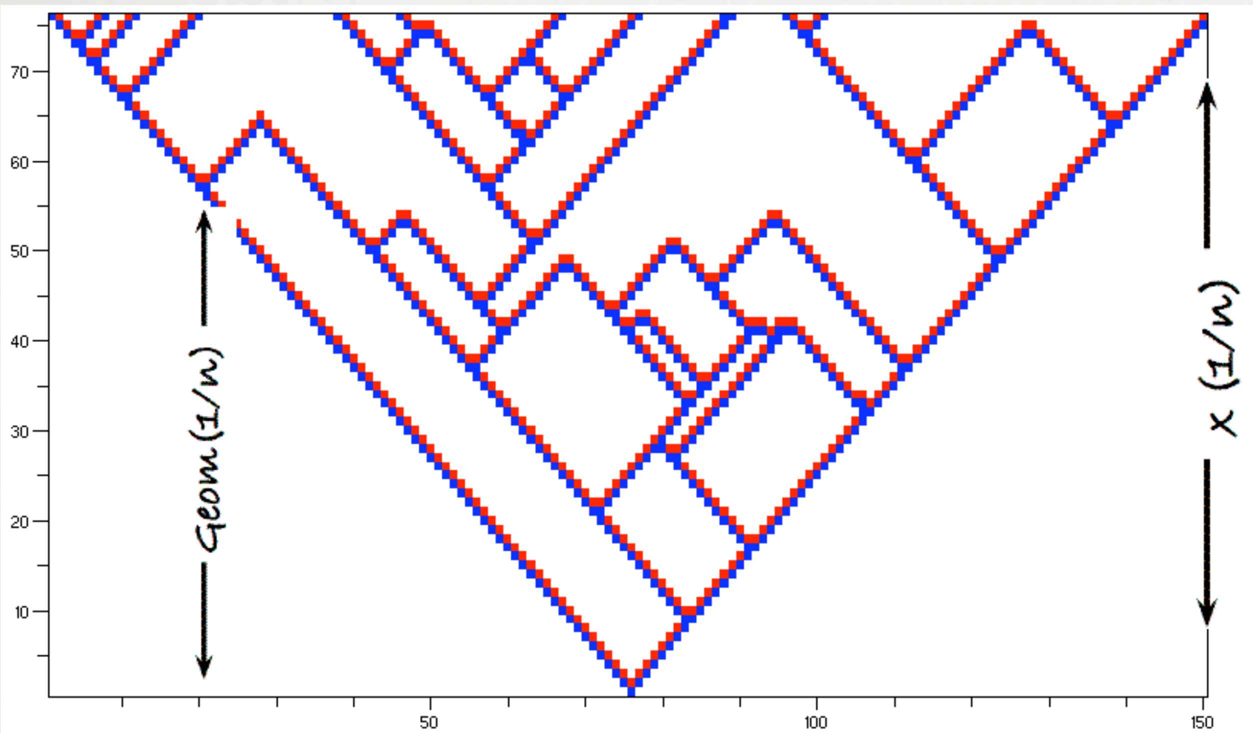

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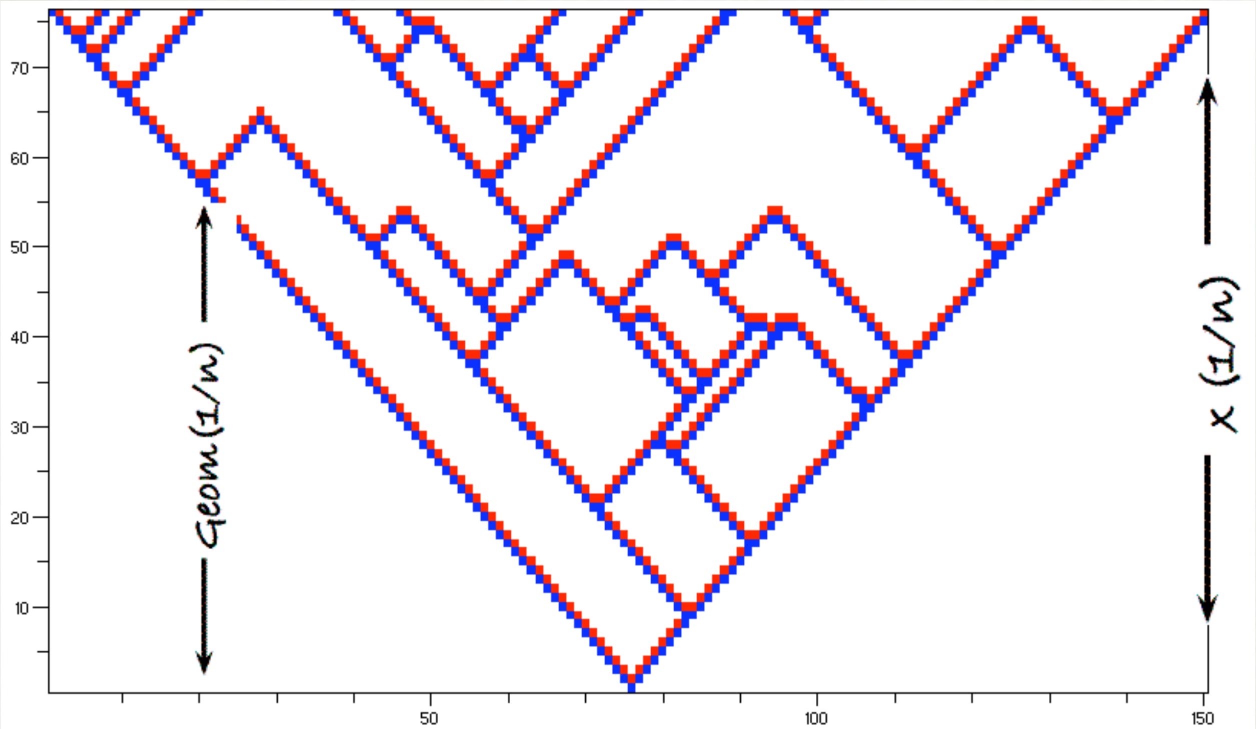

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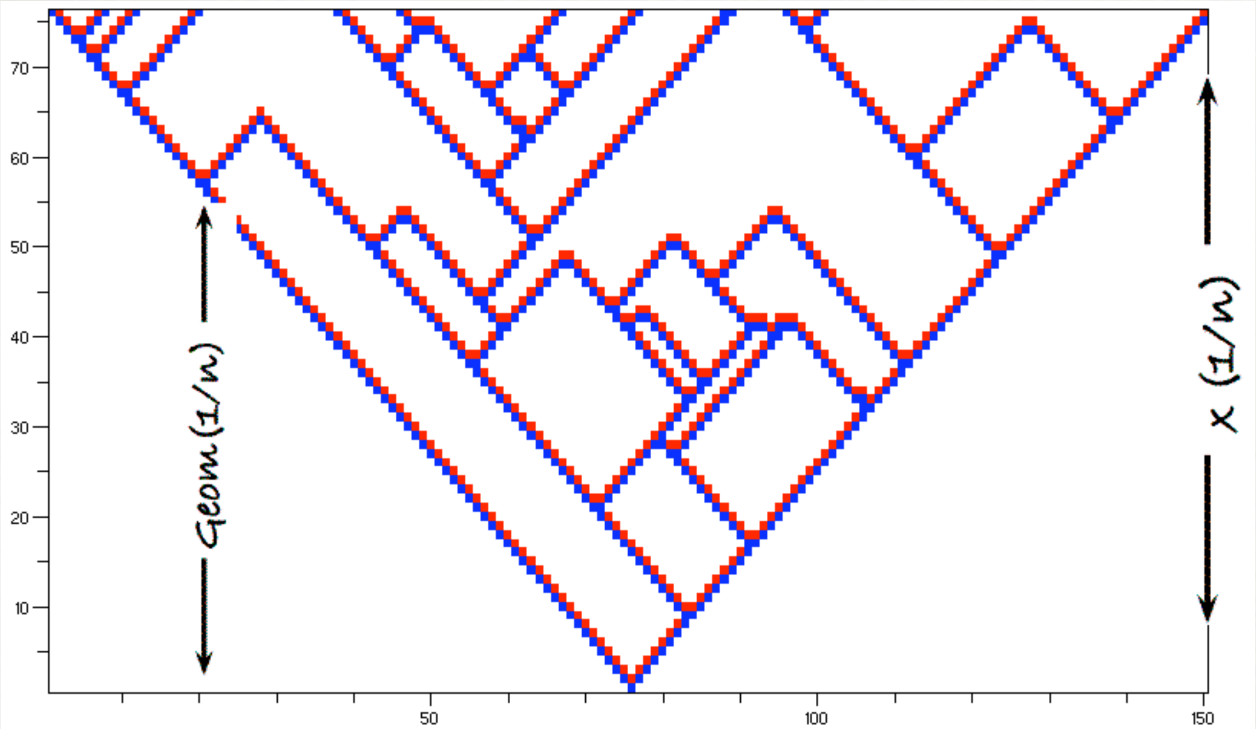

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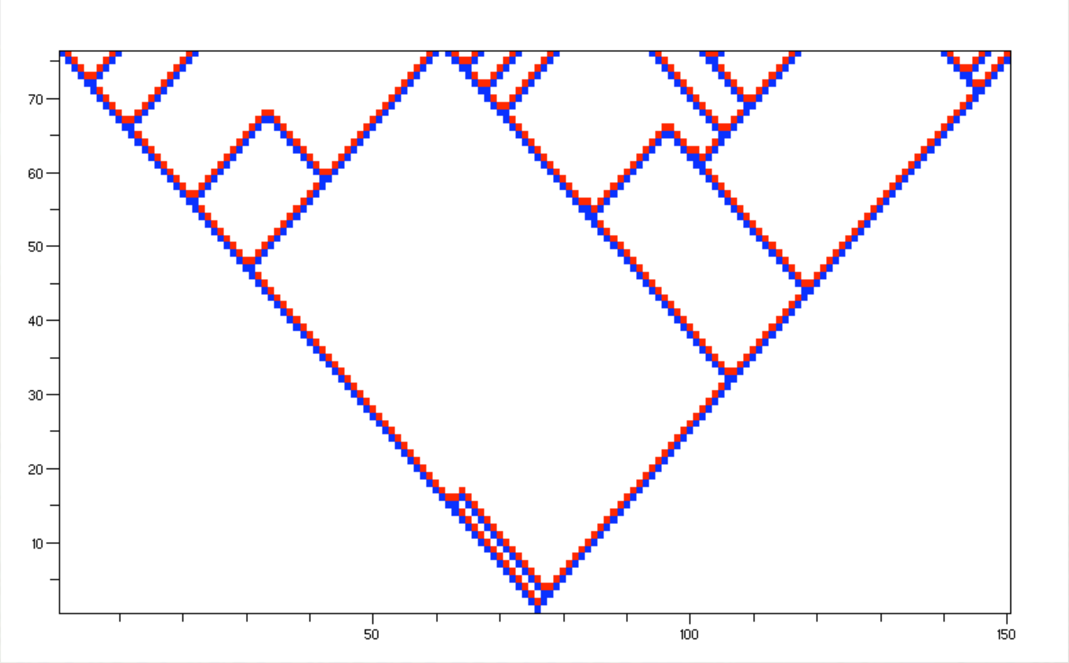

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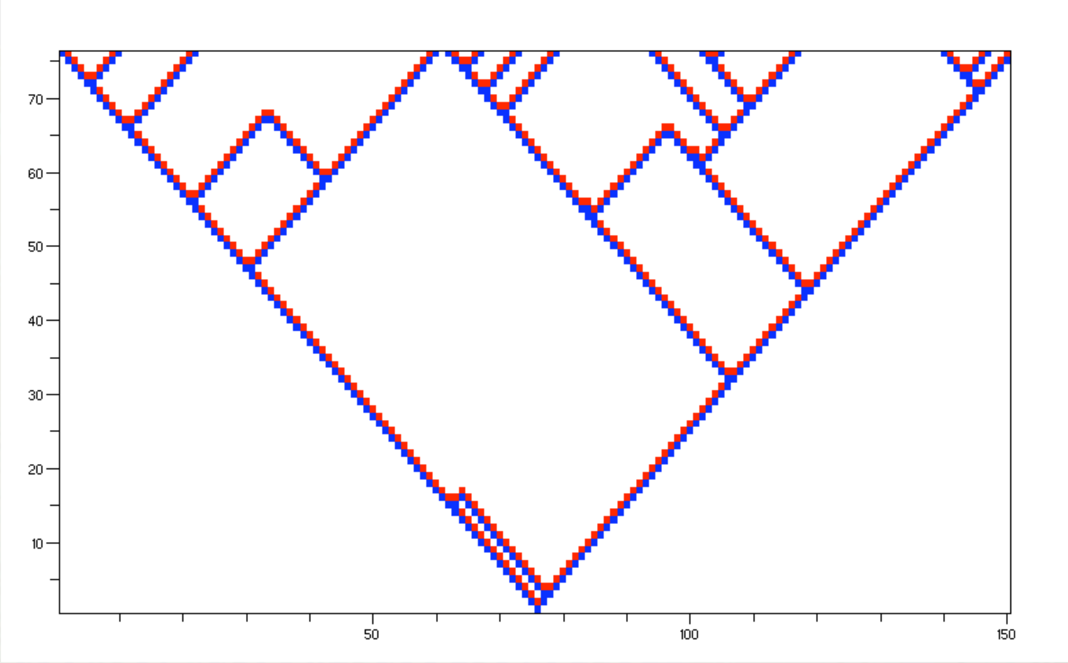

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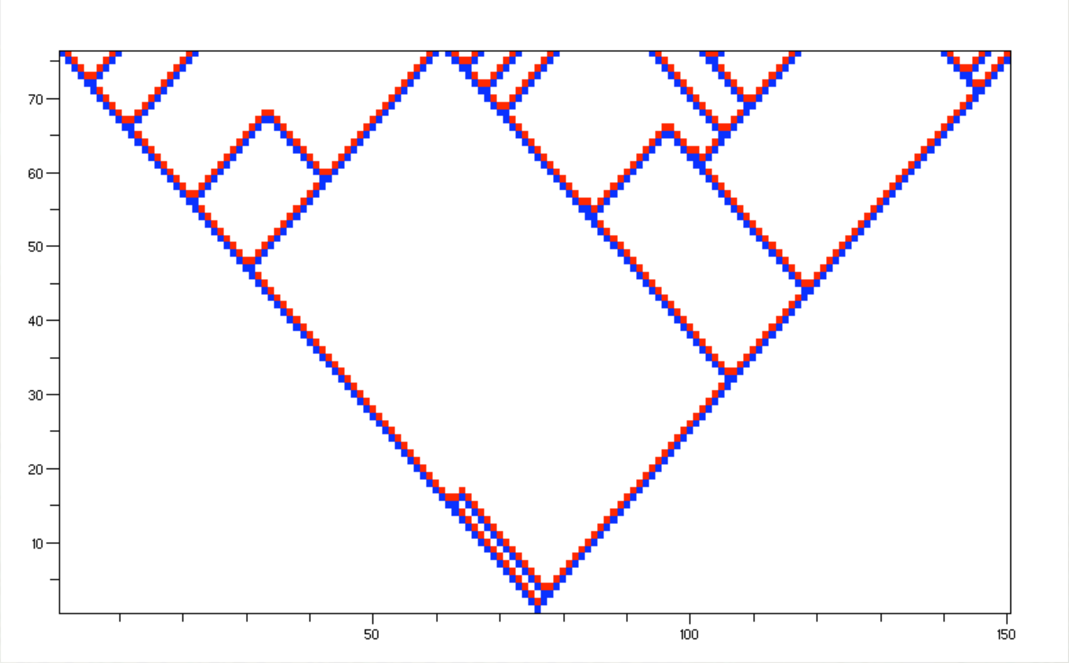

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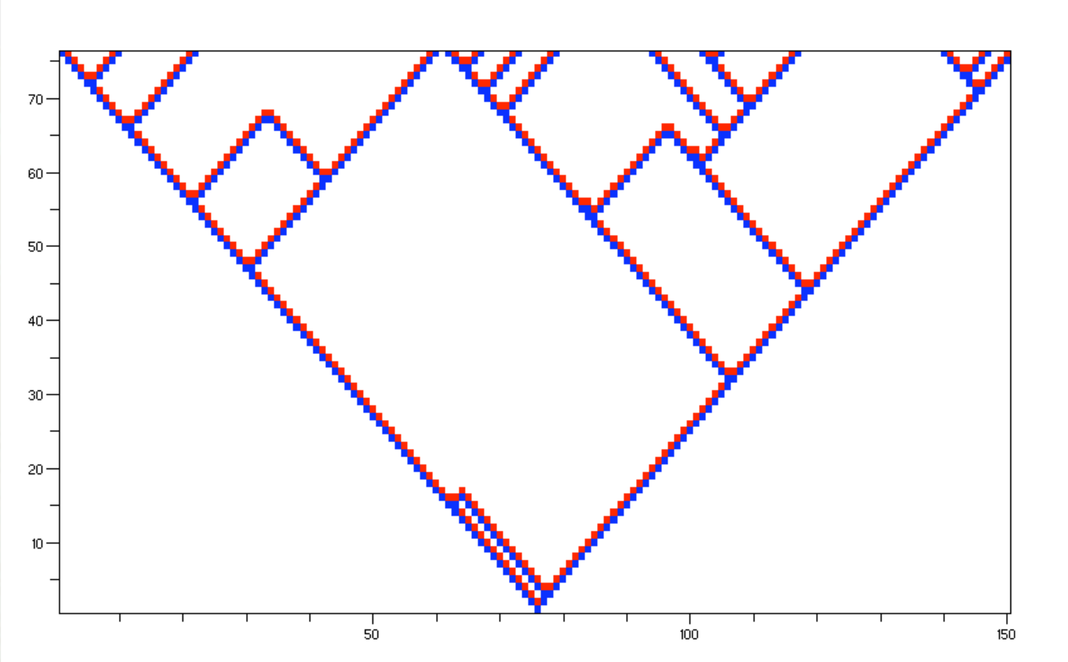

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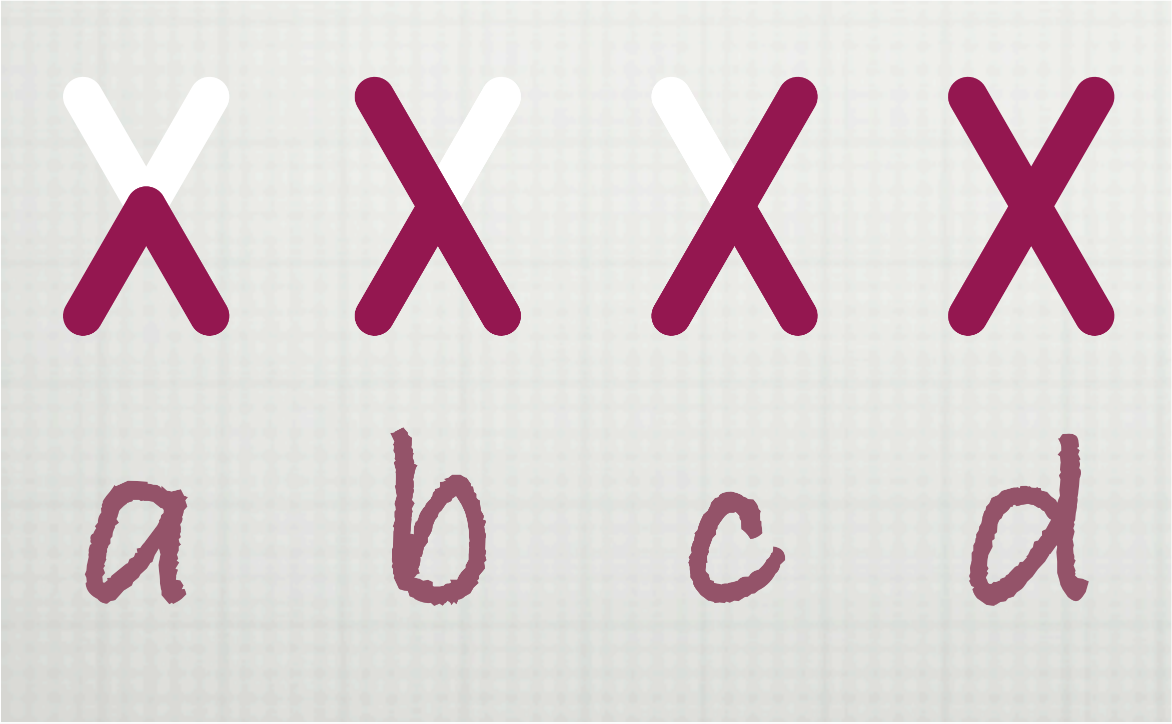

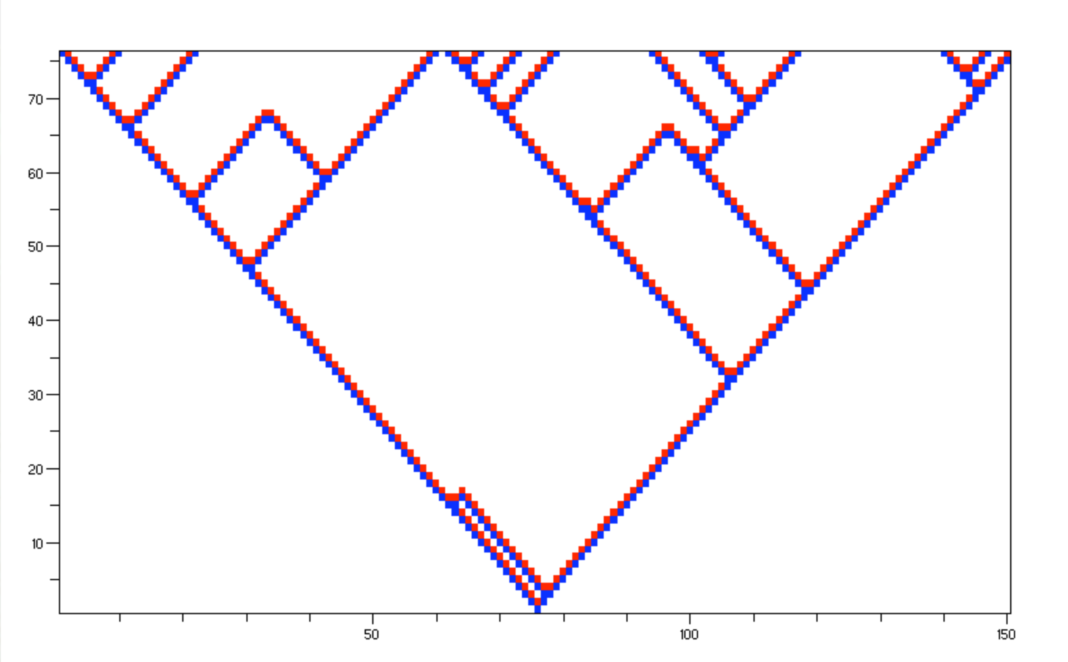

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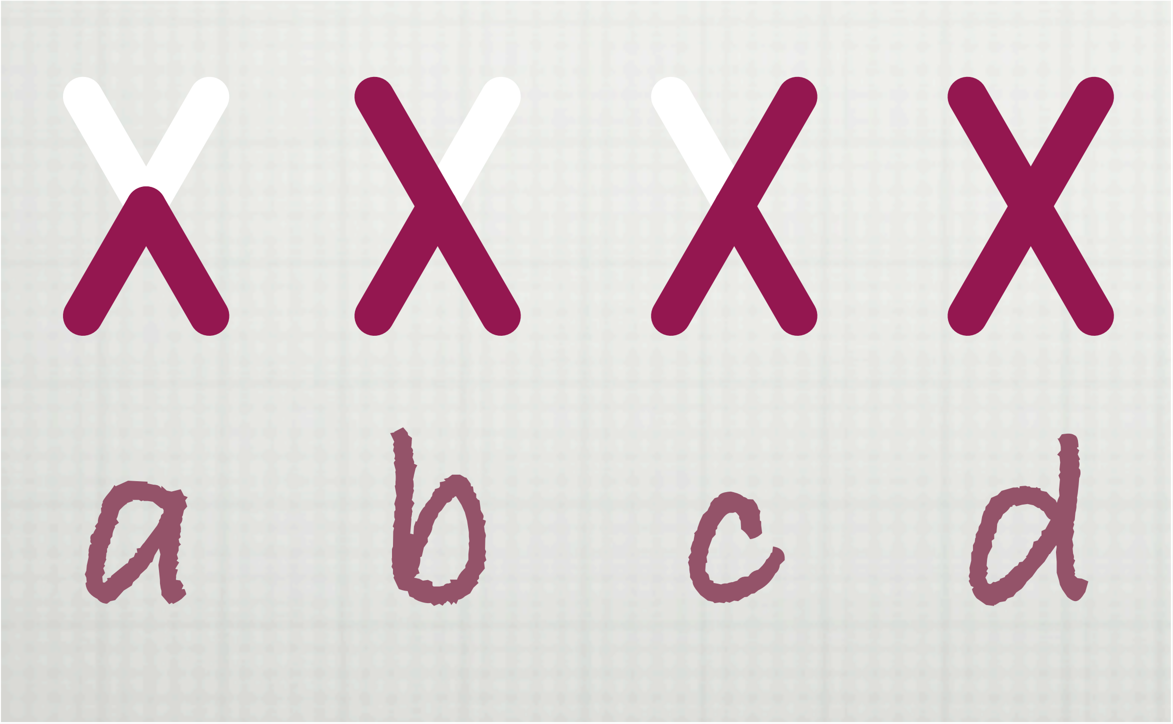

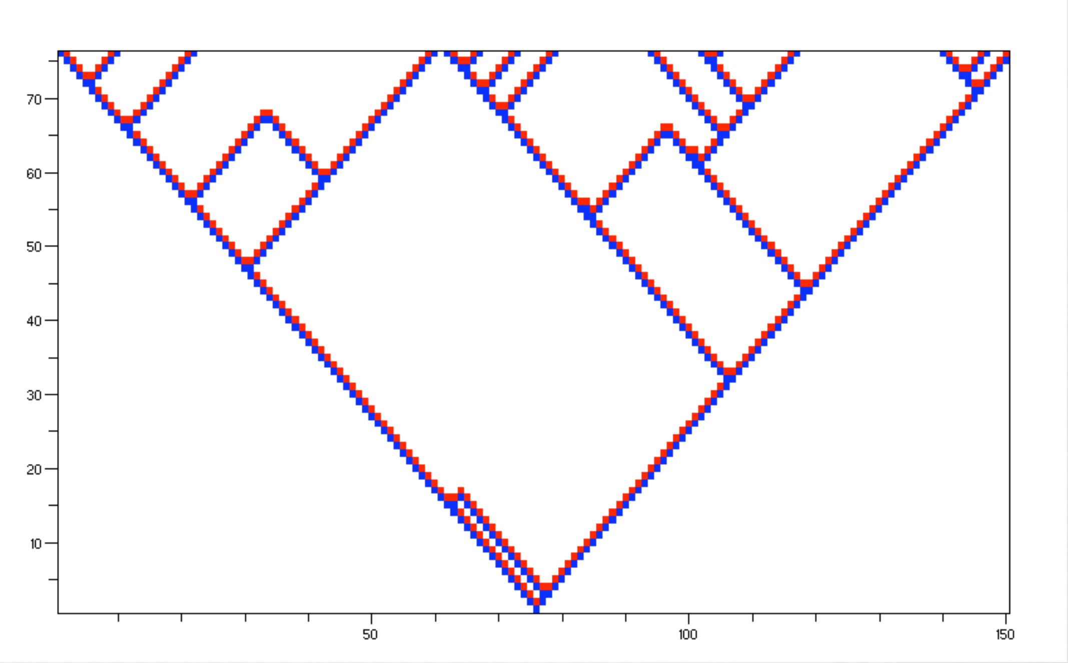

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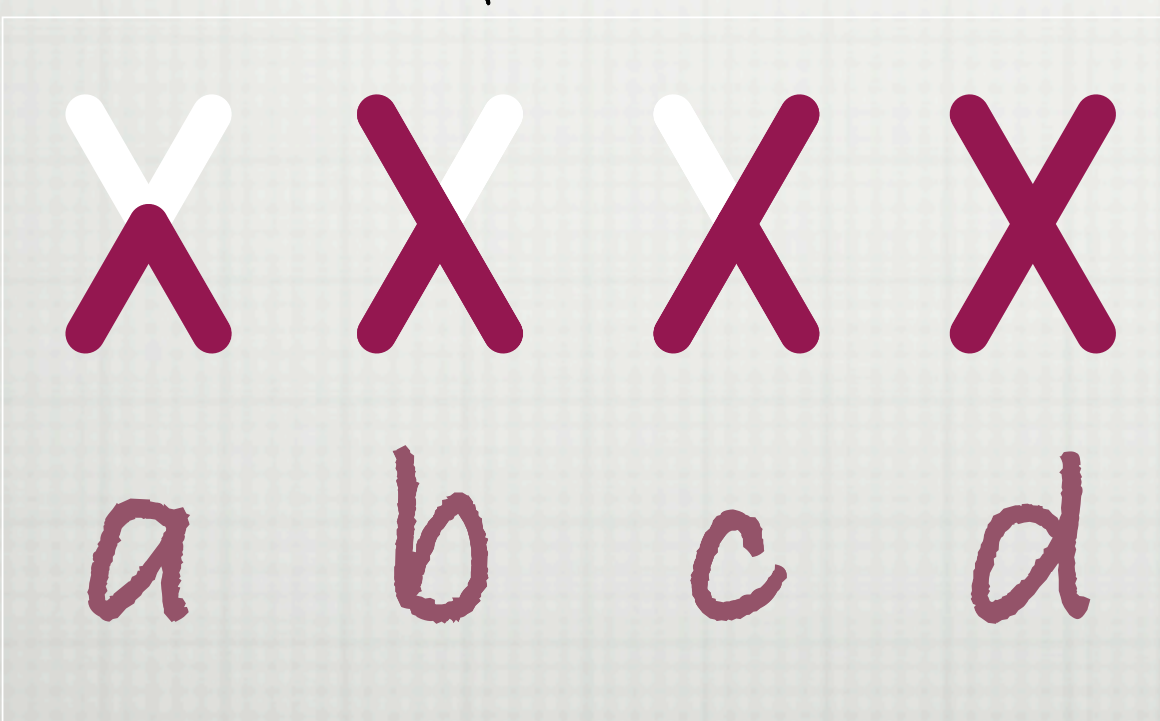

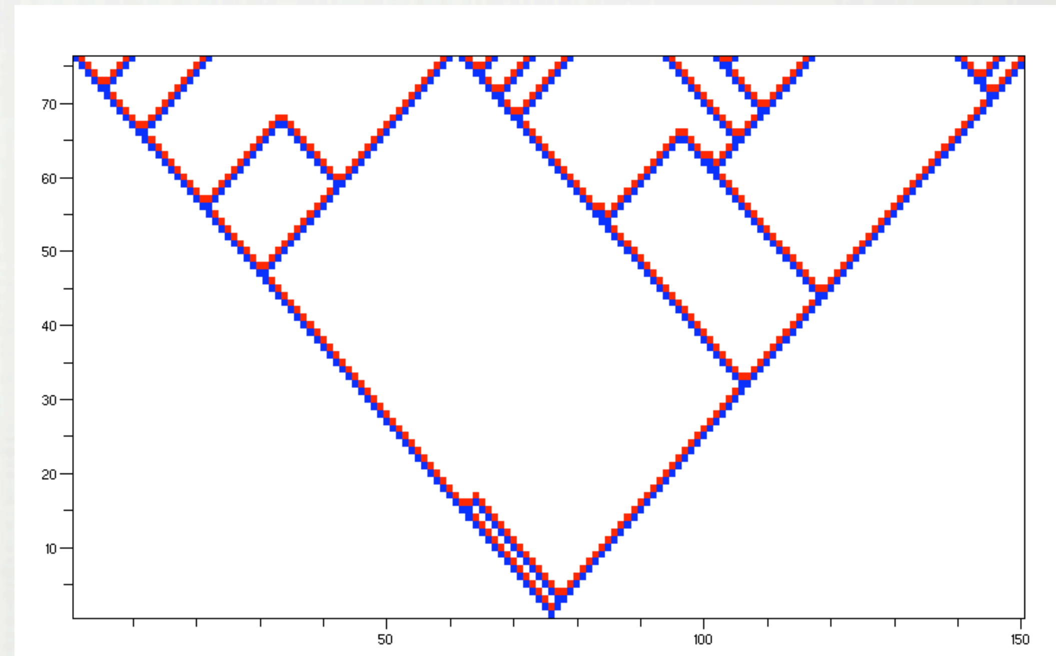

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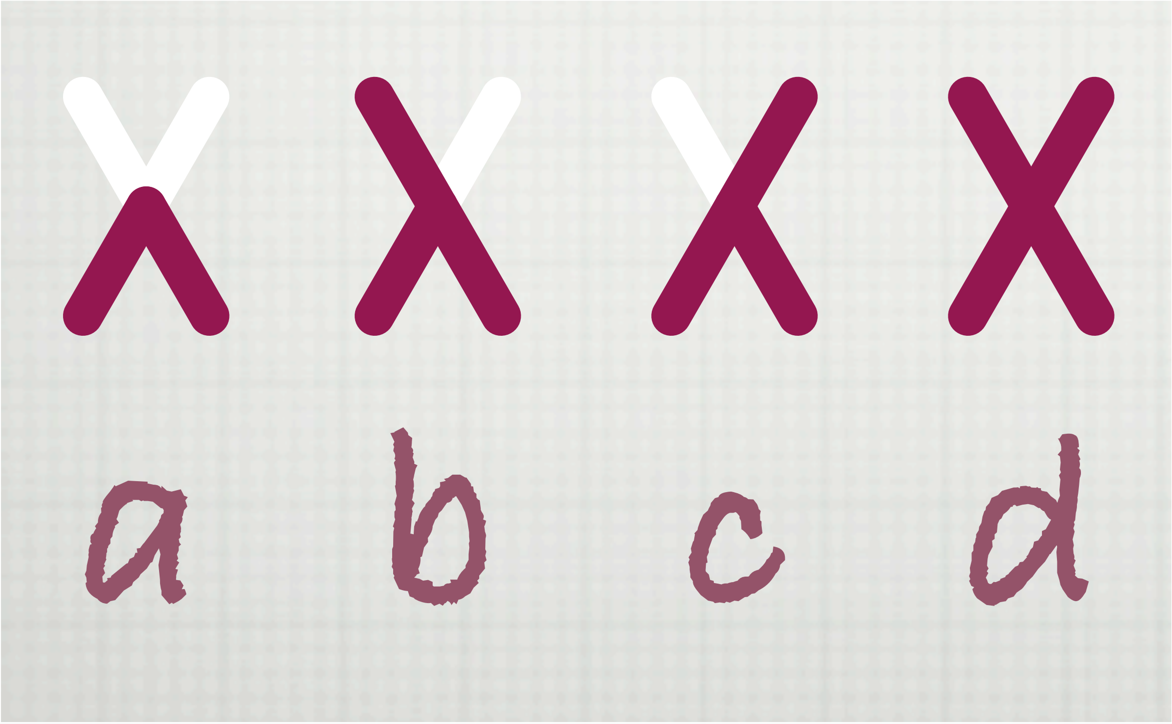

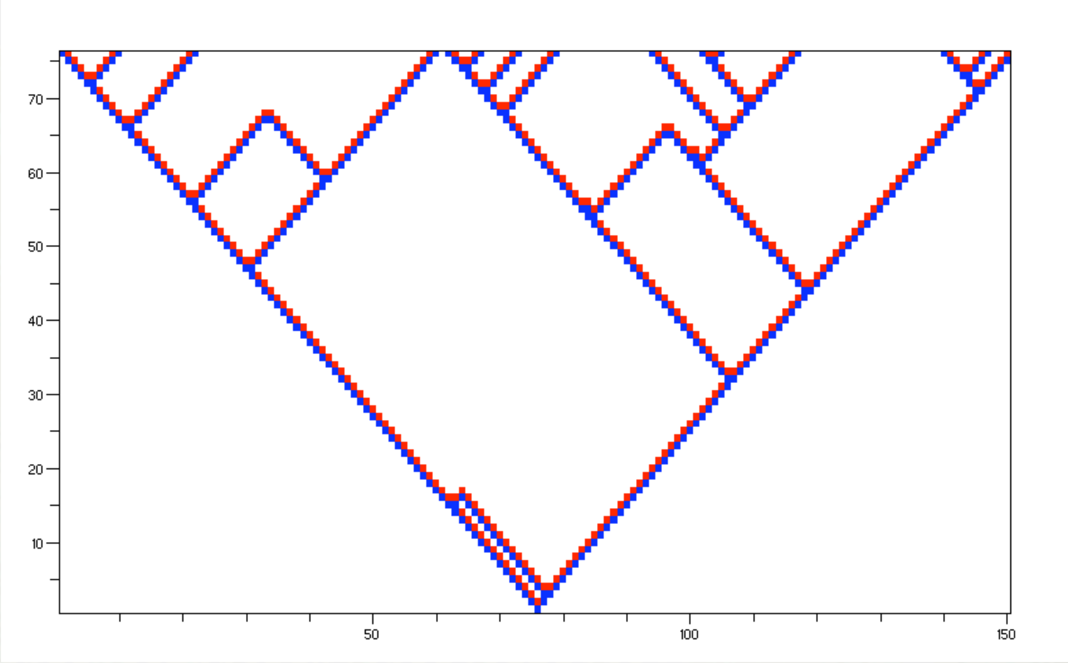

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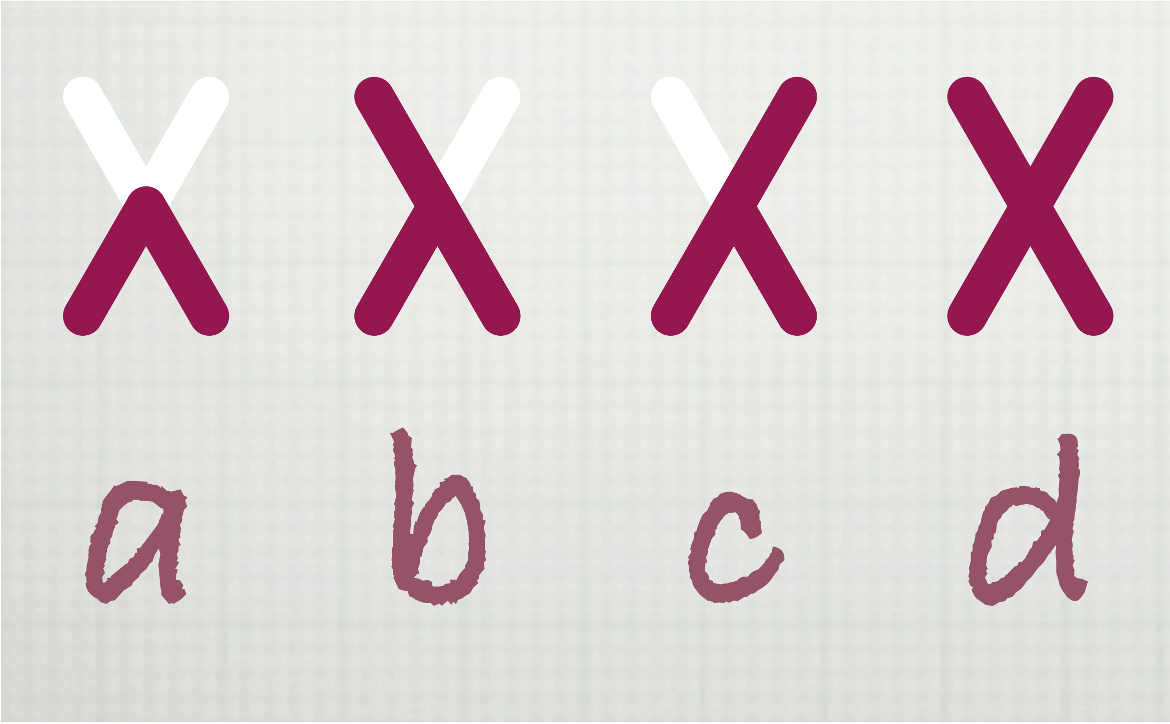

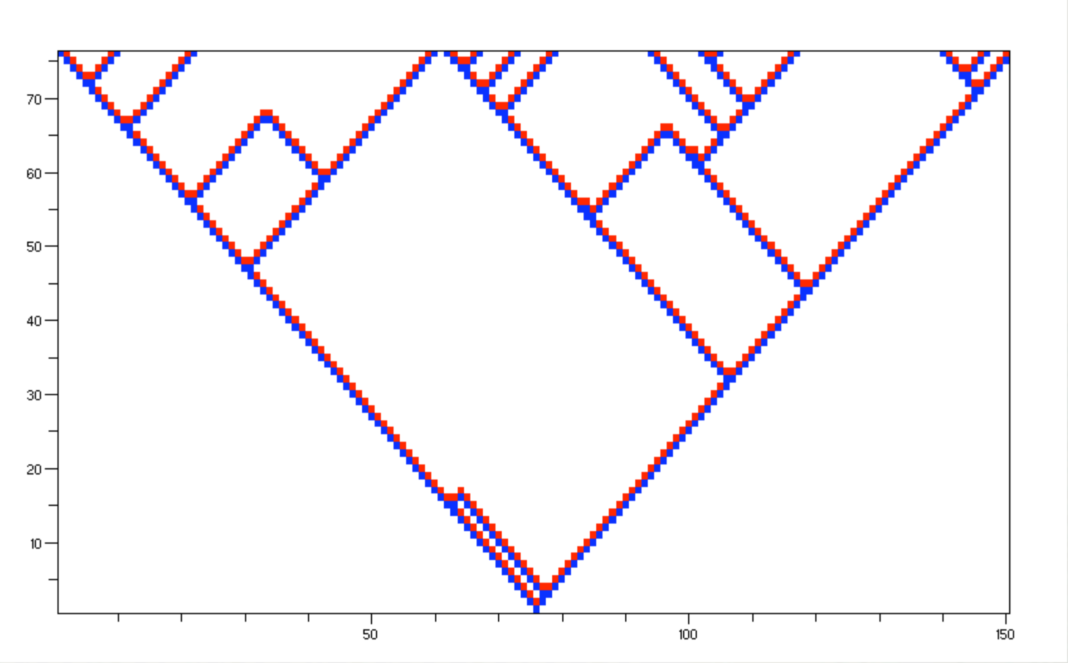

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  - pure branching: (0,0,0,1),
  - symmetric: (0,0.5,0.5,0),
  - annihilation: (1,0,0,0).

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Assume the system starts with 1 positive particle at 0, and set :

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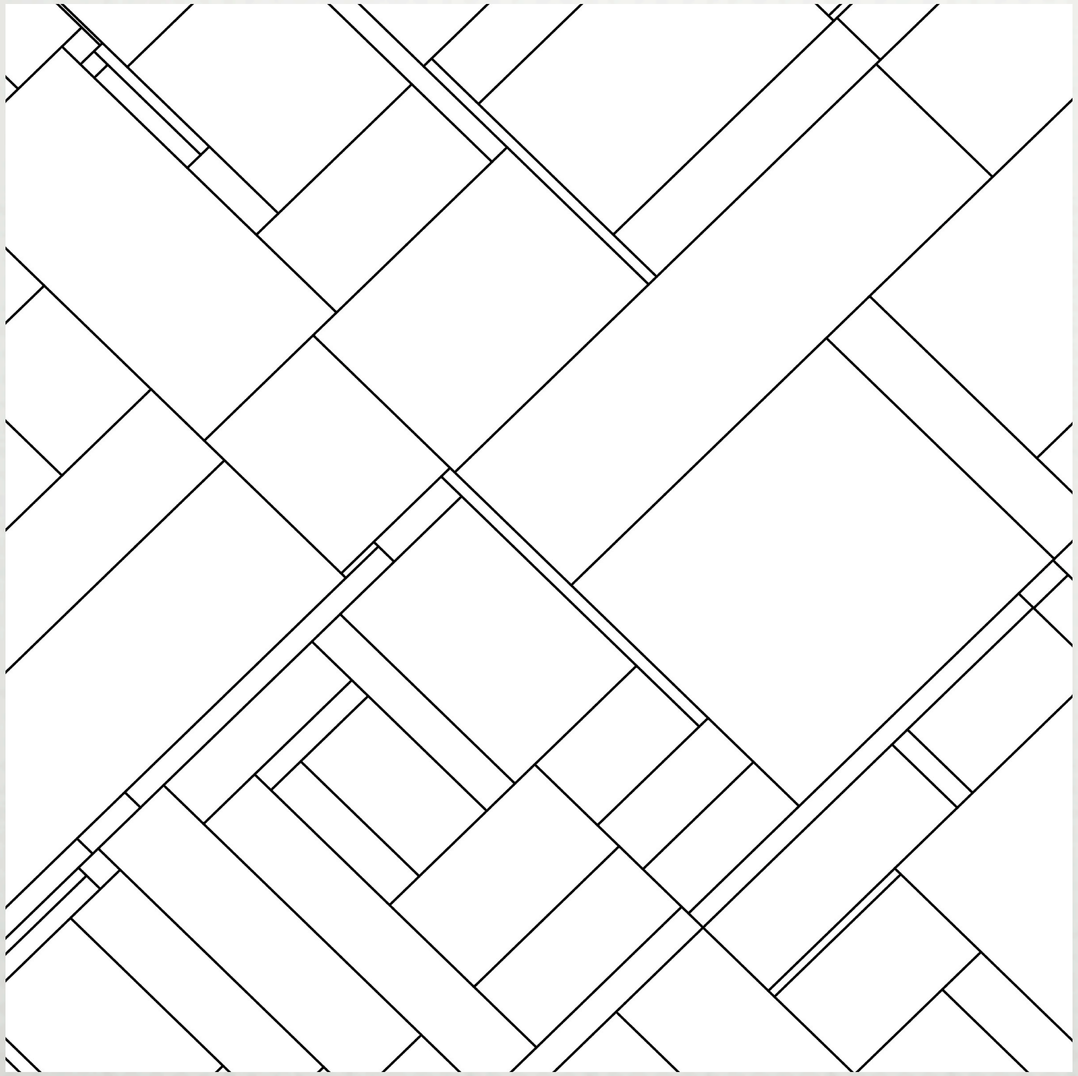
- ☐ Most of it is supported by an $O(\sqrt{t})$-wide interval around 0.

SIMULATIONS

(0,0.5,0.5,0)



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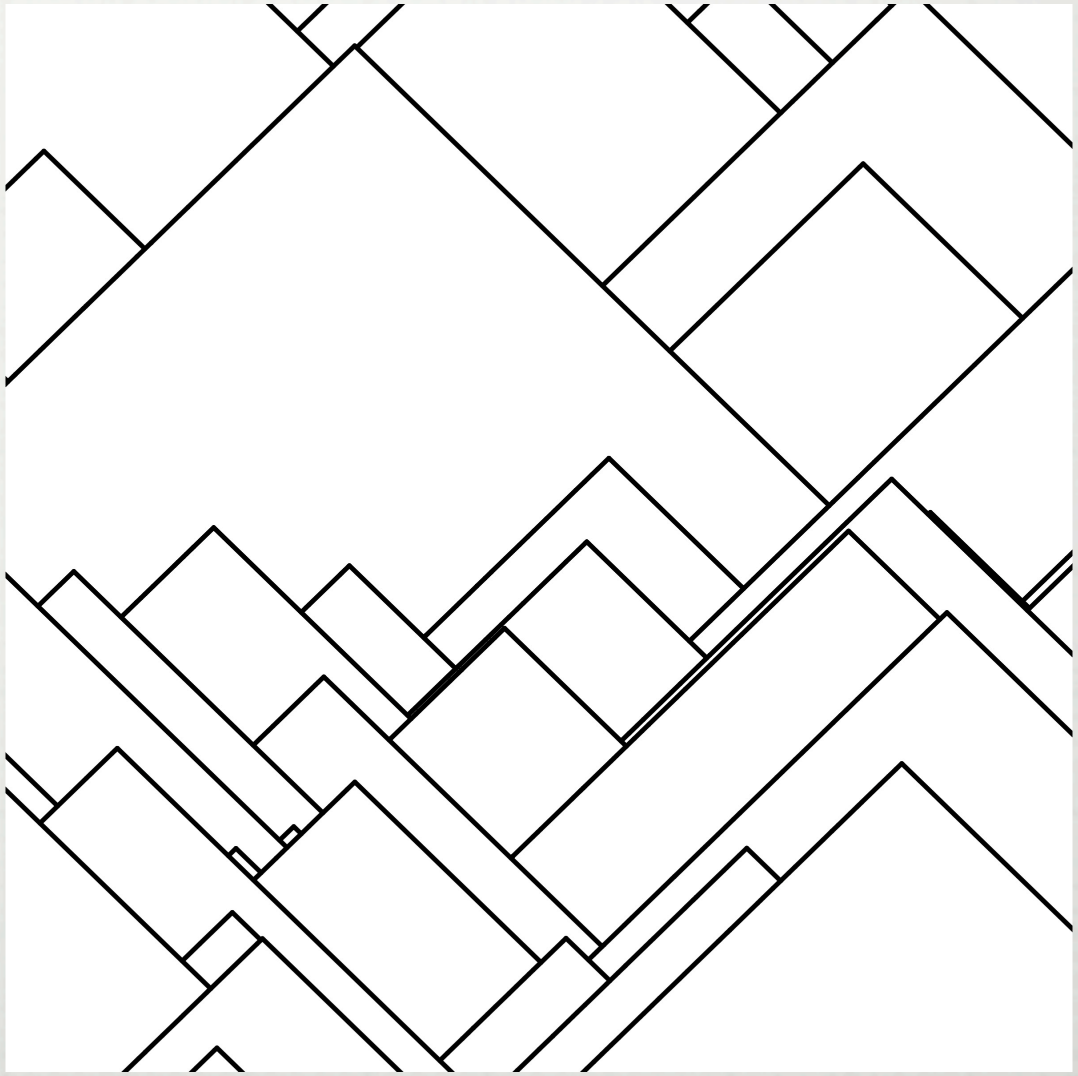

THE TILTED PROCESS

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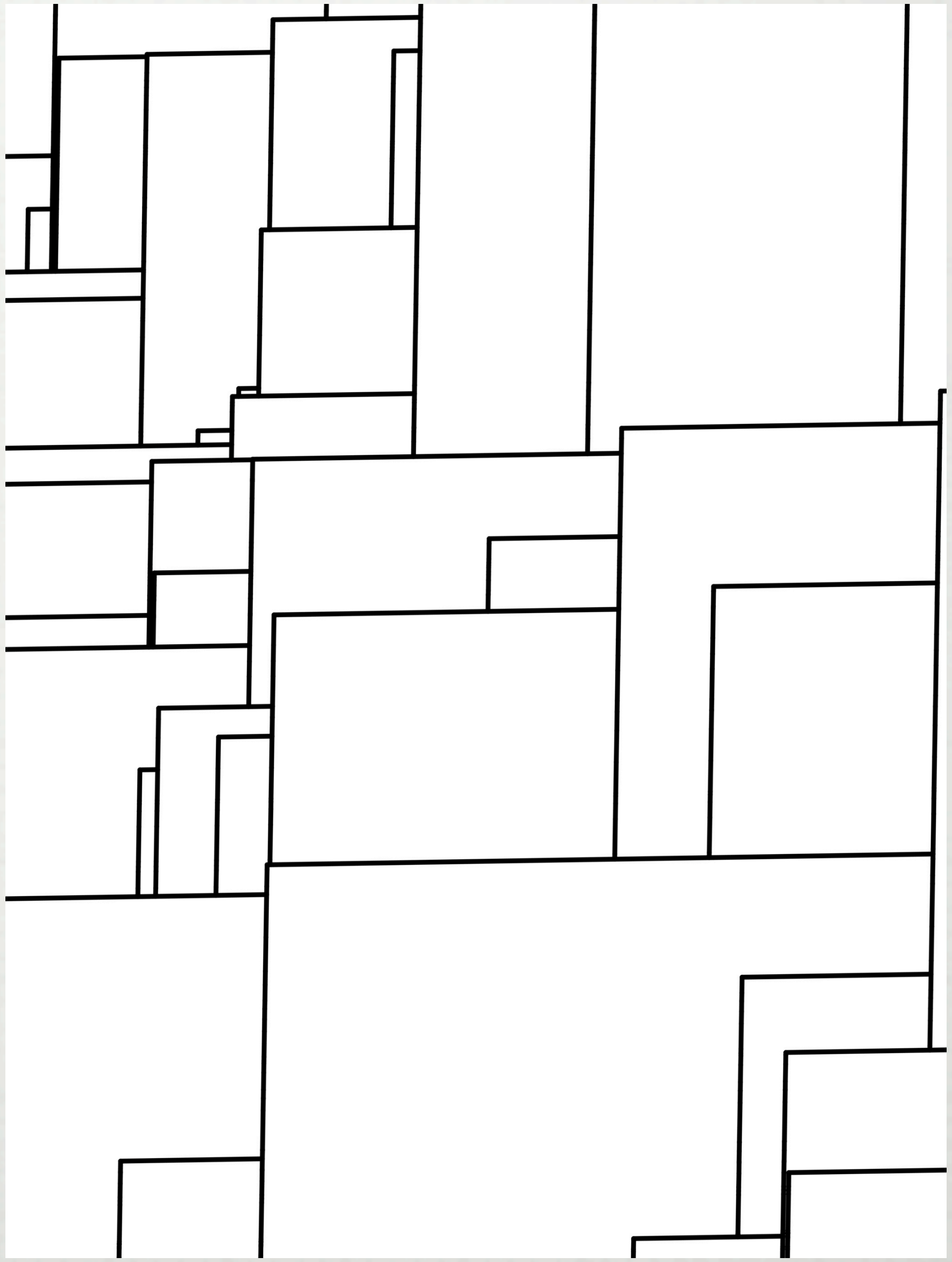

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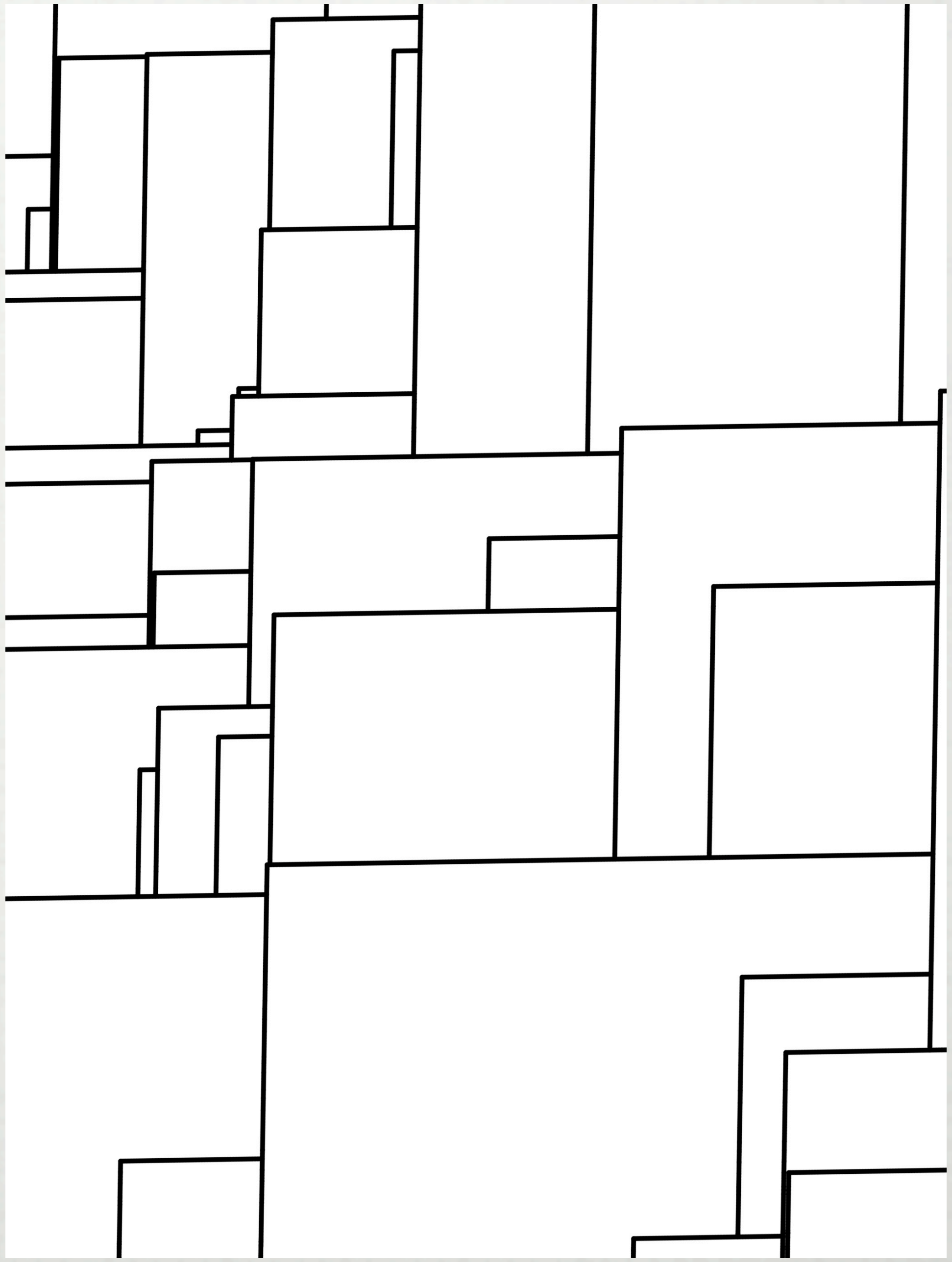

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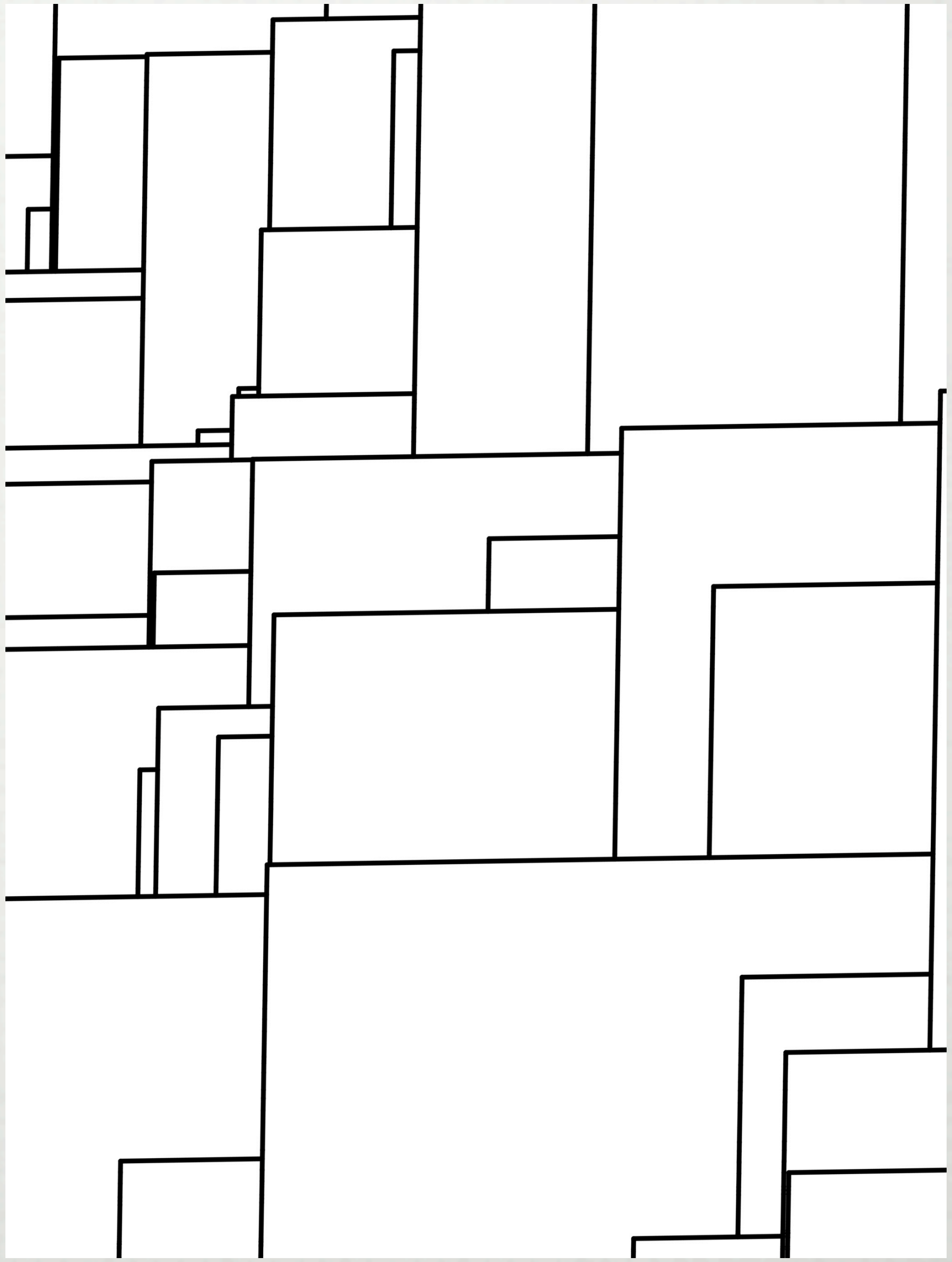
- Each particle, with rate $1/\sqrt{2}$, shoots and kills its nearest neighbour on the left ;



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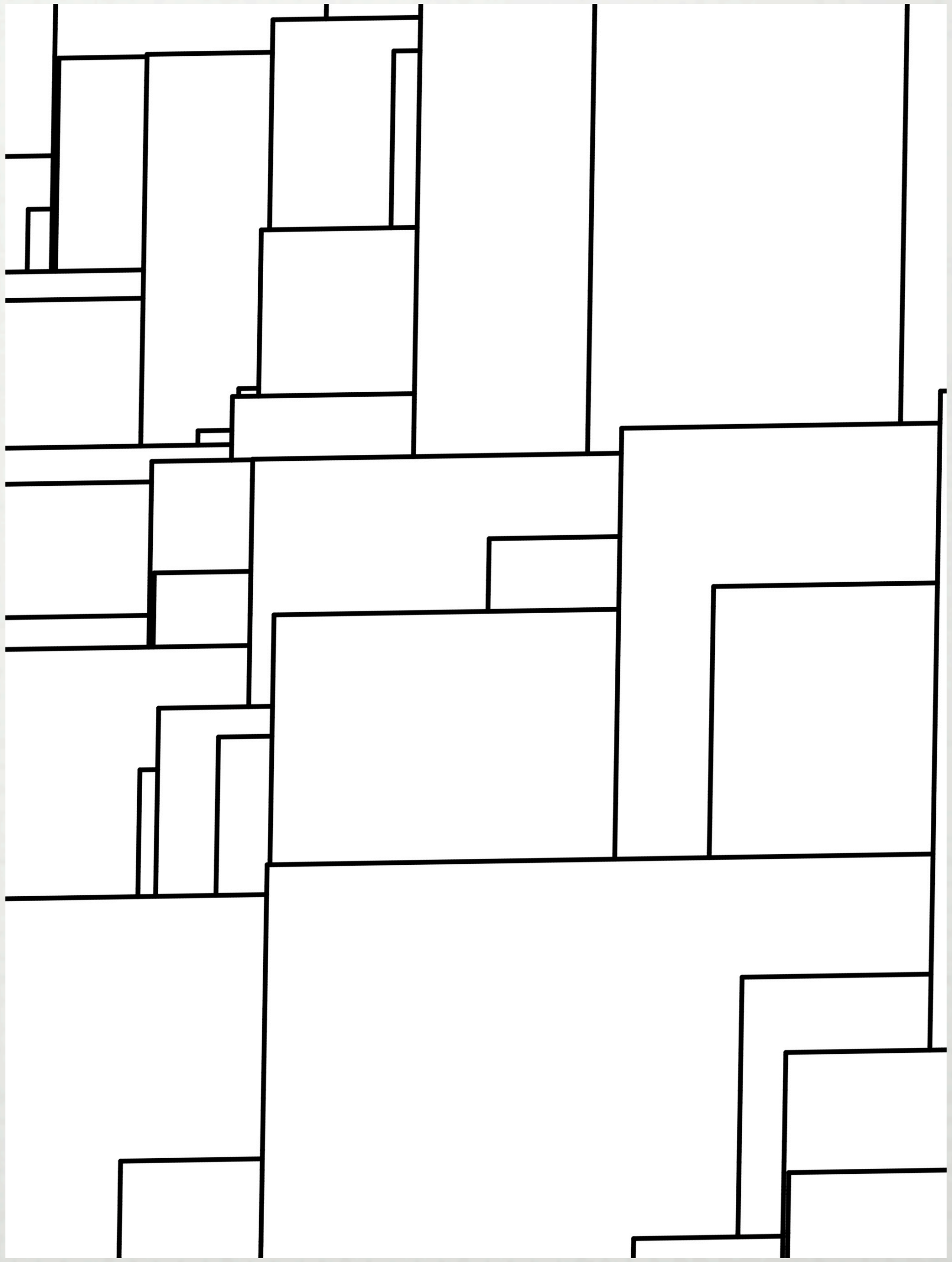
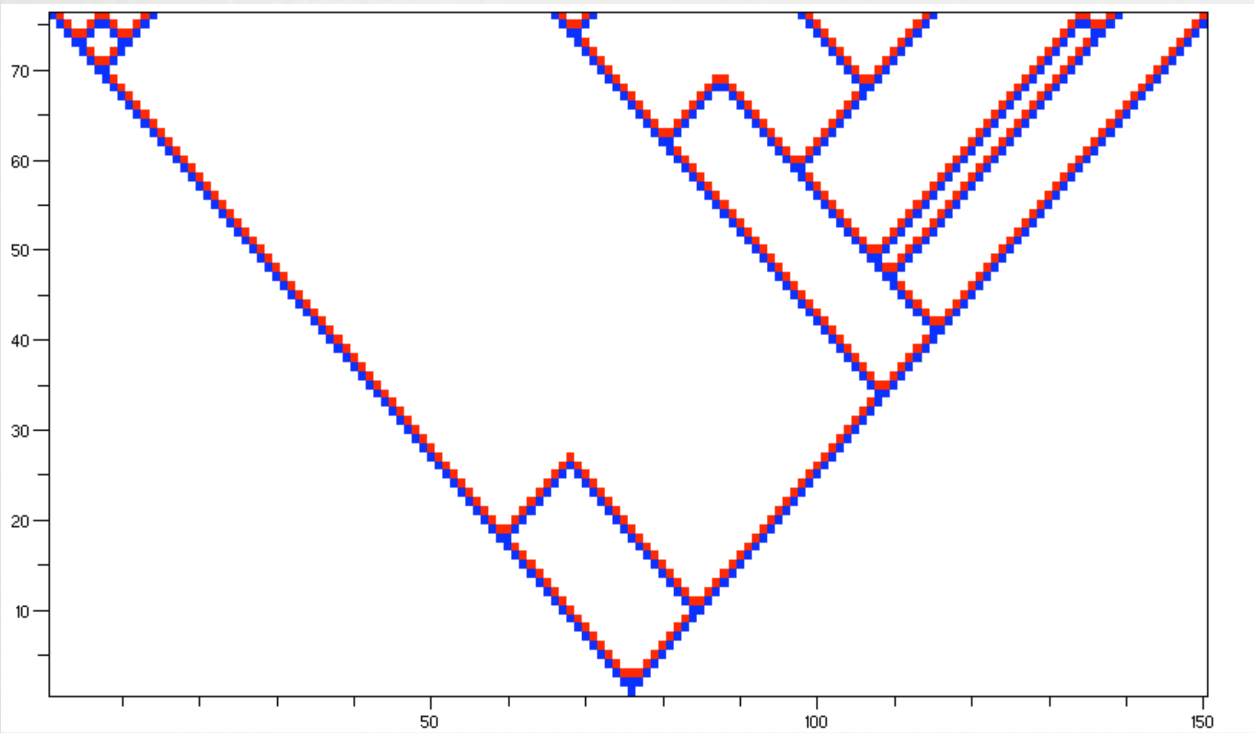
- The interval between the two particles is filled with an independent copy of a PPP($1/\sqrt{2}$).



GLOSSARY: POINT PROCESSES ON THE LINE



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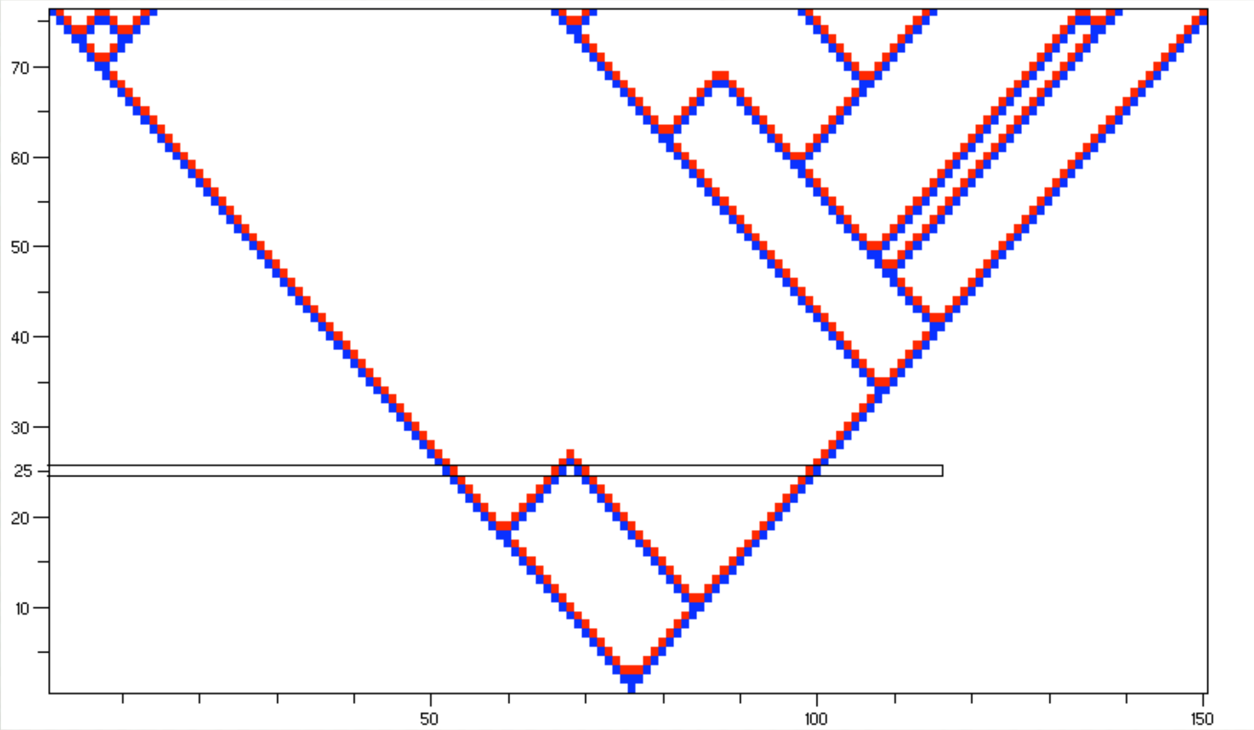

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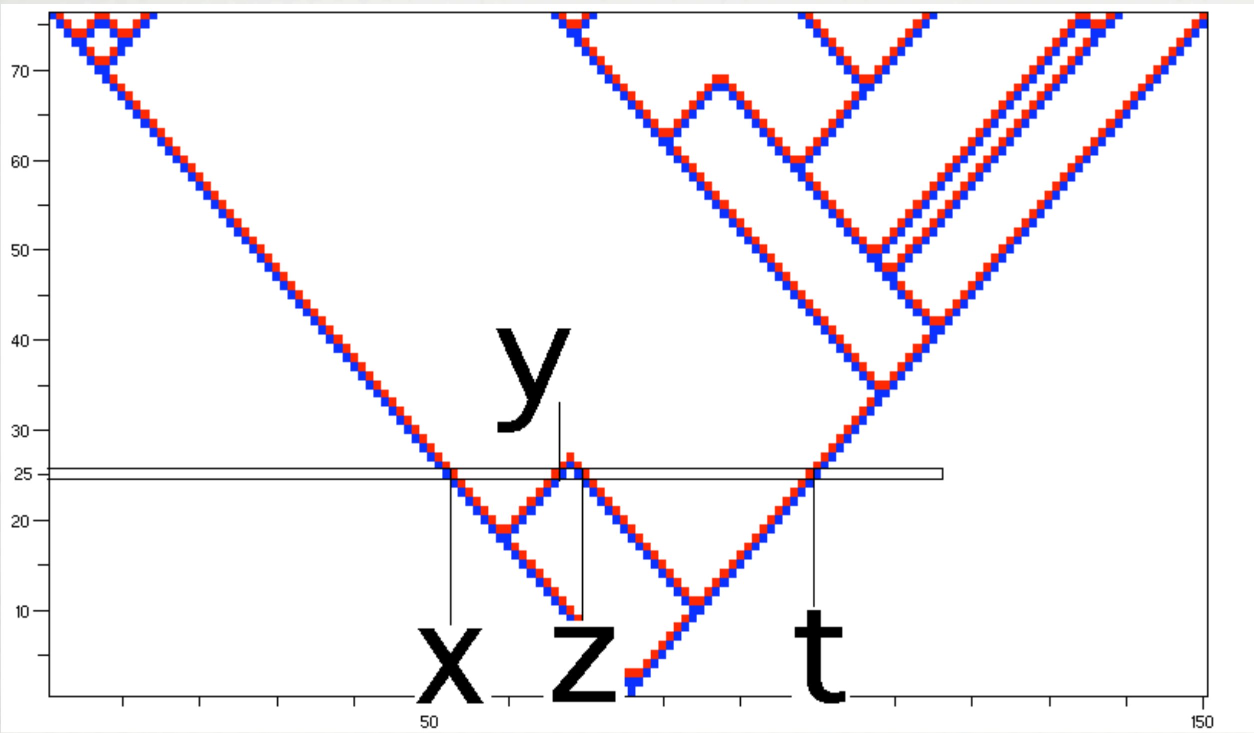

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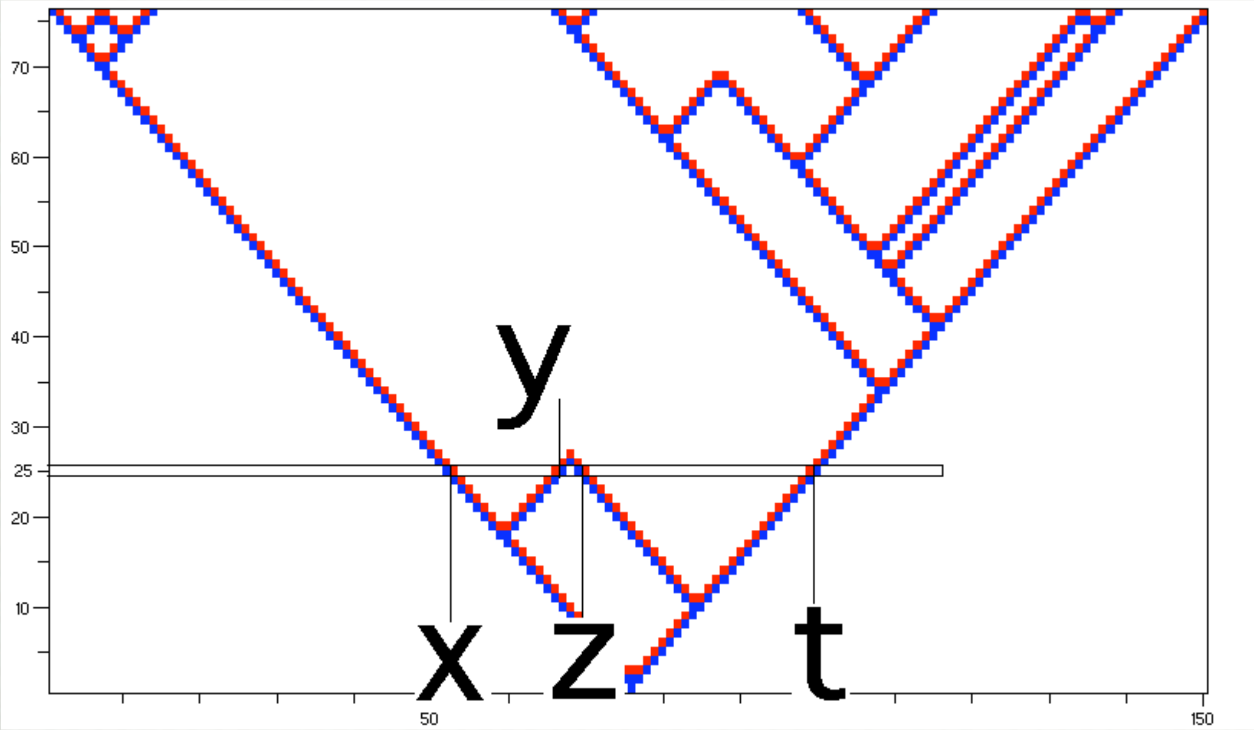

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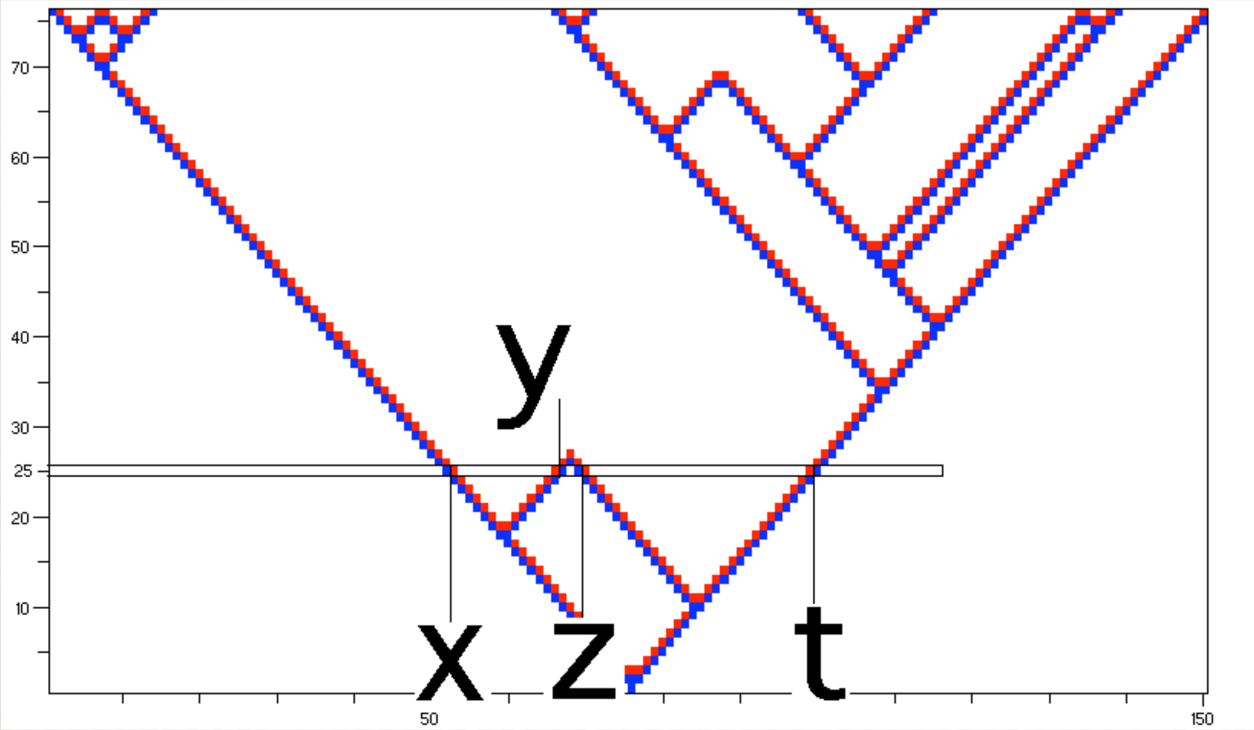

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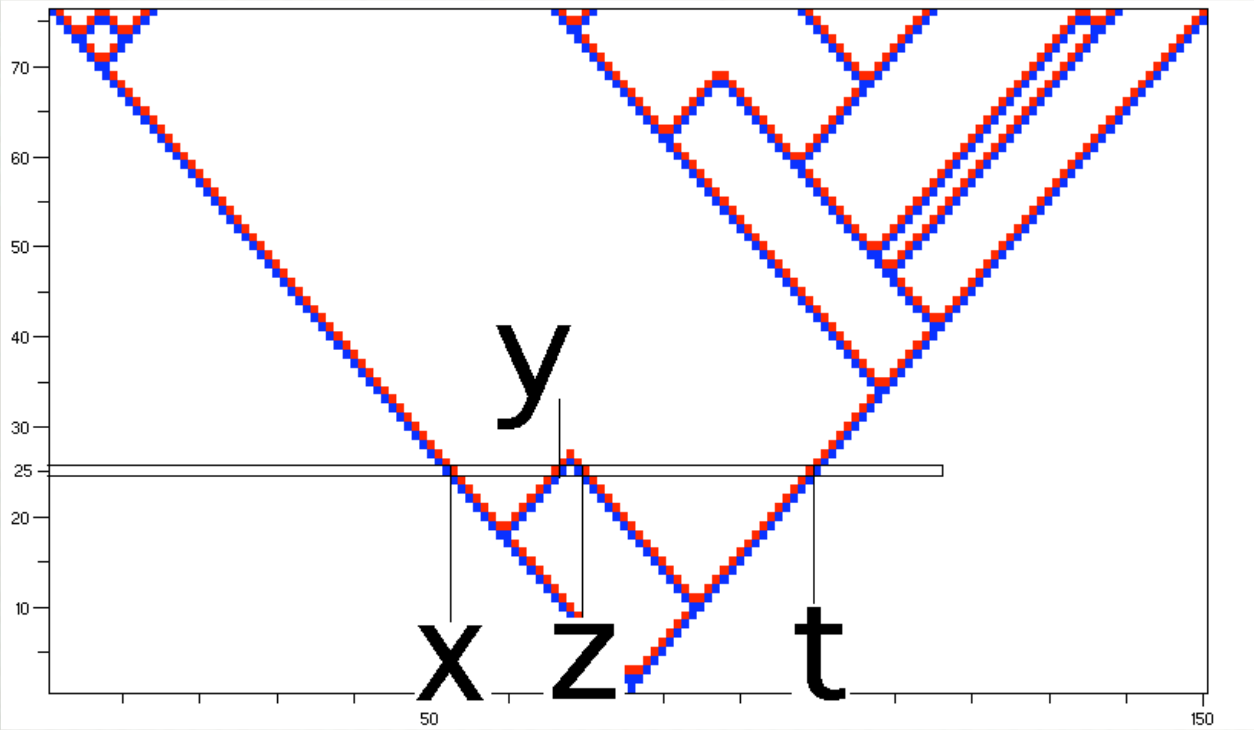

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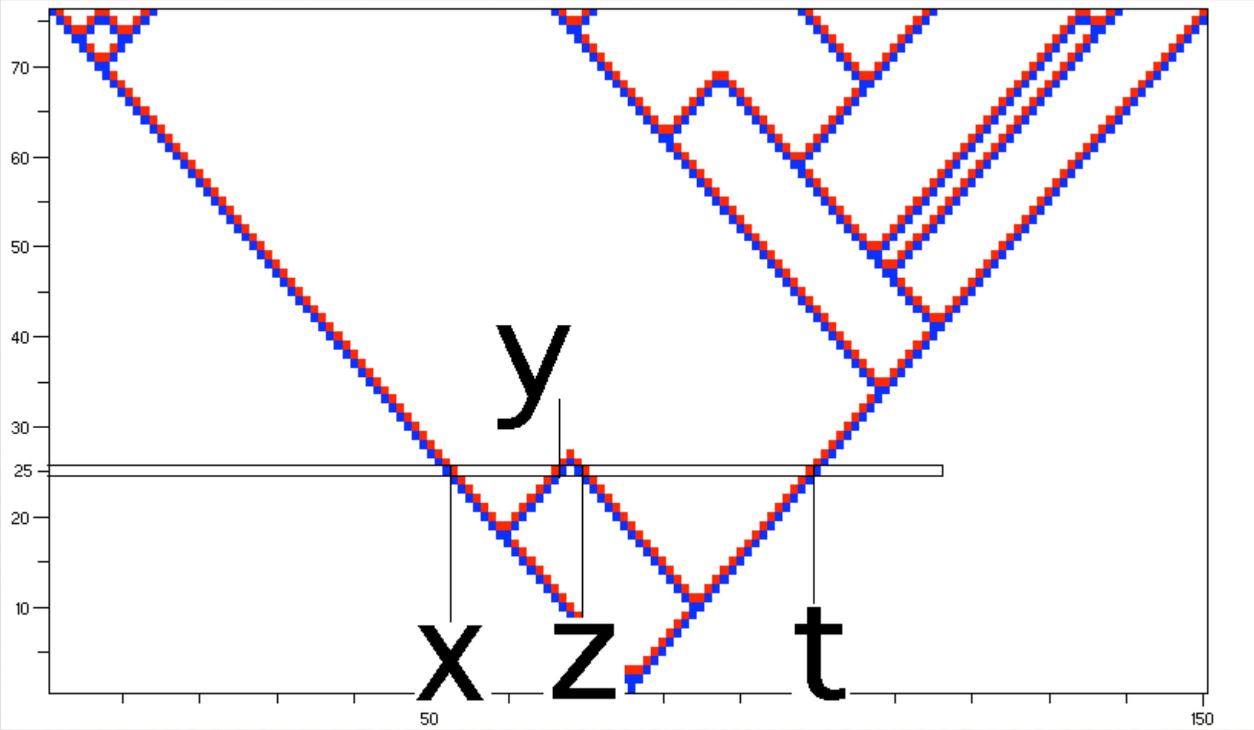

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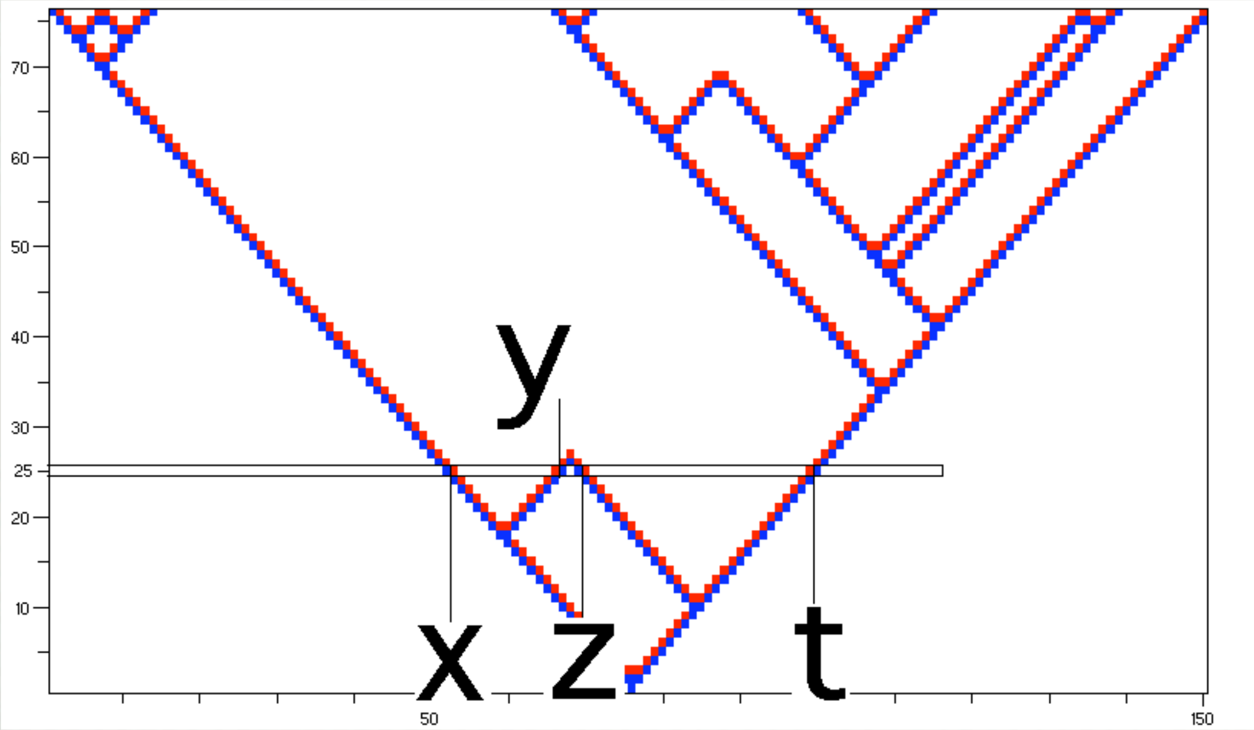

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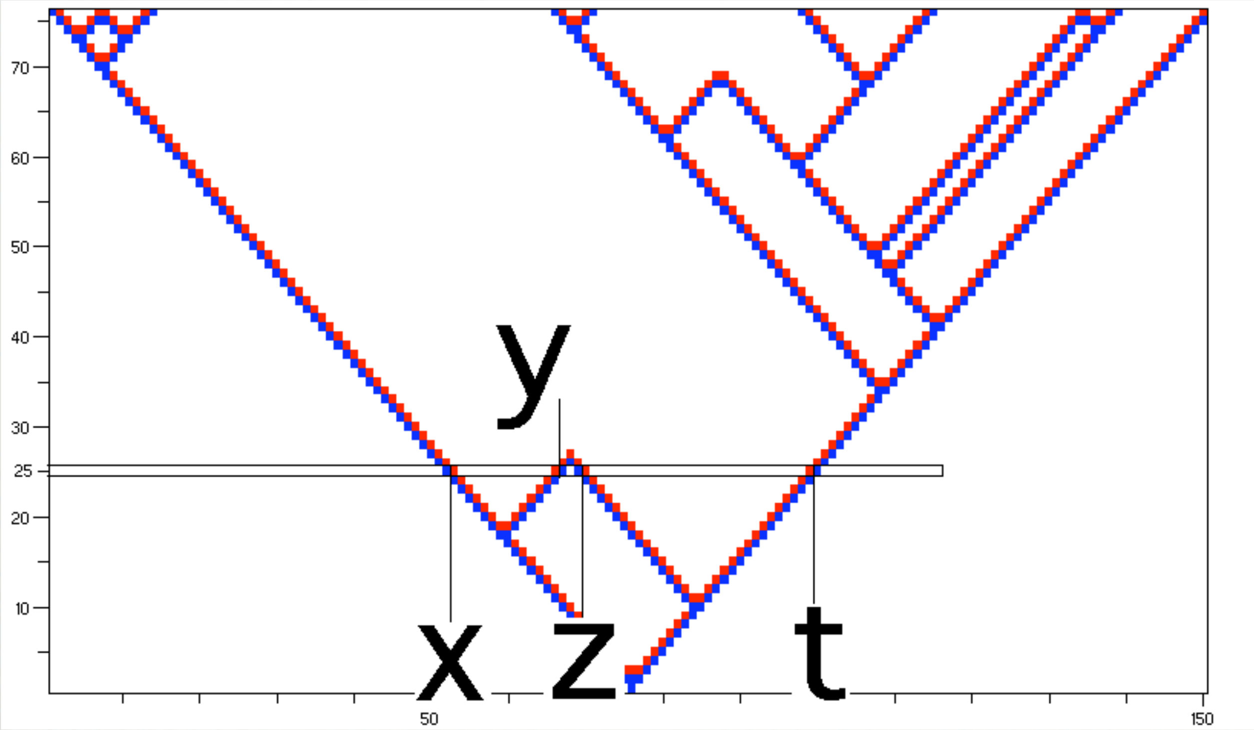

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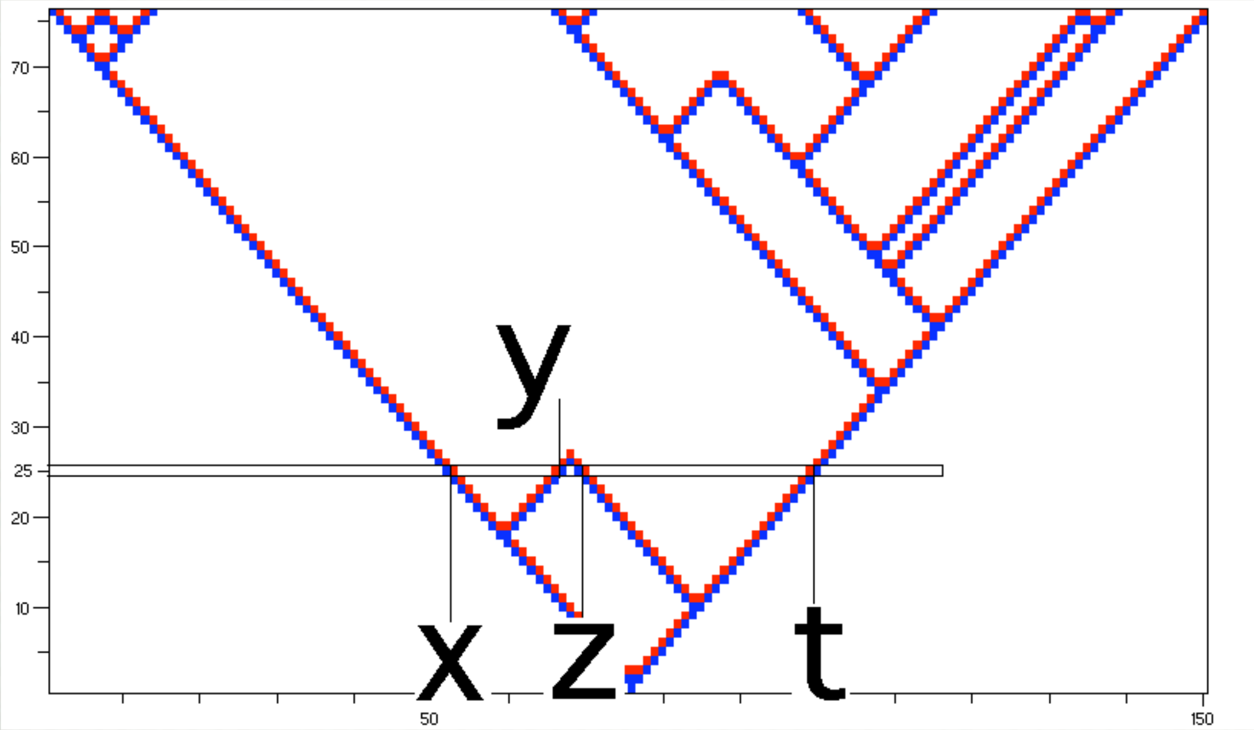

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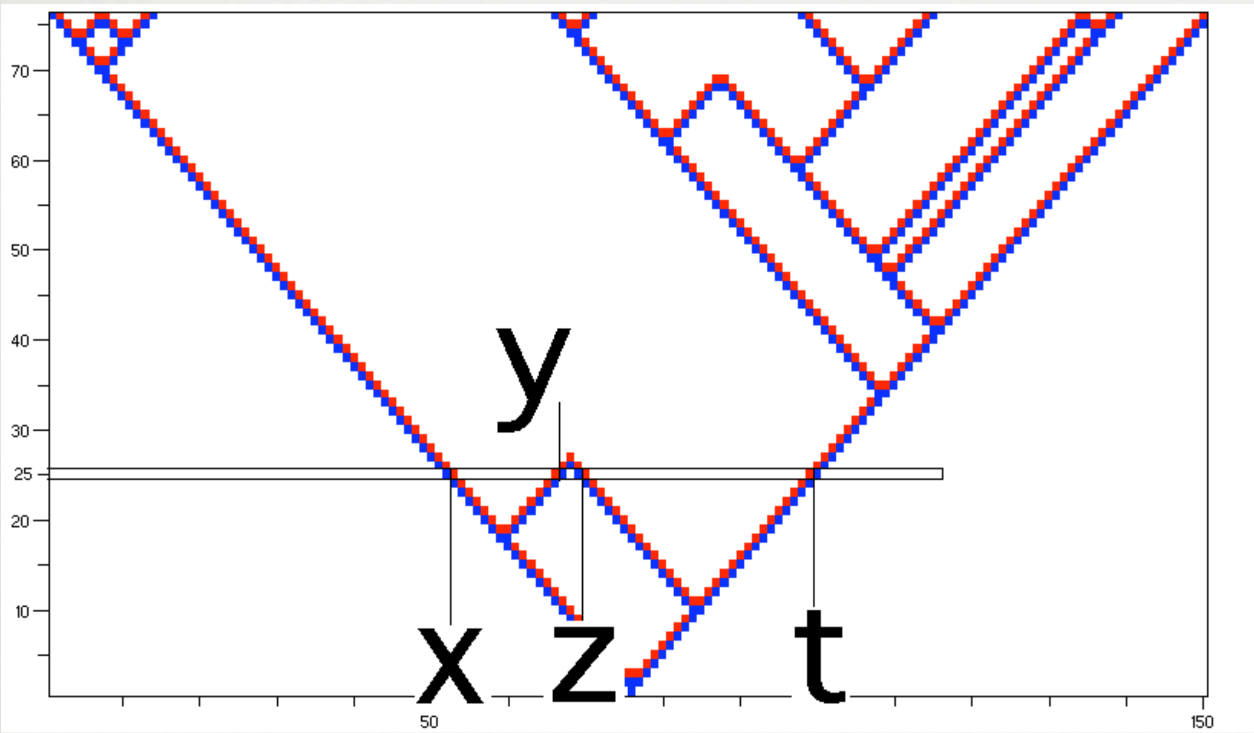

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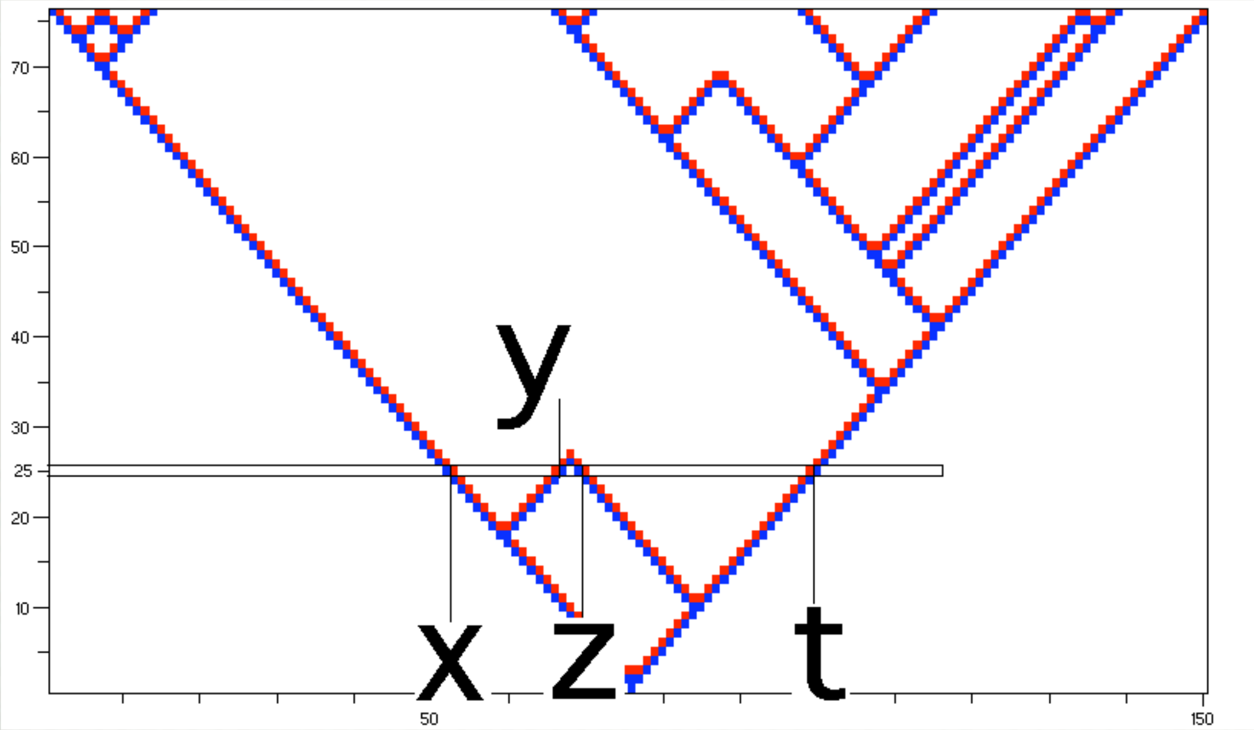

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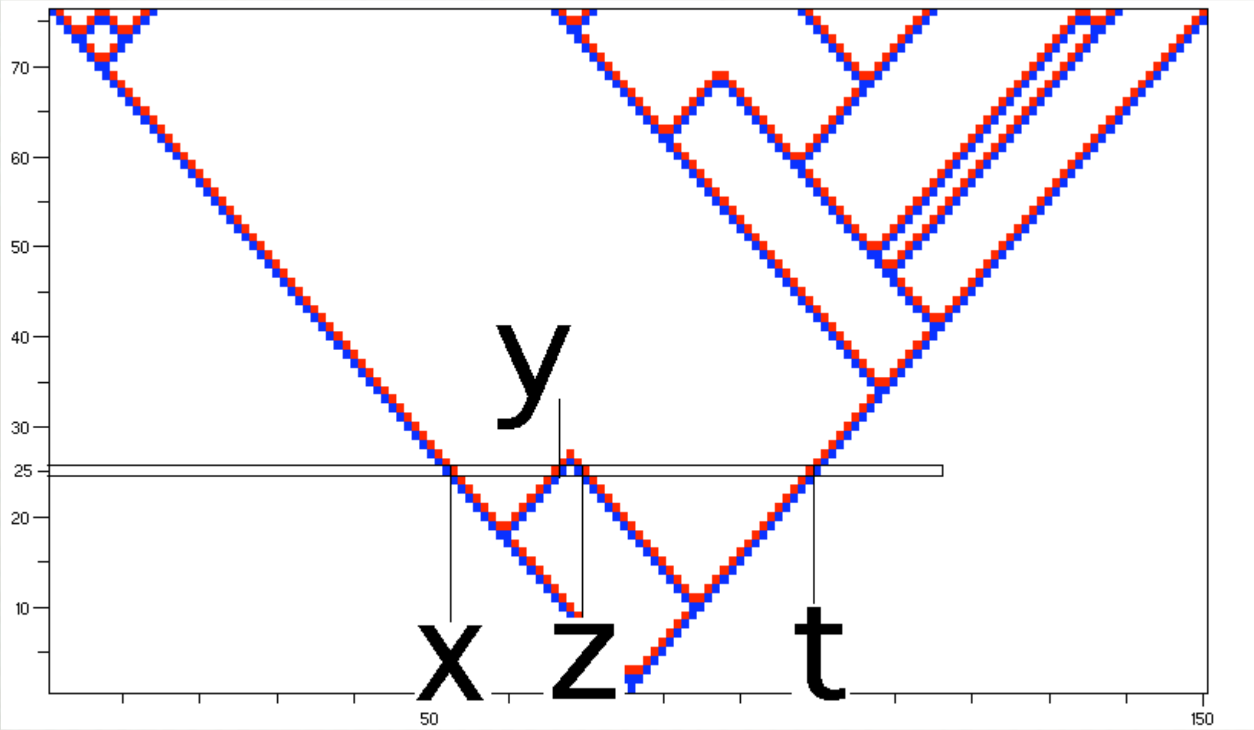

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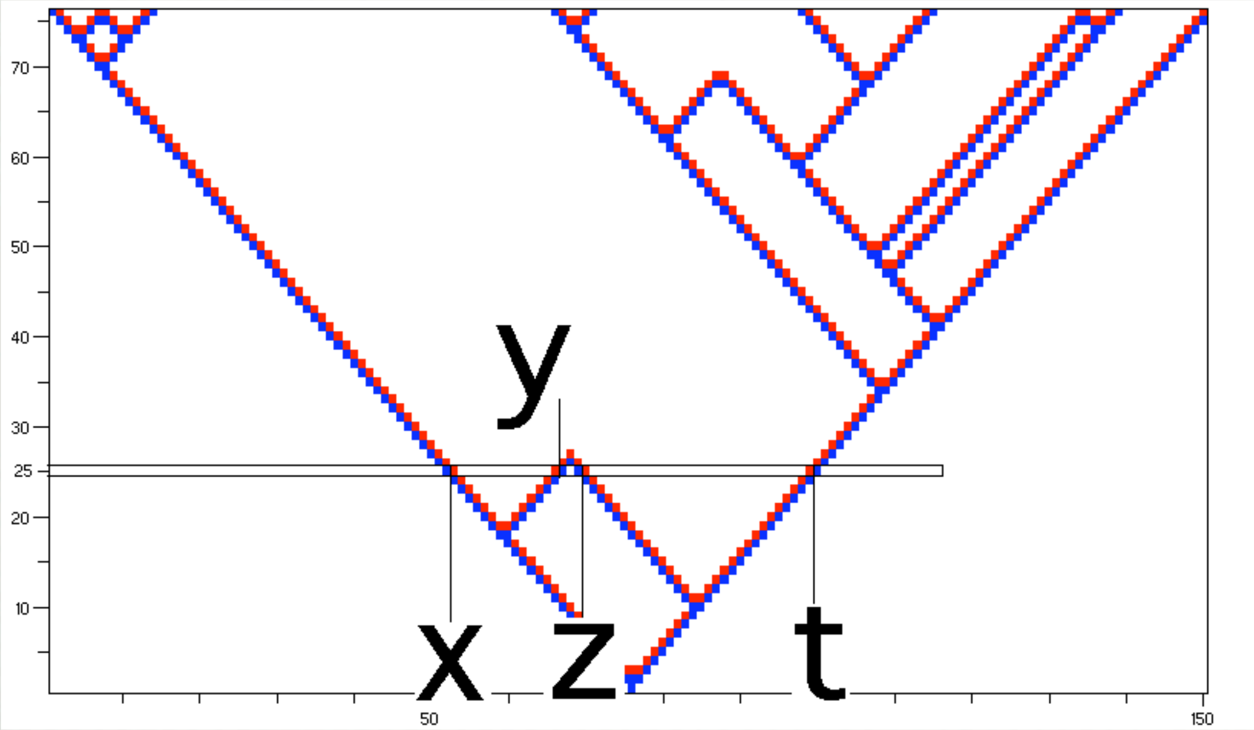

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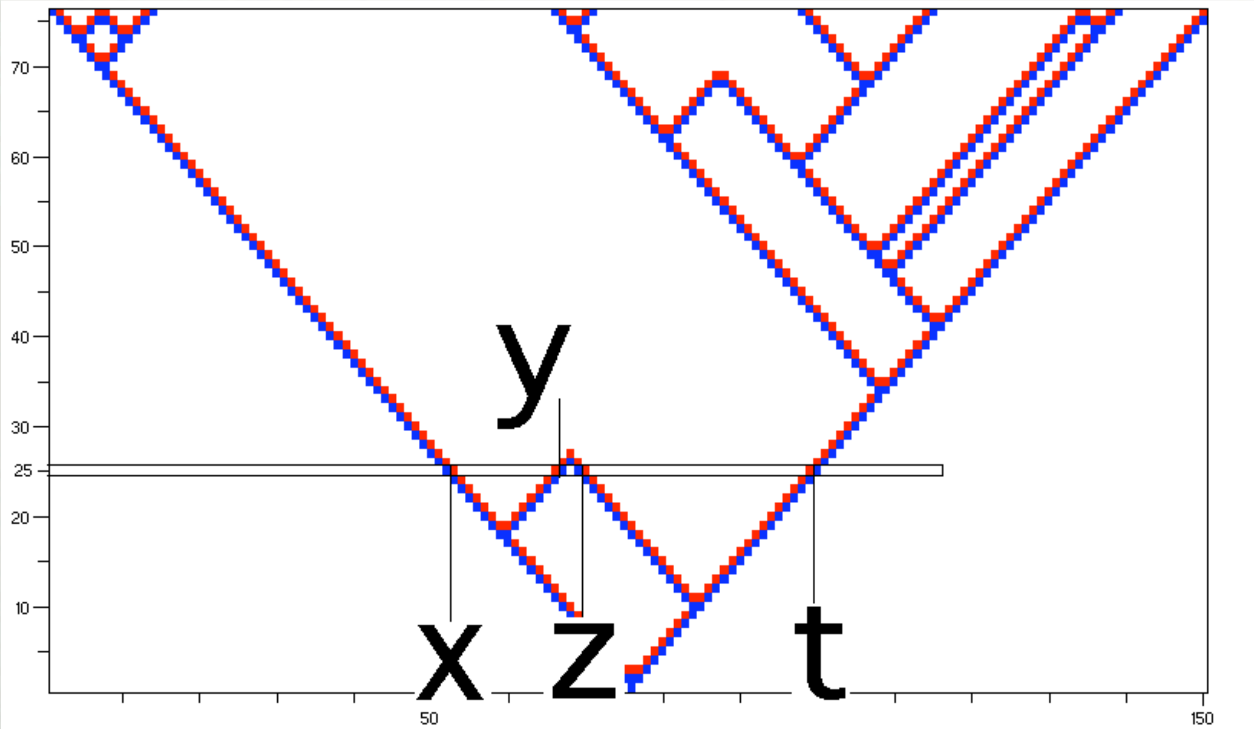

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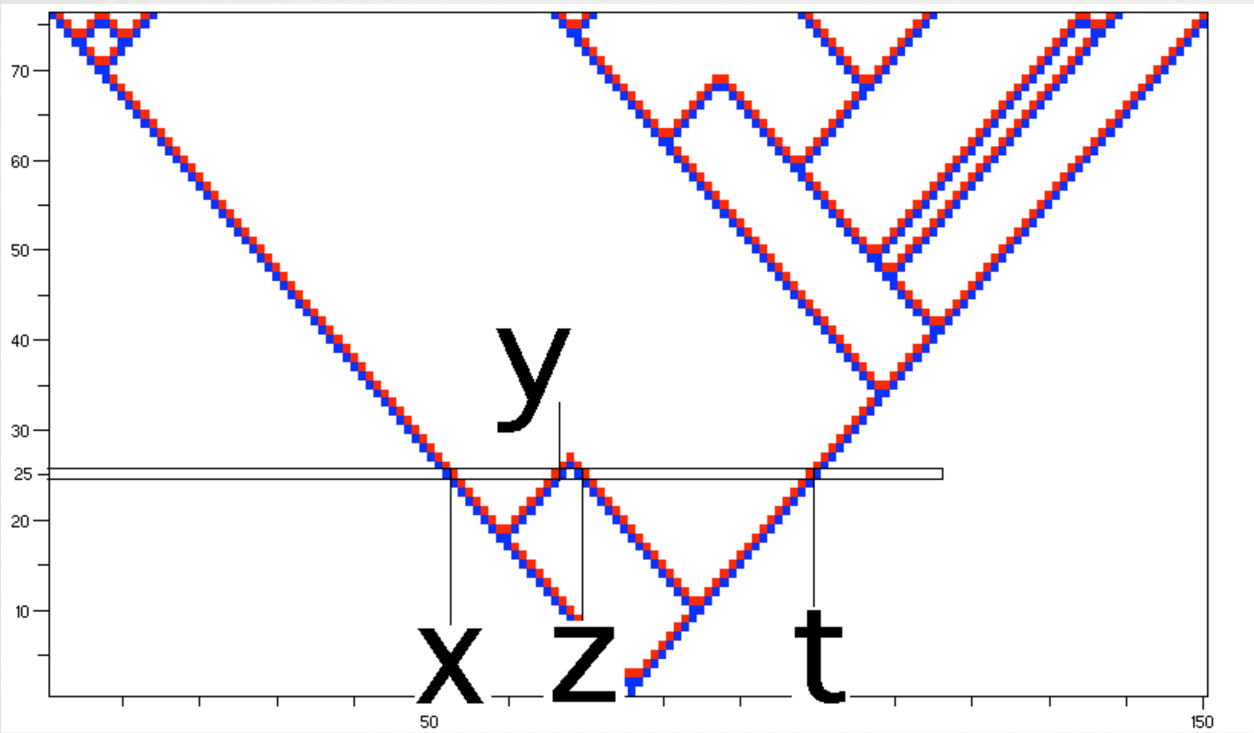

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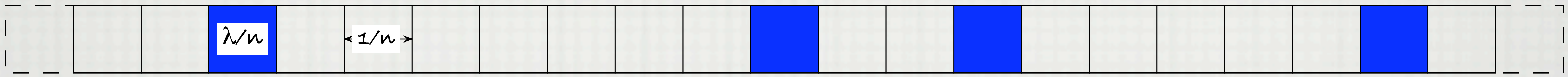

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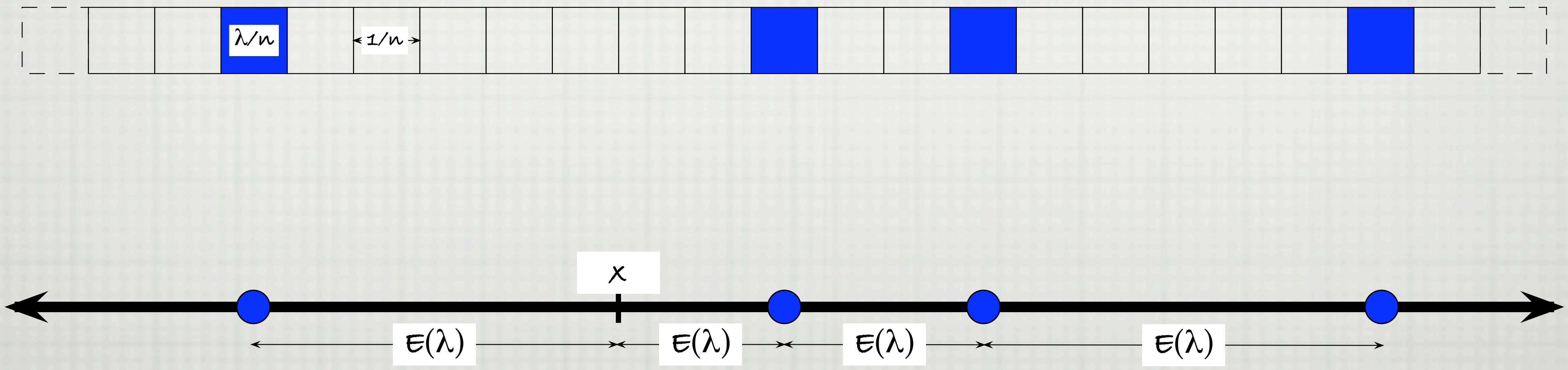
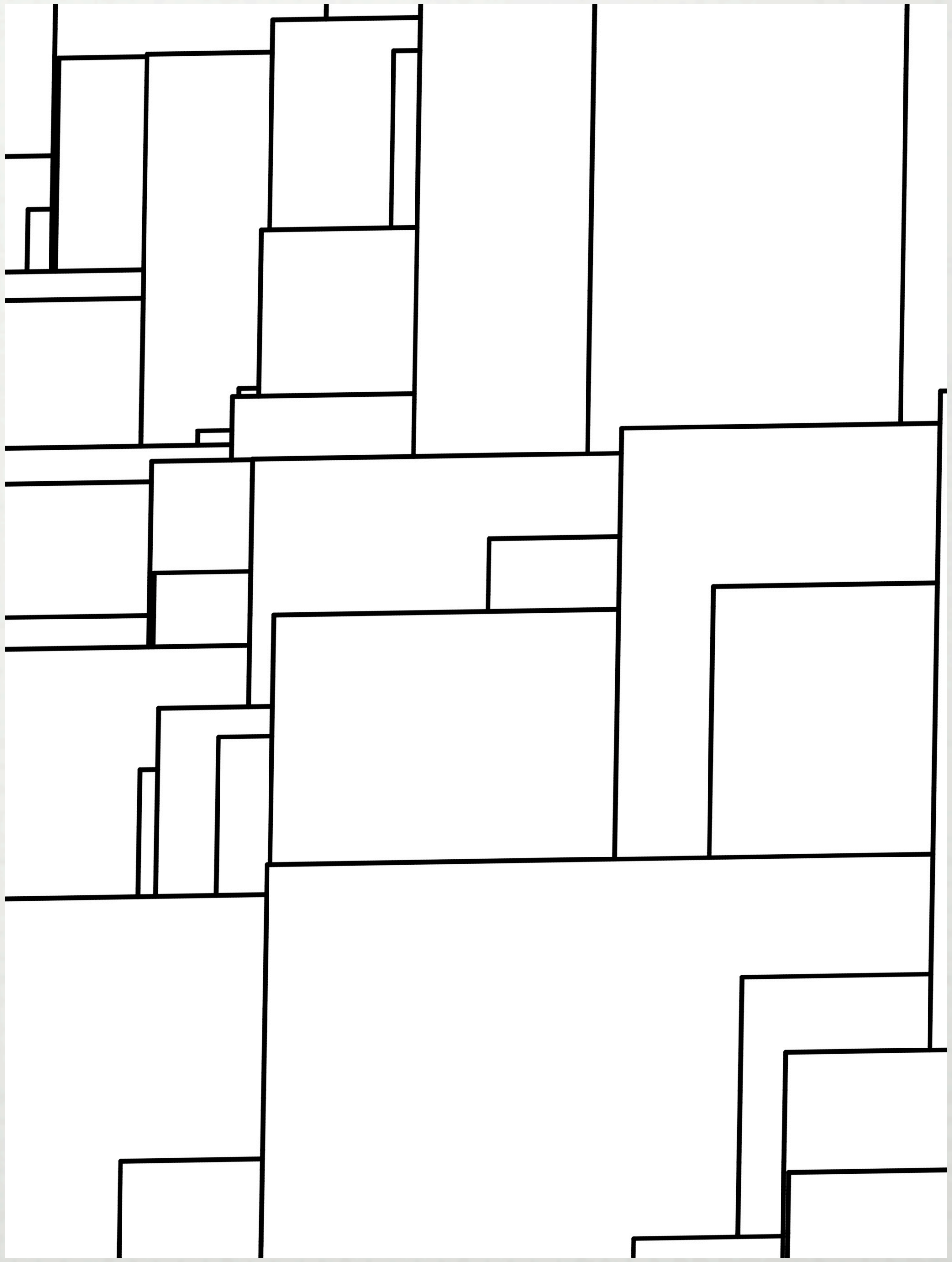


STATIONARY MEASURE OF TILTED PROCESSES #1



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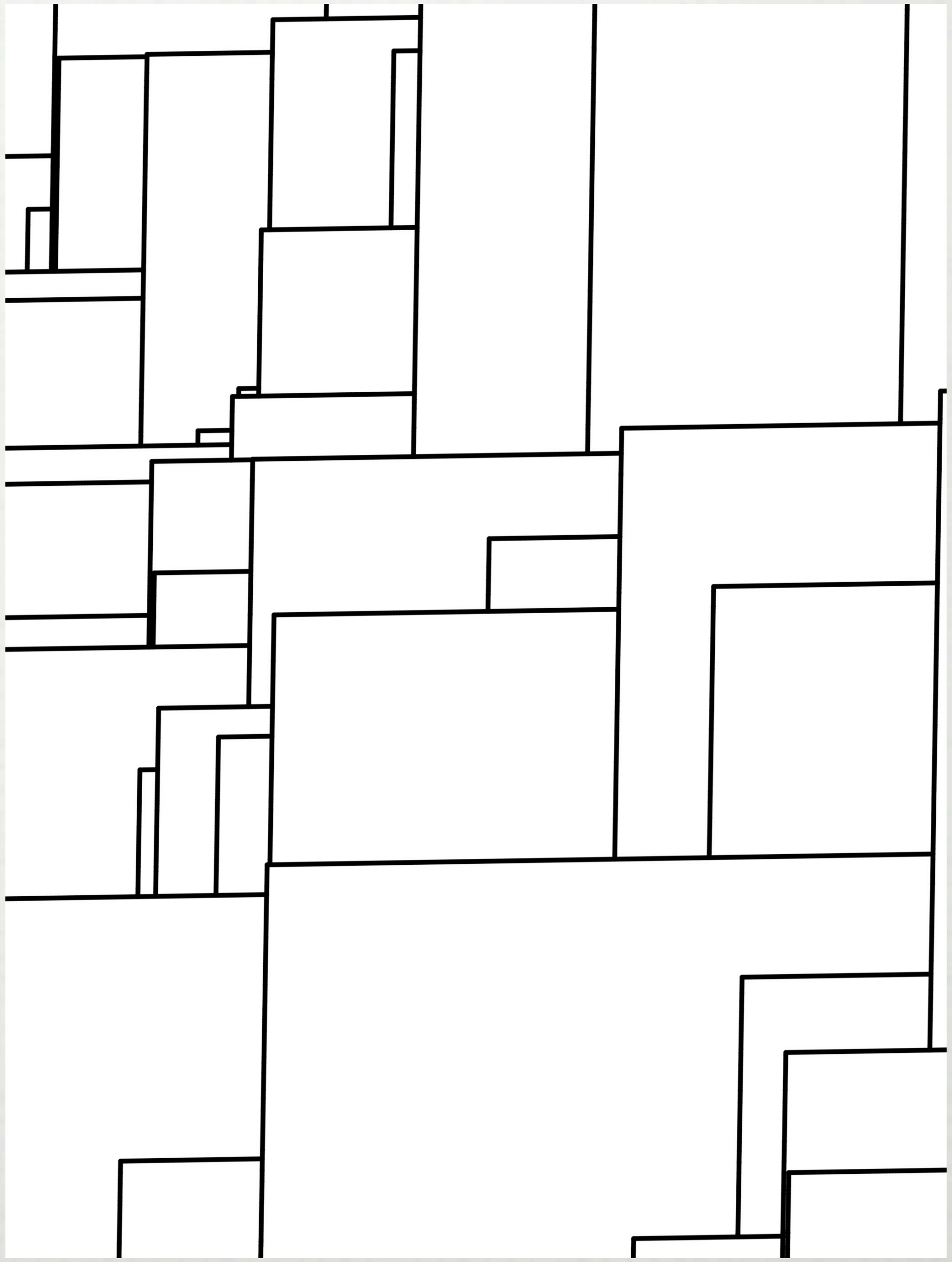

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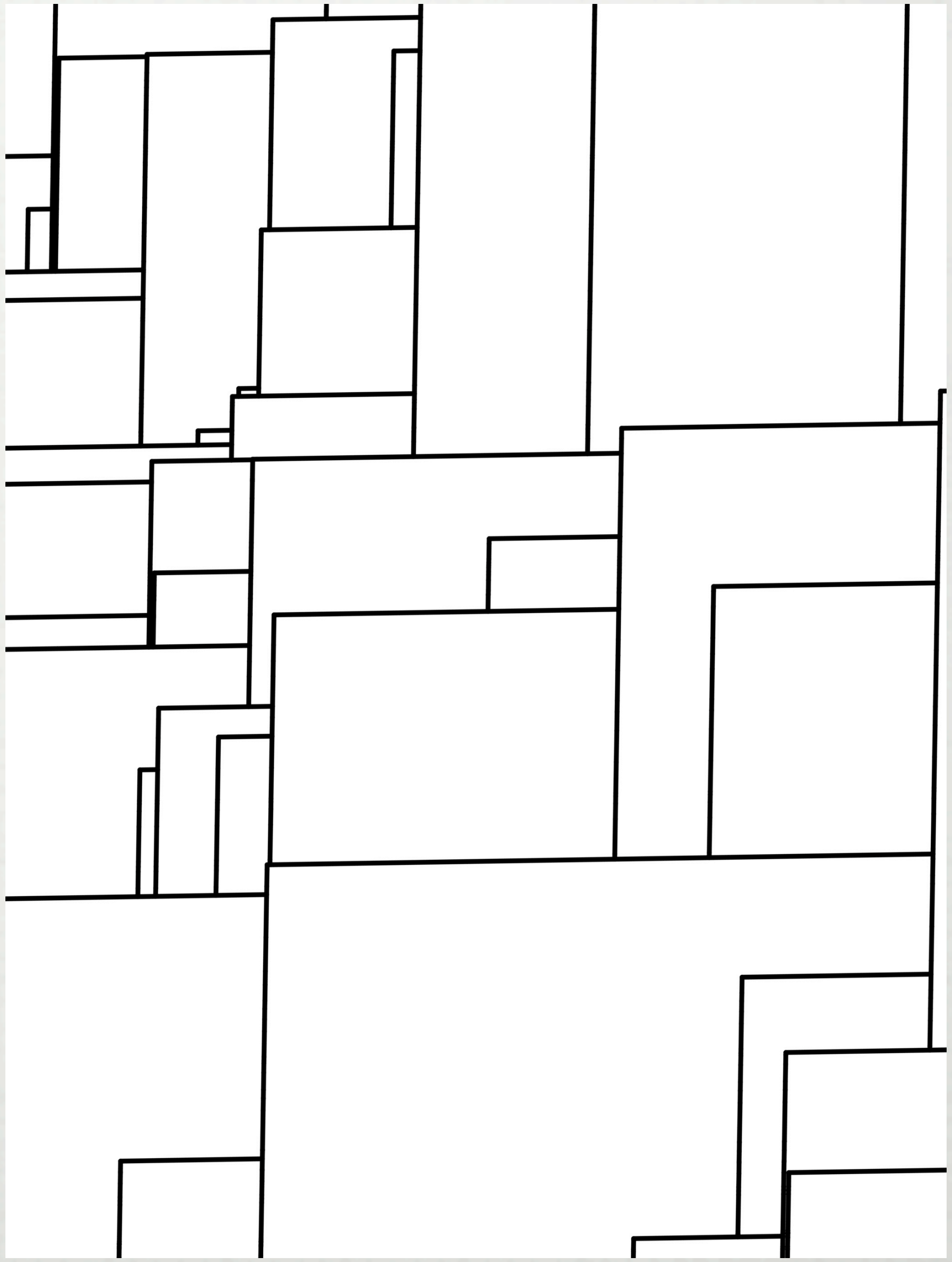

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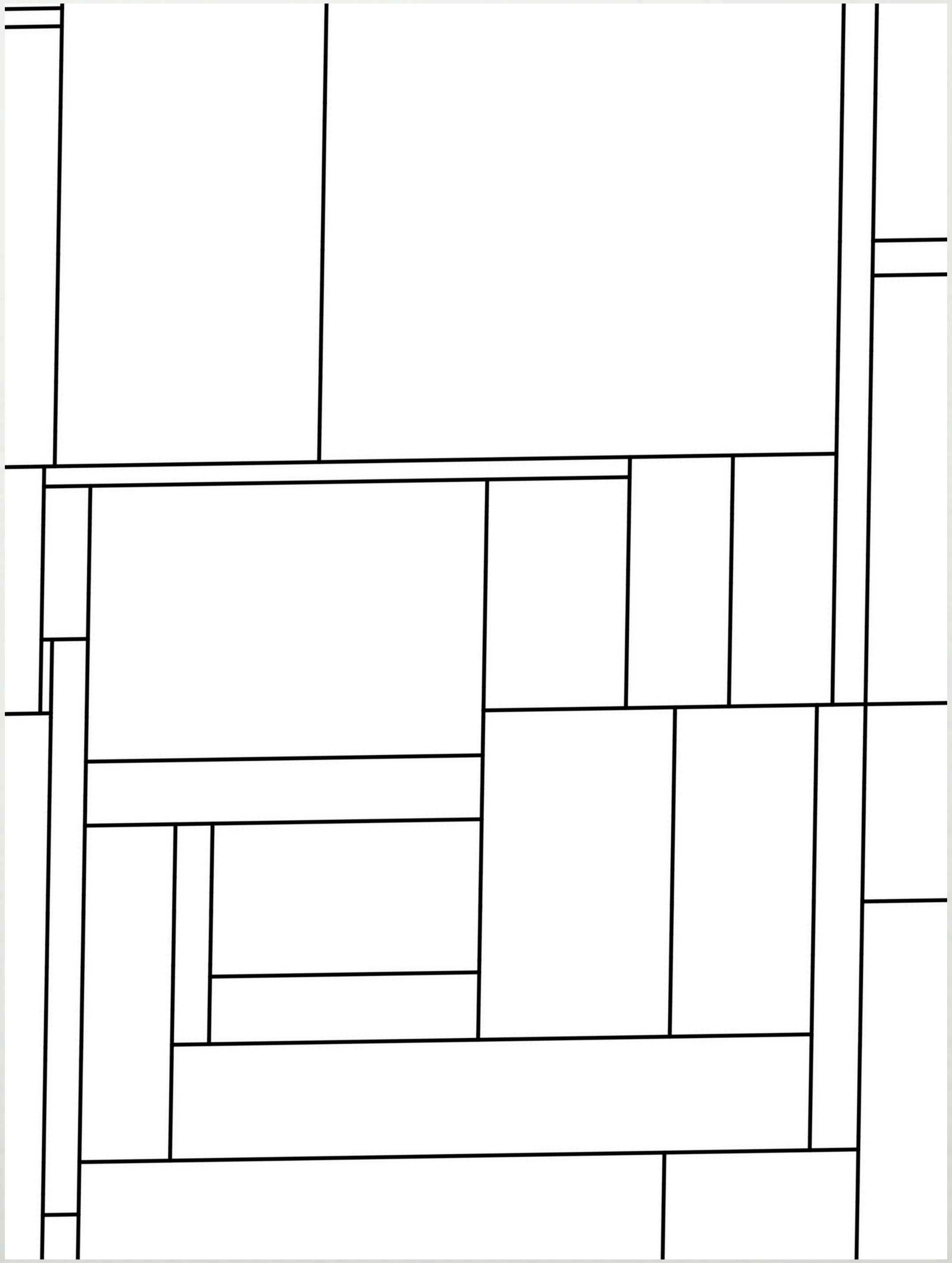

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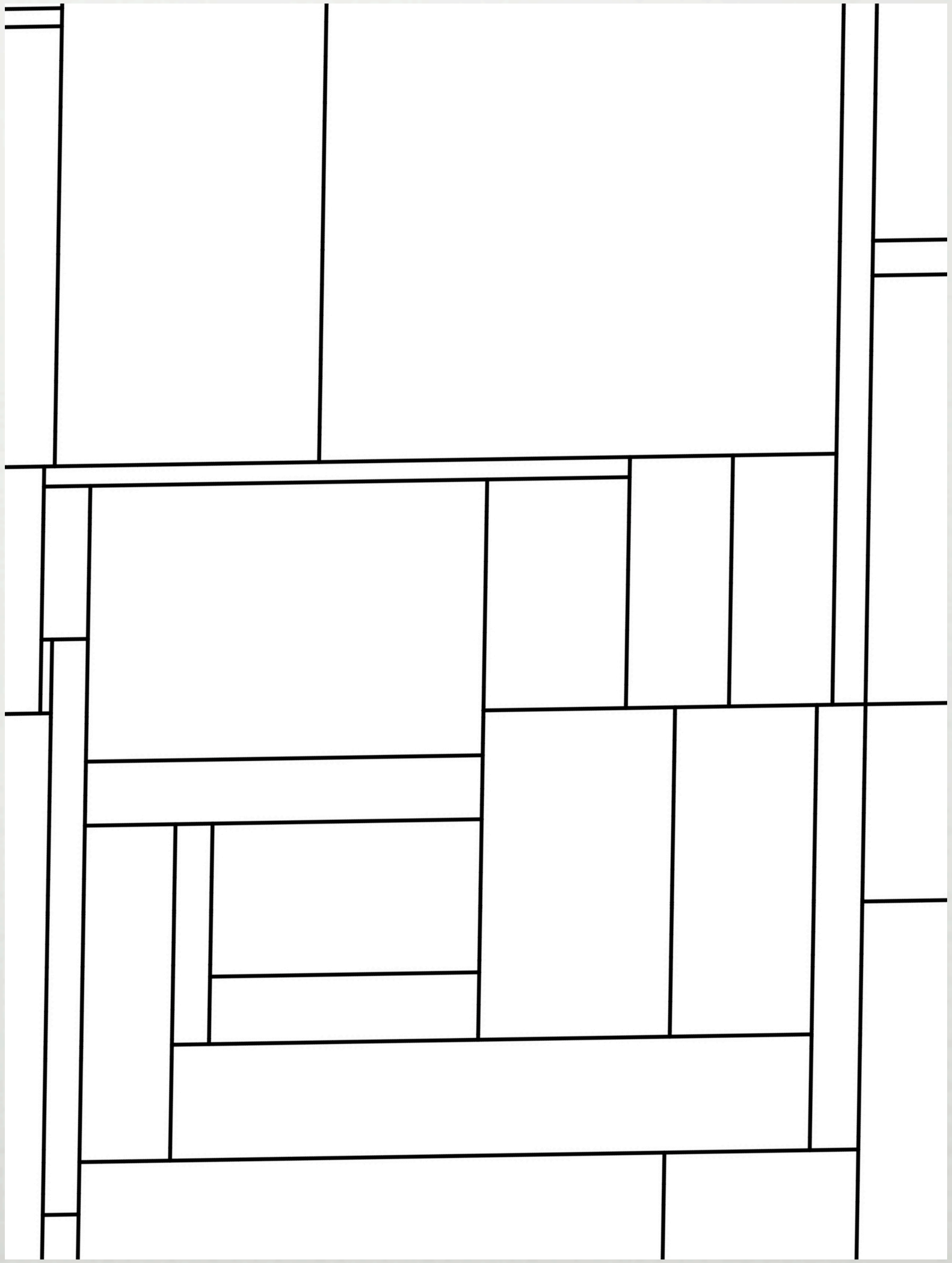

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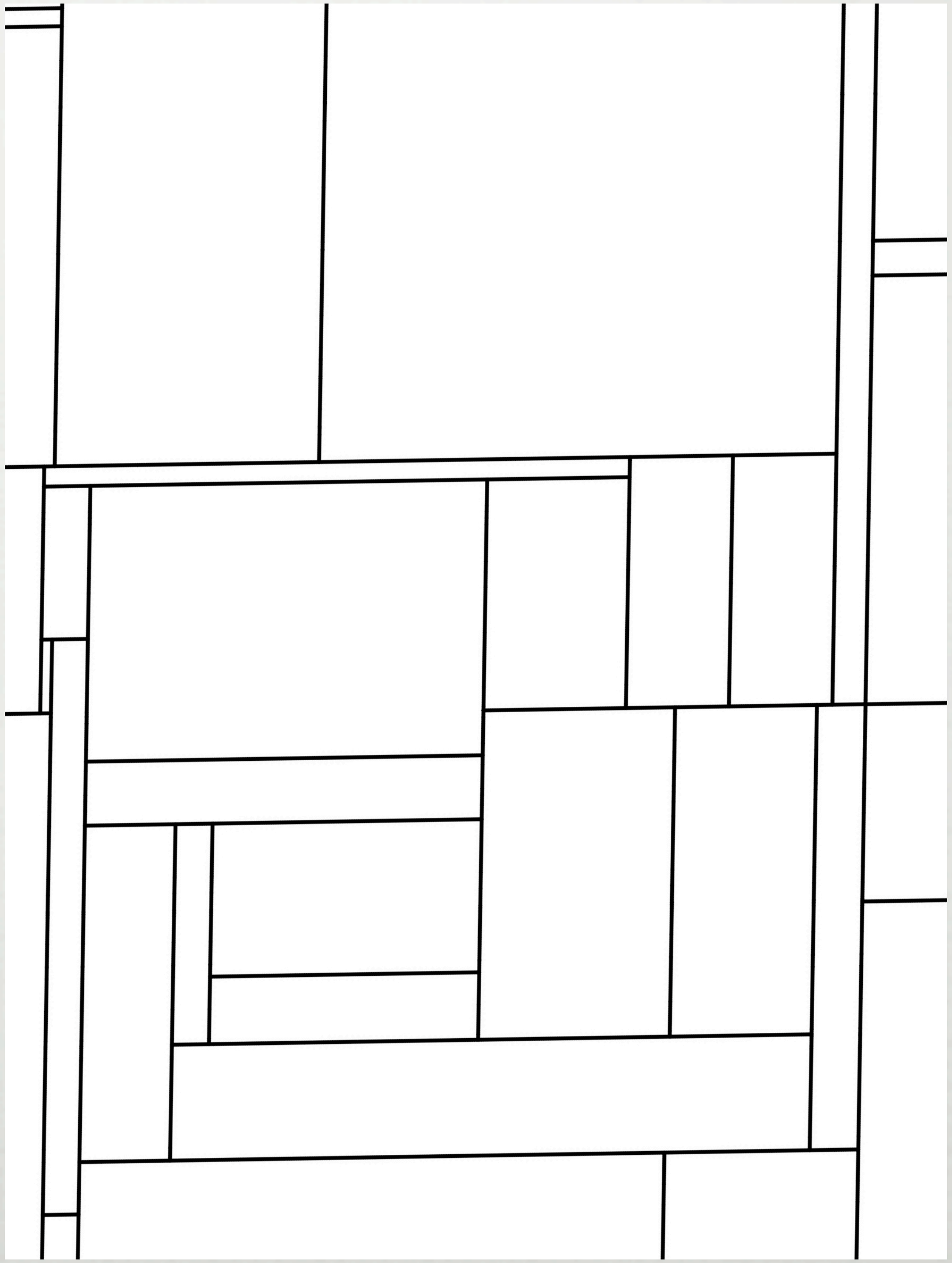

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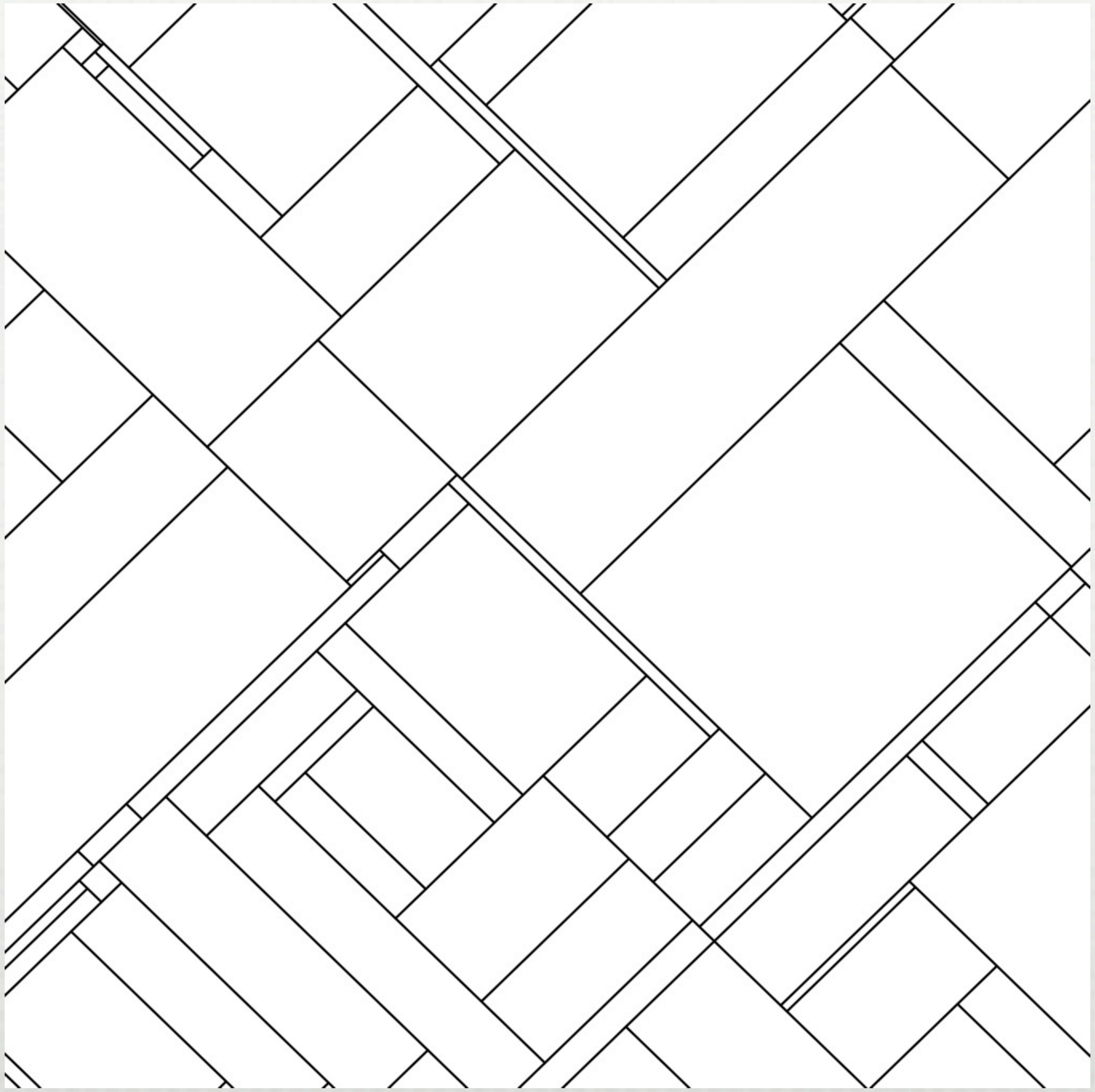

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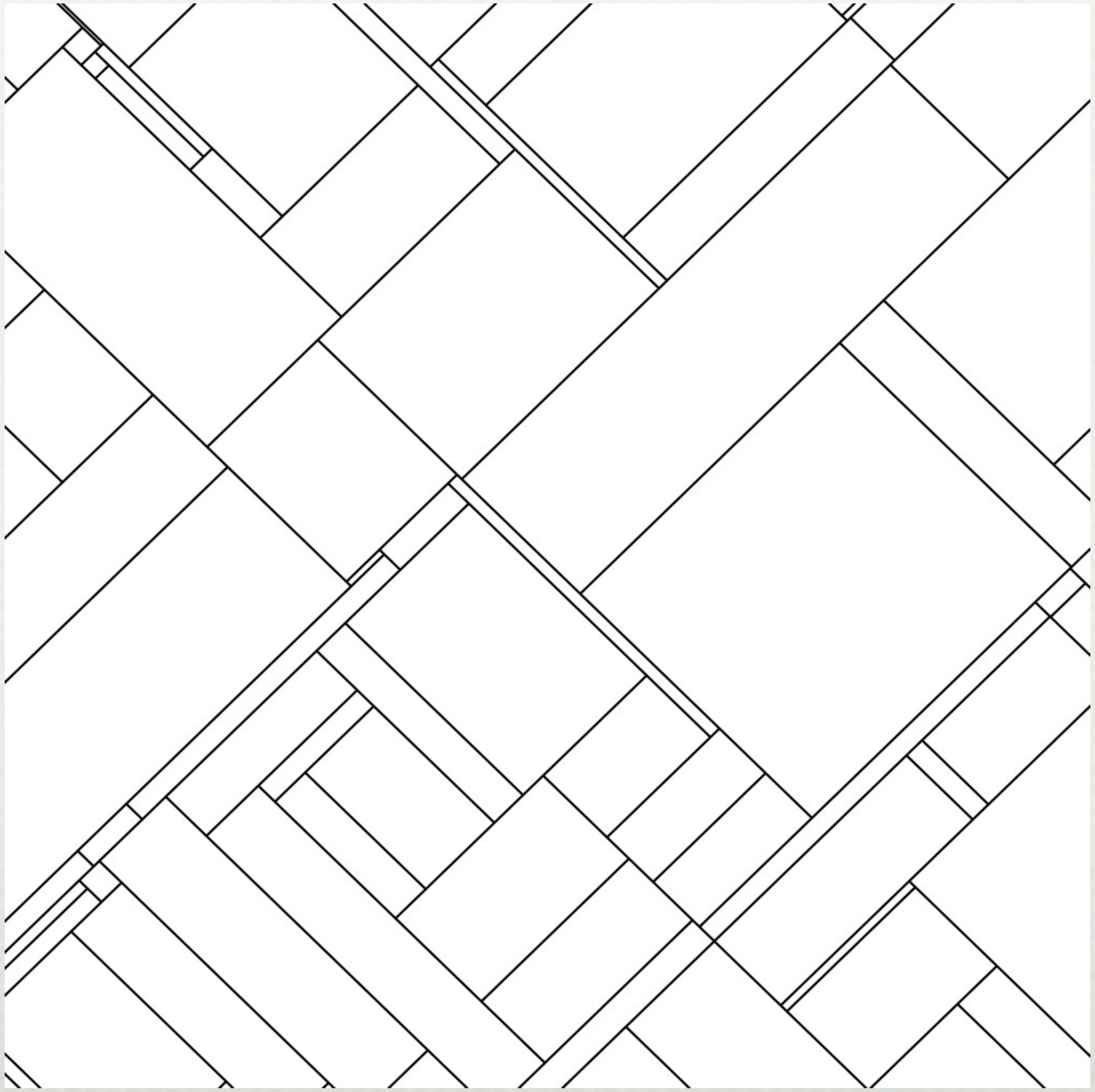

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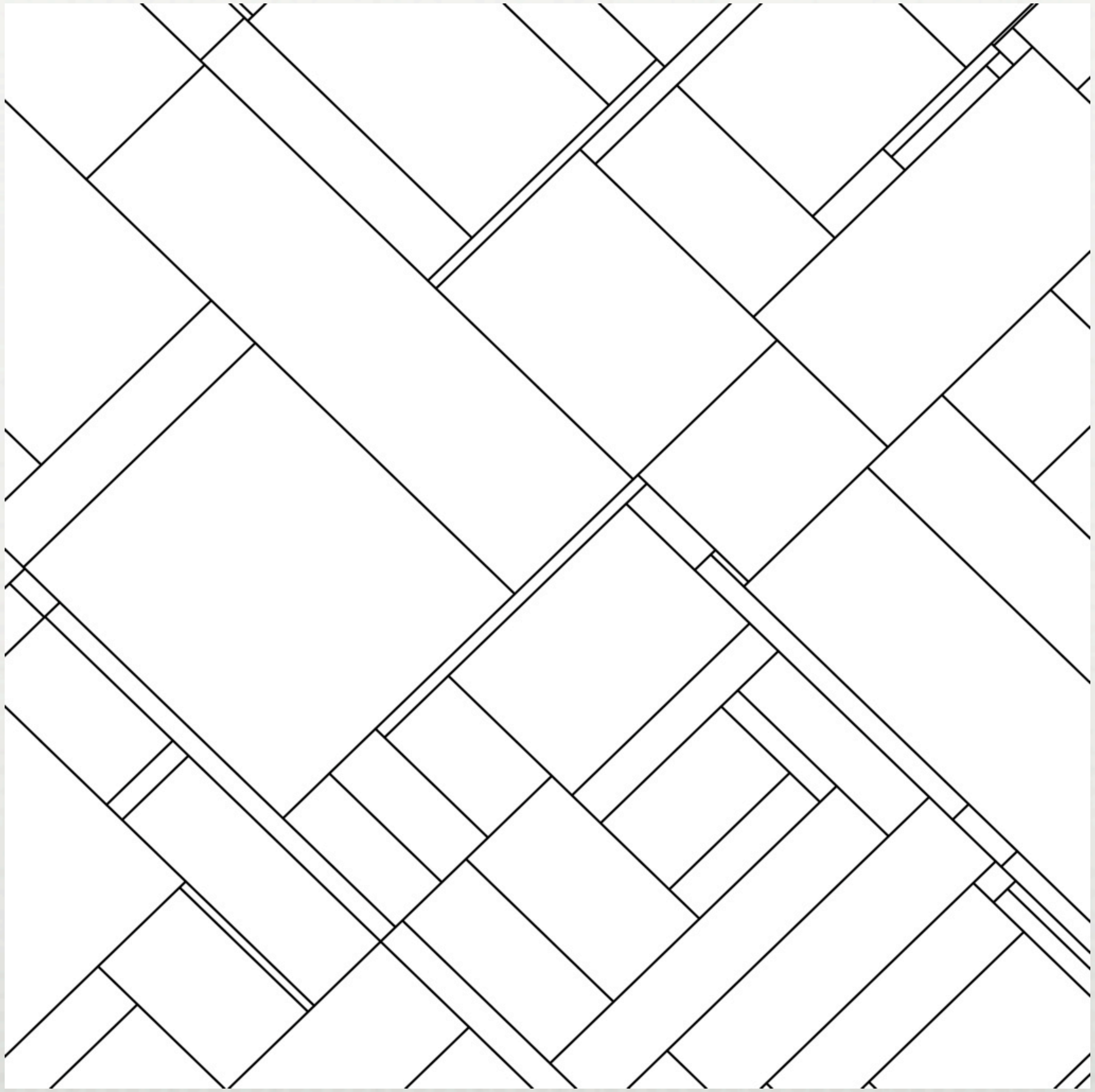

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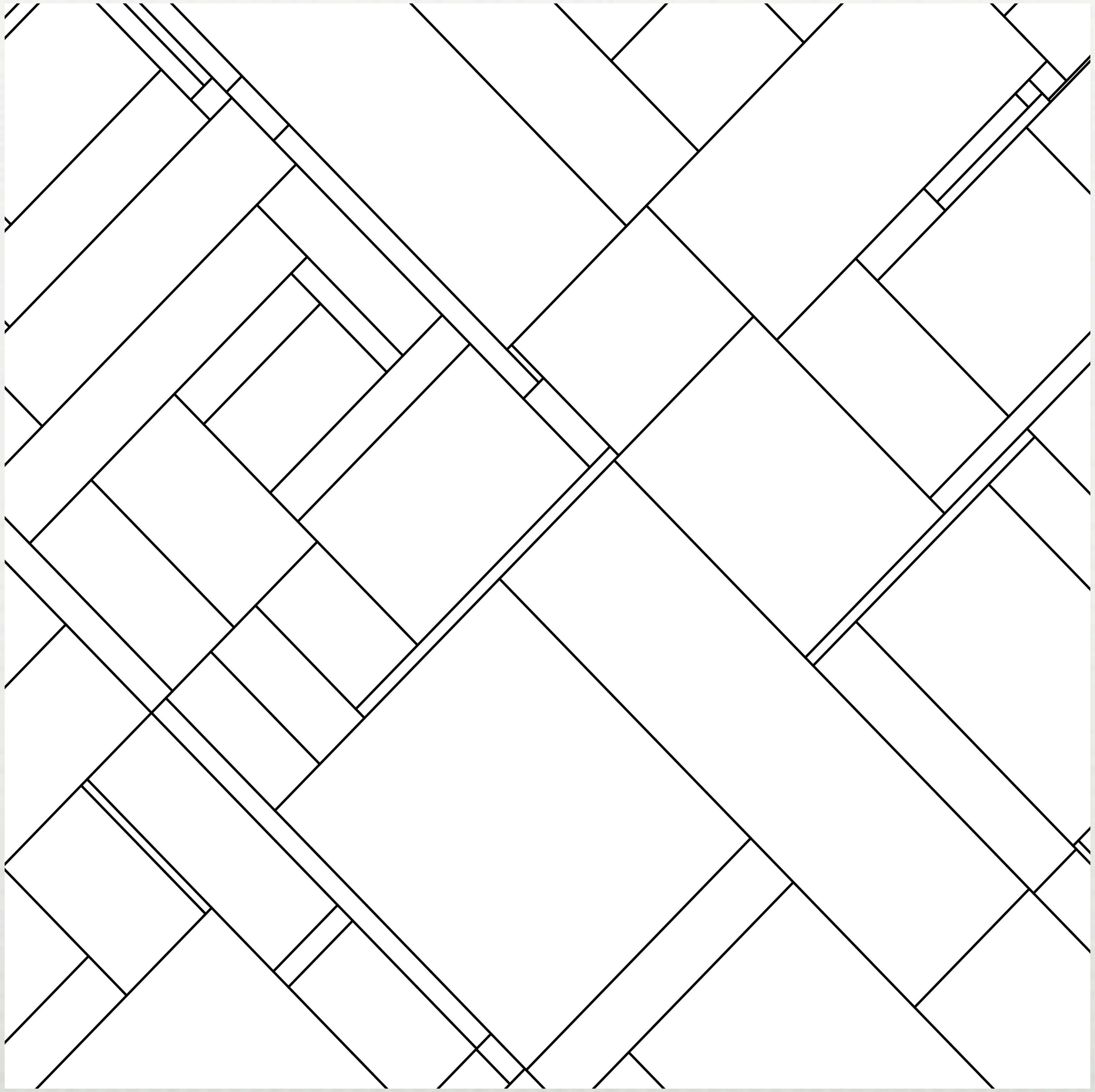

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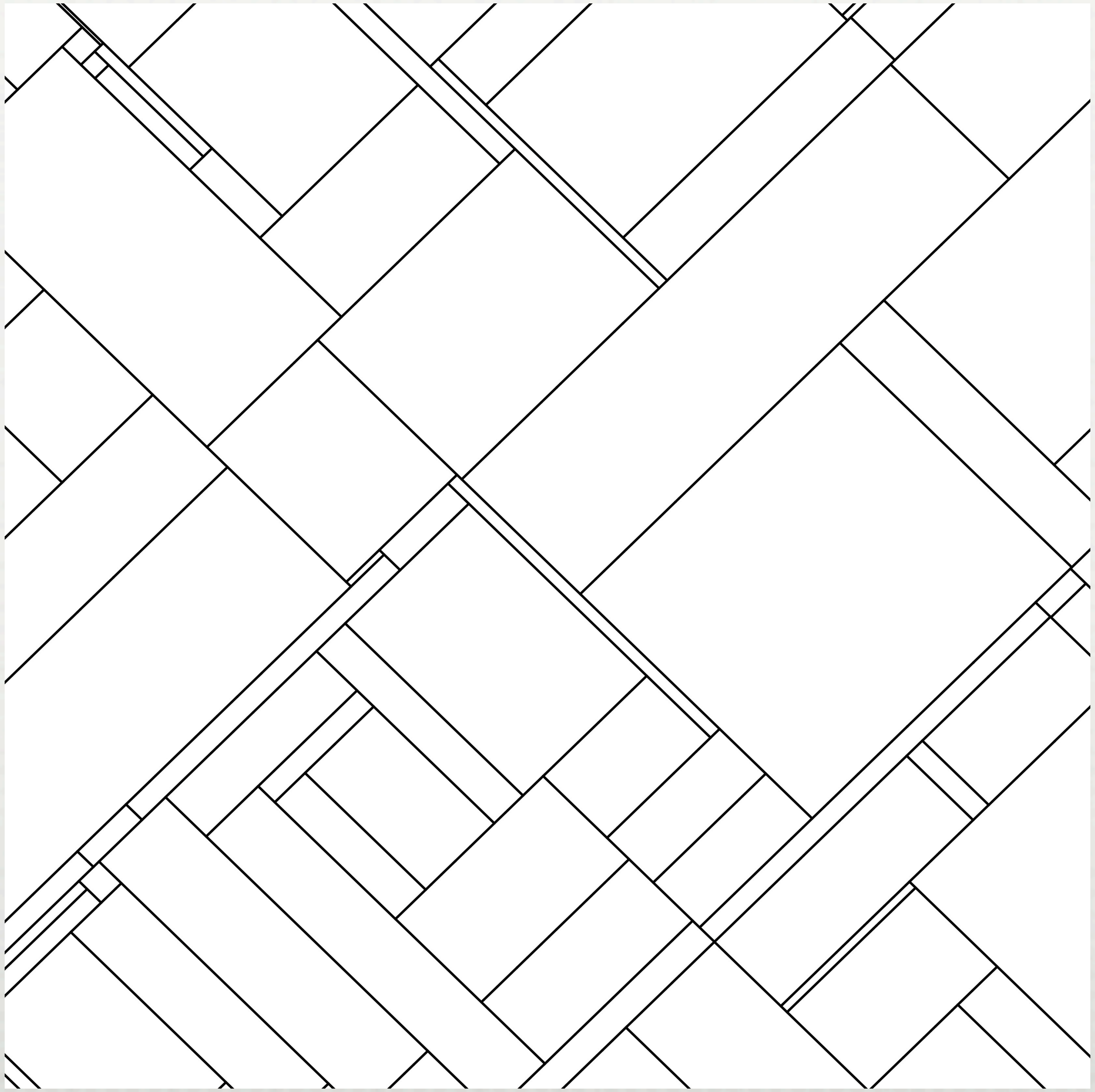

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✴ **Open problem:** to describe the full stationary distribution, i.e. the joint law of positions of positive and negative particles.

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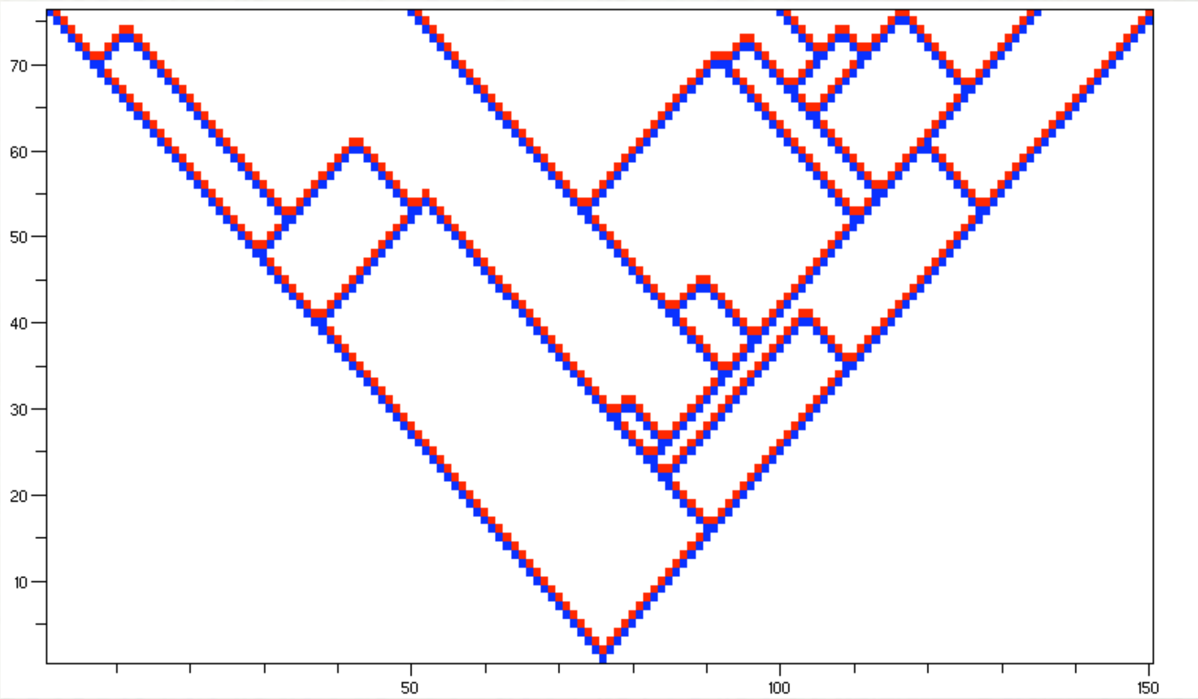

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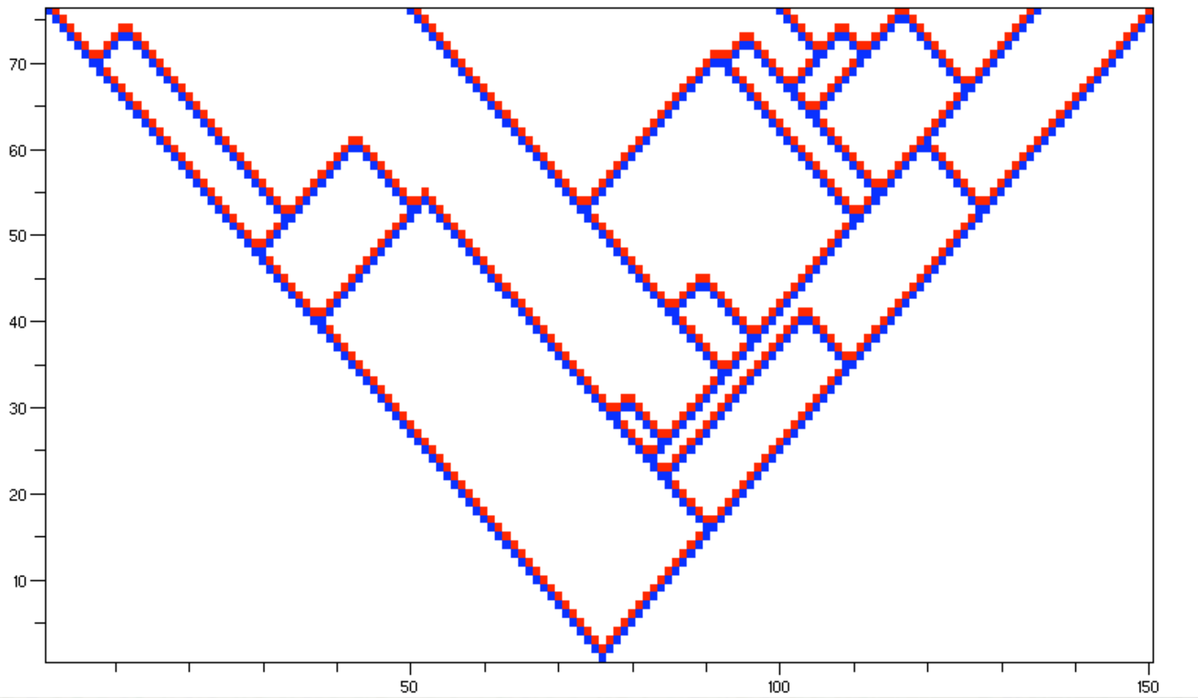

✶**Theorem.** The distribution in the quarter-plane is the restriction of the stationary distribution in the plane.

STATIONARY MEASURE : ANALYTIC APPROACH

Measure-valued Markov process :

Let X_s be the set of positions of positive particles on the line $\{x+t=s\sqrt{2}\}$. Assume the initial state $X_0=x$ to be a good multiset. Then, for ϕ positive, continuous, and with compact support,

$$G\left(e^{-\langle\varphi,\cdot\rangle}\right)(x) \;:=\; \lim_{s\downarrow 0}\; s^{-1}\left(\mathbb{E}\left[e^{-\langle\varphi,X_s\rangle}\right]-e^{-\langle\varphi,x\rangle}\right)$$

is well defined.

If furthermore, a random multiset X (a.s. good) satisfies, for every ϕ positive, continuous, and with compact support,

$$\mathbb{E}\left[G\left(e^{-\langle\varphi,\cdot\rangle}\right)(X)\right]=0,$$

then the law of X is a stationary measure of the Markov process.

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Proof. The generator of the tilted process is given by

$$Ge^{\langle \phi,.\rangle}(\pi) = e^{\langle \phi,\pi\rangle} \sum_{\{y,x\}\subset\pi,\, y<x} 2^{-\#[y,x)\cap\pi} \left(e^{-\langle \phi \mathbf{1}_{(y,x)},\pi\rangle + \mu\int_y^x (e^{\phi(u)}-1)du} - 1\right).$$

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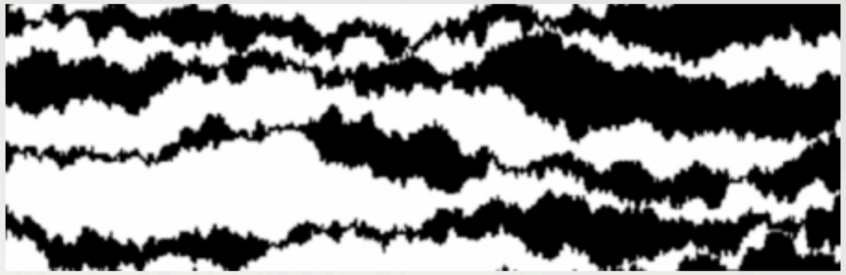
One has to check that for any symmetry T of the real axis (of the form T(x)=a-x) :

$$Ge^{\langle\phi\circ T,.\rangle}(\pi) = Ge^{\langle\phi,.\rangle}\circ T(\pi).$$

RELATED RESULTS



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THANK YOU !

CONSTRUCTION OF THE PROCESS

Theorem . For each 1-Lipshitz fonction f and for each locally finite set of points on

$$\partial f := \{(x,t) : t = f(x)\},$$

the process in the half plane

$$f_+ := \{(x,t) : t > f(x)\},$$

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- the only problem arises when $f(x)$ has slope ± 1 on an infinite interval : the past of a point of the half plane can contain infinitely many branching points.

MARKOV PROPERTY

- **Property.** Consider the process built from a configuration on a border ∂g. Then, for a border ∂f (s.t. $g<f$), the parts of the process below and under ∂f are independent, conditionally, given the process on ∂f.

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- **Conjecture** : true for the 2 components inside and outside a Jordan curve, conditionally, given the process on the curve.