Mean-field behavior for long- and finite-range percolation in high dimensions

Remco van der Hofstad

TU/e technische universiteit eindhoven

EURANDOM

Above the critical dimension, IHP, December 7-11, 2009

Joint work with:
- Markus Heydenreich (Amsterdam)
- Tim Hulshof (TU/e)
- Akira Sakai (Hokkaido)
- Antal Járai (Bath)

Percolation on $\mathbb{Z}^d$

Bonds join x to y for $x, y \in \mathbb{Z}^d$. Make bonds (x, y) independently

occupied	with probability $pD(y-x)$,
vacant	with probability $1 - pD(y-x)$,

where $p \in [0, 1/\|D\|_\infty]$ is percolation parameter.

Key examples:

- nearest-neighbor percolation: $D(x) = \mathbb{1}_{\{|x|=1\}}/(2d)$;
- spread-out percolation: $D(x) = \mathbb{1}_{\{0<\|x\|_\infty \leq L\}}/[(2L+1)^d - 1]$.

Special attention to spread-out long-range percolation:

$$D(x) = \frac{1}{Z_L}\Big(\frac{|x|}{L} + 1\Big)^{-(d+\alpha)},$$

where $\alpha > 0$ and Z_L is normalizing constant.

Phase transition

Percolation has a phase transition, i.e, there is a critical probability $p_c = p_c(d, L) \in (0, \infty)$, such that

- For $p < p_c$, a.s. no infinite cluster exists.
- For $p > p_c$, a.s. a unique infinite cluster.
- For $p = p_c$, behavior not understood and dimension dependent.

No percolation at criticality for $d = 2$,
and for nn $d \geq 19$ and spread-out model with $d > 6$ (Hara-Slade 90).

Berger (02): For long-range percolation and $\alpha \in (0, d)$, there is no percolation at criticality for $d \geq 2$.

Robust infrared bound

Define two-point function

$$\tau_p(x) = \mathbb{P}_p(0 \longleftrightarrow x),$$

let $\hat{\tau}_p(k)$ be its Fourier transform.

Theorem 1. (Percolation infrared bound, HS90+BCHSS05b+HvdHS08.) For $d \gg 6$ in the nearest-neighbor case, and $d > 3(\alpha \wedge 2)$ and L sufficiently large in spread-out (long- or finite range) case,

$$\hat{\tau}_p(k) = \frac{1 + O(\beta)}{p_c - p + p[1 - \hat{D}(k)]}$$

uniformly for $p \in (p_c/2, p_c]$, where $\beta = 1/d$ or $\beta = L^{-d}$, respectively, and $\hat{D}(k)$ is Fourier transform of D.

Results true under general and simple random walk condition.

Large critical clusters

Central question:

What is structure of large critical clusters?

Here we can think of

- Dimension of large clusters;
- Local structure of large clusters.

Go by name of incipient infinite cluster (IIC), which is

infinite cluster that is on verge of appearing at criticality.

2d-Incipient Infinite Cluster

Kesten (1986) has constructed IIC for percolation on $\mathbb{Z}^2$.
IIC describes local structure of large critical clusters.

Constructions Kesten:
(a) Condition 0 to be in infinite component for $p > p_c$, and take limit as $p \downarrow 0$.
(b) Condition on $0 \longleftrightarrow \partial B_n$ at $p = p_c$, and take limit as $n \to \infty$.

For events E, define

$$\mathbb{P}_\infty(E) = \lim_{p \downarrow p_c} \mathbb{P}_p(E | 0 \longleftrightarrow \infty).$$

Similar for Construction (b). RSW theory plays an important role.

Járai (03, 04) gives several more constructions for IIC: robust object!

Definition long-range IIC

For cylinder events E, define

$$\mathbb{P}_\infty(E) = \lim_{p \uparrow p_c} \frac{1}{\chi(p)} \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(E \cap \{0 \longleftrightarrow x\}),$$

where $\chi(p) = \sum_x \mathbb{P}_p(0 \longleftrightarrow x) = \mathbb{E}_p|\mathcal{C}(0)|$ is expected cluster size.

Theorem 2. (Heydenreich-vdH-Hulshof) Under conditions Theorem 1, above limit exists for every cylinder event E. Moreover, $\mathbb{P}_\infty$ extends to a probability measure on full sigma-algebra of events, and $\mathbb{P}_\infty(|\mathcal{C}(0)| = \infty) = 1$.

Properties IIC

Let $Q_R = \{y\colon |y| \leq R\}$ be Euclidean ball of radius R.

Theorem 3. (Heydenreich-vdH-Hulshof) Under conditions Theorem 1,
(i) there are positive constants $c_1 = c_1(d, L)$ and $c_2 = c_2(d, L)$ such that for $R \geq 1$

$$c_1 R^{2(2\wedge\alpha)} \leq \mathbb{E}_\infty[|\mathcal{C}(0) \cap Q_R|] \leq c_2 R^{2(2\wedge\alpha)}.$$

(ii) We say that y is in backbone of IIC when $0 \longleftrightarrow y$ and $y \longleftrightarrow \infty$ occur disjointly. Denote IIC backbone by $\mathcal{B}(0)$.

Then, there are positive constants $c_3 = c_3(d, L)$ and $c_4 = c_4(d, L)$ such that for $R \geq 1$

$$c_3 R^{(2\wedge\alpha)} \leq \mathbb{E}_\infty[|\mathcal{B}(0) \cap Q_R|] \leq c_4 R^{(2\wedge\alpha)}.$$

(Indicates that IIC backbone is $(2 \wedge \alpha)$ and IIC $2(2 \wedge \alpha)$ dimensional).

Alternative definition IIC

For cylinder events E, define

$$\mathbb{Q}_\infty(E) = \lim_{|x|\to\infty} \mathbb{P}_{p_c}(E | 0 \longleftrightarrow x).$$

Theorem 4. **(vdH-Járai (04))** For spread-out finite-range percolation with L sufficiently large and $d > 6$, or nearest-neighbour percolation for d sufficiently large, the above limit exists for every cylinder event E, and $\mathbb{Q}_\infty = \mathbb{P}_\infty$.

Results make essential use of asymptotics critical two-point function (HHS (03), Hara (08))

$$\tau(x) = \mathbb{P}_{p_c}(0 \longleftrightarrow x) \sim |x|^{-(d-2)}.$$

Recent work Sakai + Chen for long-range percolation.
Asymptotics hard to prove in general, unnecessary for construction IIC.

IIC is robust and natural object!

Proof existence IIC

Proof relies on various forms of lace expansion for two-point function $\tau_p(x) = \mathbb{P}_p(0 \longleftrightarrow x)$.

Lace expansion can be used to show that when E only depends on finitely many bonds,

$$\mathbb{P}_p(E \cap \{0 \longleftrightarrow x\}) = \psi_p(E; x) + \sum_y \pi_p(E; y)\tau_p(x - y).$$

Sum out over x and divide through by $\chi(p) = \mathbb{E}_p|\mathcal{C}(0)| = \sum_x \tau_p(x)$ to arrive at

$$\frac{1}{\chi(p)} \sum_x \mathbb{P}_p(E \cap \{0 \longleftrightarrow x\}) = \sum_y \pi_p(E; y) + \frac{1}{\chi(p)} \sum_x \psi_p(E; x).$$

Proof existence IIC (Cont.)

Recall

$$\frac{1}{\chi(p)}\sum_x \mathbb{P}_p(E\cap\{0\longleftrightarrow x\}) = \sum_y \pi_p(E;y) + \frac{1}{\chi(p)}\sum_x \psi_p(E;x).$$

Lace expansion coefficients $\sum_y \psi_p(E;y), \sum_y \pi_p(E;y)$ are bounded uniformly for $p<p_c$ and $\chi(p)\to\infty$ as $p\uparrow p_c$, so that

$$\mathbb{P}_\infty(E) = \lim_{p\uparrow p_c}\frac{1}{\chi(p)}\sum_x \mathbb{P}_p(E\cap\{0\longleftrightarrow x\}) = \sum_y \pi_{p_c}(E;y).$$

Relatively explicit formula for IIC allows to prove equalities for different constructions.

Random walks on high-dimensional IIC

Let $(S_n)_{n\geq 0}$ be simple random walk on IIC, and let

$$p_n(x,y) = \mathbb{P}^x(S_n = y)$$

be probability that random walk started at x is at y at time n.

Spectral dimension:

$$d_s(\mathrm{IIC}) = -2 \lim_{n\to\infty} \frac{\log p_{2n}(x,x)}{\log n}.$$

Volume-growth dimension:

$$d_f(\mathrm{IIC}) = \lim_{r\to\infty} \frac{\log |B(x,r)|}{\log r},$$

where $B(x,r)$ consists of vertices in IIC at graph distance at most r.

Random walks on IIC on tree

IIC on tree has been constructed by Kesten 86.

Consists of
- unique infinite line of descent (immortal particle);
- critical clusters attached at every vertex on infinite line.

Kesten proved

$$d_s = 4/3, \qquad d_f = 2.$$

"Random walk trap model."
Strongest results: Barlow + Kumagai (06).

IIC is not full dimensional, i.e., expect $d_s(\mathrm{IIC}), d_f(\mathrm{IIC}) < d$:

anomalous diffusion.

Random walks on high-dimensional IIC

Theorem 5. (Kozma-Nachmias 09) Fix $d > 6$ and sufficiently spread-out model, or $d \geq 19$ and nearest-neighbor model. Then,

$$d_s(\mathrm{IIC}) = 4/3, \qquad d_f(\mathrm{IIC}) = 2,$$

and, with τ_r hitting-time of ball $B_{\mathrm{IIC}}(0, r)$ and W_n range of random walk, i.e., number of distinct vertices visited at time n, and in probability,

$$\lim_{r\to\infty} \frac{\log E^0[\tau_r]}{\log r} = 3, \qquad \lim_{n\to\infty} \frac{\log |W_n|}{\log n} = 2/3,$$

where E^0 denotes conditional law of RW on IIC.

RW on high-dimensional long-range IIC

Theorem 6. (Heydenreich-vdH-Hulshof) Fix $d > 3(\alpha \wedge 2)$ for the sufficiently spread-out model. Then, results in Theorem 5 again hold:

$$d_s(\mathrm{IIC}) = 4/3, \qquad d_f(\mathrm{IIC}) = 2,$$

and

$$\lim_{r\to\infty} \frac{\log E^0[\tau_r]}{\log r} = 3, \qquad \lim_{n\to\infty} \frac{\log |W_n|}{\log n} = 2/3.$$

Proof follows that of Nachmias + Kozma (09), adapting for
(a) different construction IIC;
(b) different control various quantities due to more general percolation model.

RW on high-dimensional long-range IIC

In previous result, distinction between finite- and long-range percolation models invisible in results.

To see distinction, we need to investigate geometric quantities:

exit times from Euclidean ball Q_R.

Let τ_{Q_R} be exit time of Euclidean ball of radius R. Then, uniformly in $R \geq 1$,

$$\mathbb{P}_{\mathrm{IIC}}\big(E^0(\tau_{Q_R}) \geq \varepsilon^{-1} R^{3(2\wedge\alpha)}\big)$$

is small when ε is small.

Believe this result to be sharp. Working on lower bound: uniformly in $R \geq 1$,

$$\mathbb{P}_{\mathrm{IIC}}\big(E^0(\tau_{Q_R}) \leq \varepsilon R^{3(2\wedge\alpha)}\big)$$

is small when ε is small.

Proof: first main ingredient

• Robust theorems implying Theorem 5, 6, by Barlow, Járai, Kumagai, Slade (08) and Kumagai, Misumi (08), assuming appropriate bounds on effective resistances and volume growth.

BJKS (08) results used to study
(a) random walk on oriented percolation IIC above 6 dimensions;
(b) random walk on IIC for percolation on tree;
(c) random walk on invasion percolation cluster on tree.
Flexible result, Taylor made for applications in various settings.

BJKS (08): Intrinsic or graph distances;
KM (08): flexible in metric, Euclidean distance is example.

Proof: second main ingredient

• Verification of conditions BJKS (08) in KN (09) for properties under graph distance:
Proof of $d_f(\mathrm{IIC}) = 2$ and effective resistance bounds.
These bounds are reduced to two main bounds for critical percolation:

$$\mathbb{E}_{p_c}[|B_{p_c}(0,r)|] \asymp r, \qquad \mathbb{P}_{p_c}(\partial B_{p_c}(0,r) \neq \varnothing) \asymp 1/r,$$

alike on tree. Second bound: $\rho_{\mathrm{Int}} = 1$.

All these results can easily be adapted to long-range setting, using results in Heydenreich, vdH, Sakai (08).

• Verification of conditions KM (08) for Euclidean distance:
Upper bounds work, lower bounds under construction.

A consistent picture

Reasonable to assume that there exists $a > 1$ such that

$$E^0[d_{\mathrm{IIC}}(0, S_n)] \asymp n^{1/a}.$$

Then,

$$p_{2n}(0,0) \asymp 1/|B_{\mathrm{IIC}}(0, n^{1/a})|, \quad E^0[\tau_r] \asymp r^a, \quad E^0[|W_n|] \asymp |B_{\mathrm{IIC}}(0, n^{1/a})|.$$

Assumption implies

$$\mathbb{E}_{p_c}[|B_{p_c}(0,r)| \mid \partial B_{p_c}(0,r) \neq \varnothing] \approx \mathbb{E}_{\mathrm{IIC}}[|B_{\mathrm{IIC}}(0,r)|] \asymp r^2.$$

Thus,

$$d_s(\mathrm{IIC}) = 4/a, \qquad \lim_{r\to\infty} \frac{\log E^0[\tau_r]}{\log r} = a, \qquad \lim_{n\to\infty} \frac{\log |W_n|}{\log n} = 2/a.$$

Remains to determine a.

Effective resistance

Proof make essential use of relation

random walks and electrical networks.

Define quadratic form

$$\mathcal{E}(f,g) = \frac{1}{2}\sum_{b\in\mathbb{B}}(f(\bar{b}) - f(\underline{b}))(g(\bar{b}) - g(\underline{b})).$$

Then, for $A, B \subseteq \mathbb{Z}^d$, effective resistance between A and B is

$$R_{\rm eff}(A,B)^{-1} = \inf\{\mathcal{E}(f,f)\colon\ f|_A = 1, f|_B = 0\}.$$

$R_{\rm eff}(A,B)$ satisfies series and parallel laws in electricity. Implies, on any graph G,

$$R_{\rm eff}(A,B) \leq d_G(A,B).$$

Equality when there is unique path between A and B.

Assumptions BJKS (08) and KM (08)

Let $J(\lambda)$ be set of $r \geq 1$ such that

$$\lambda^{-1} r^2 \leq |B_{\text{IIC}}(0,r)| \leq \lambda r^2, \qquad \lambda^{-1} r \leq R_{\text{eff}}(0, B_{\text{IIC}}(0,r)^c) \leq \lambda r.$$

• Assumption BJKS (08): There exists $r^* \geq 1$ and $c_1, q_0 > 0$ s.t.

$$\mathbb{P}_G(r \in J(\lambda)) \geq 1 - c_1 \lambda^{-q_0}.$$

Implies that volume balls grow as radius squared, and effective resistance grows as radius, like it does on tree.

• Assumption KM (08): Similar, but for balls in different metrics, and (possibly) different volume growth and effective resistance growth. Correct scaling:

$$|\text{IIC} \cap Q_R| \sim R^{2(2\wedge\alpha)}, \qquad R_{\text{eff}}(0, (\text{IIC} \cap Q_R)^c) \sim \lambda R^{2\wedge\alpha}.$$

Effective resistance

Estimate

$$E^0[\tau_r] \asymp R_{\text{eff}}(0, B_{\text{IIC}}(0,r)^c)|B_{\text{IIC}}(0,r)| \asymp r \cdot r^2 = r^3,$$

so that $a = 3$.

- Upper bound always valid;
- Lower bound when path between 0 and $B_{\text{IIC}}(0,r)^c$ is essentially unique.

Implies results Theorems 5-6, subject to assumptions

- volume growth critical balls; and
- intrinsic one-arm exponent.

Questions

• Can IIC also be constructed in $p \downarrow p$ limit that Kesten (86) uses?
Problem: Weaker control over super-critical percolation.

• Can IIC also be constructed by conditioning on $0 \longleftrightarrow \partial Q_R$ and taking $R \to \infty$ and using recent one-arm result of Kozma+Nachmias (09)?

• Can IIC also be constructed using invasion percolation?
Scaling $|y|^{-(d-4)}$ conjectured for invasion percolation two-point function, important open problem.

• What is scaling limit of IIC?

Literature

[1] Barlow, Járai, Kumagai, and Slade. Random walk on the incipient infinite cluster for oriented percolation in high dimensions. *Comm. Math. Phys.*, **278**(2):385–431, (2008).

[2] Heydenreich, van der Hofstad, and Sakai. Mean-field behavior for long- and finite range Ising model, percolation and self-avoiding walk. *Journ. of Stat. Phys.*, **132**: 1001–1049, (2008).

[3] van der Hofstad and Járai. The incipient infinite cluster for high-dimensional unoriented percolation. *J. Statist. Phys.*, **114**(3-4):625–663, (2004).

[4] Kozma and Nachmias. The Alexander-Orbach conjecture holds in high dimensions. *Inventiones Math.*, **178**(3):635–654, (2009).