Giant vacant component left by a random walk in a random d-regular graph

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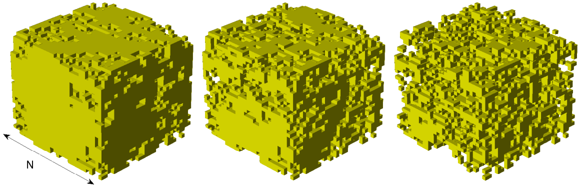
Random walk on large graphs

- Let $G_n = (V_n, \mathcal{E}_n)$ be a sequence of graphs with $|V_n| = n$.
- Consider a random walk $(X_i)_{i \geq 0}$ starting uniformly on V_n.

Corrosion by a random walk trajectory (H.J. Hilhorst)



- Fix a real parameter $u \geq 0$.
- Consider the vacant set left by the random walk up to un, i.e.

$$\mathcal{V}_n^u = V_n \setminus \{X_1, \dots, X_{\lfloor un \rfloor}\}$$

- Let $\mathcal{C}_{\max}$ be the largest component of $\mathcal{V}_n^u$.
- How does $|\mathcal{C}_{\max}|$ behave for large n? The behavior depends on u?

The sequence of graphs we consider

Assumptions on G_n:

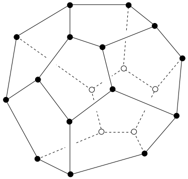
$$G_n \text{ is } d\text{-regular.} \tag{A0}$$

$$\exists \alpha > 0 \text{ such that, } \forall x, n,\ B(x, \alpha \log(n)) \text{ has at most one cycle.} \tag{A1}$$

$$\text{The spectral gap } \lambda_{G_n} \geq \beta > 0. \tag{A2}$$

Examples of such graphs

- Random d-regular graphs (quenched results).
- d-regular, large-girth expanders (eg. Lubotzky-Phillips-Sarnak graphs)



Why these graphs?

- Finite approximations of trees.
- Increasing interest among physicists and computer scientists.
- Could serve as a 'mean field model' for the corresponding problem in the discrete torus.

Consequences of (A0)-(A2)

- Most points have tree-like neighborhood:

$$\#\left\{x; B\left(x, \frac{\alpha}{2}\log(n)\right) \text{ is a tree}\right\} \sim n.$$

- Bounded Cheeger's constant

$$\inf\left\{\frac{|\partial A|}{|A|}; n \geq 1, A \subset V_n, |A| \leq n/2\right\} > 0.$$

Main result

- Recall that $\mathcal{V}_n^u = V_n \setminus \{X_1, \ldots, X_{\lfloor un \rfloor}\}$.
- $\mathcal{C}_{\max}, \mathcal{C}_{\text{sec}}$ denote the largest and second largest clusters of $\mathcal{V}_n^u$.
- Let $u_* = d(d-1)\log(d-1)/(d-2)^2$.

Theorem (Č.T.W. 09)

Under (A0), (A1) and (A2),

- (sub-critical phase) If $u > u_*$, there is $\kappa(u, d, \alpha, \beta)$ such that,

$$P\left[|\mathcal{C}_{\max}| \geq \kappa \log(n)\right] \to 0. \qquad \text{(fast)}$$

- (super-critical phase) If $u < u_*$, there is $\rho(u, d, \alpha, \beta)$ such that,

$$P\left[|\mathcal{C}_{\max}| \geq \rho n\right] \to 1. \qquad \text{(fast)}$$

- (uniqueness) If $u < u_*$, for every $\varepsilon > 0$,

$$P\left[|\mathcal{C}_{\text{sec}}| \geq \varepsilon n\right] \to 0.$$

Remarks

- A similar phase transition is observed in Bernoulli percolation.
- The critical value in the Bernoulli case is $p_c = 1/(d-1)$ (the same as for the infinite tree).
- Recalling that **typical** points have tree-like neighborhoods: ‘the critical value is local’ for Bernoulli percolation.

Where did our critical value u_* come from?

Random interlacements

A dependent percolation process in a given lattice.

Theorem (Sznitman 07, T. 09)

For fixed $u \geq 0$, there exists a unique measure Q^u giving a random subset $\mathcal{V}^u$ of $\mathbb{T}^d$ such that,

$$Q^u[\mathcal{V}^u \supset K] = \exp\{-u \cdot \mathrm{cap}(K)\} \text{ for all finite } K \subset \mathbb{T}^d.$$

Where $\mathrm{cap}(K)$ is the capacity of a set $K \subset \mathbb{T}^d$.

Theorem (Teixeira 09)

Under the law Q^u,

$$\begin{array}{c}\text{the set } \mathcal{V}^u \text{ has almost surely} \\ \text{infinite connected components}\end{array} \Leftrightarrow u > u_*.$$

Where u_* is the value appearing in our main result.
This reinforces the local character of the critical value because...

Local Picture

- Let $x \in V_n$ be such that $B(x, \alpha/2 \log(n))$ is isomorphic to a tree (majority of sites).
- Define $\tilde{\mathcal{C}}_x^u$ to be the cluster of $B \cap \mathcal{V}_n^u$ containing x. Where $B = B(x, \alpha/100 \log(n))$.

Lemma

Assume (A0)-(A2) and take x, B as above. Then we can couple $Q^{u(1+\varepsilon)}, Q^{u(1-\varepsilon)}$ and P in a way that (up to isomorphisms):

$$\mathbb{P}\left[\mathrm{C}_{u(1+\varepsilon)} \subset \tilde{\mathcal{C}}_x^u \subset \mathrm{C}_{u(1-\varepsilon)}\right] \geq 1 - c_{u,\varepsilon} n^{-\alpha/100}.$$

Here C_u stands for $\mathcal{V}^u \cap B(0, \alpha/100 \log(n))$.

- Local picture relates to critical value.

The corresponding question for the torus

Let G_n be the d-dimensional discrete torus $(\mathbb{Z}/n\mathbb{Z})^d$.
One can similarly define the vacant set left by a random walk $\mathcal{V}^u_{n^d}$.

Theorem (Windisch 08)

The local picture of $\mathcal{V}^u_{n^d}$ converges to random interlacements on $\mathbb{Z}^d$.

Theorem (Sznitman 07, Sidoravicius-Sznitman 08)

There is a critical value $u_* \in (0, \infty)$ for the existence of an infinite cluster in the vacant set $\mathcal{V}^u$ of random interlacements on $\mathbb{Z}^d$.

Does $\mathcal{V}^u_{n^d}$ (the vacant set of the torus) undergo a phase transition at the same value u_* appearing in the vacant set $\mathcal{V}^u$ of $\mathbb{Z}^d$?

Main obstructions in the proof

Sub-critical

- Convergence of local picture $\Rightarrow |\mathcal{C}_{\max}|$ is $o(n)$ for $u > u^*$.
- If we want $|\mathcal{C}_{\max}| \lesssim \kappa_u \log(n)$, we should be aware that $\kappa_u \to \infty$ as $u \downarrow u^*$.
- We have to exit the 'local picture ball', since $|\mathcal{C}_{\max}| \gg \operatorname{diam}(G_n)$.

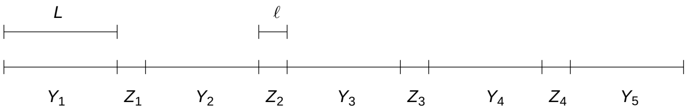
Super-critical

- Local picture $\Rightarrow$ For $u < u_*$, of order n points in V_n^u belong to 'intermediate components' (size of order n^δ).
- Usual way to join these components is using 'sprinkling'.
- It is not clear how to perform sprinkling in this context, since $\{X_{un}, X_{un+1}, \dots, X_{(u+\varepsilon)n}\}$ is highly dependent on the 'intermediate components'.

Piecewise independent measure

We need to extract independence from the random walk trajectory. Compare the law P with Q defined as follows:

1. Consider i.i.d r.w. $(Y_i)_{i \geq 1}$ of length $L = n^\gamma$.
2. Denote by a_i and b_i the start and end points of Y_i.
3. Let the Z_i's be random walk bridges from b_i to a_{i+1} with length $\ell = \log^2 n$.



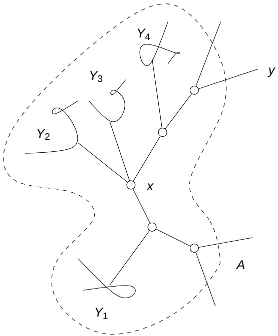
Lemma

The laws Q and P up to time un are very close in total variation.

The proof uses that the mixing time $\ll \log^2(n)$.

Sub-critical regime: The exploration process

- Breath first search algorithm.
- Explore the vacant component of x.
- Once we meet a segment Y_i,
  - call this segment ‘tied’
    (the non-tied are called ‘free’),
  - continue on other branch.



- We have a pool of ‘free’ segments.
- The probability that y intersects a ‘free’ segment is

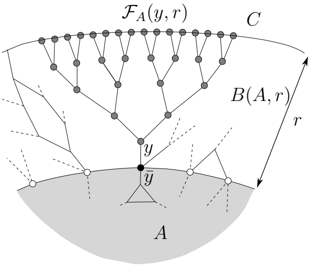
$$P\left[H(A \cup \{y\}) < L \middle| H(A) \geq L\right],$$

where A is the explored set up to the current time.

Bounding the conditional hitting

One needs the following conditions:

- $|A| < K \log(n)$.
- There are no cycles in the close future of y (seen from A).
- Only one neighbor of y in A.
- The close future of y does not meet A (no cycles to A).



Proposition

Under these conditions we can prove that

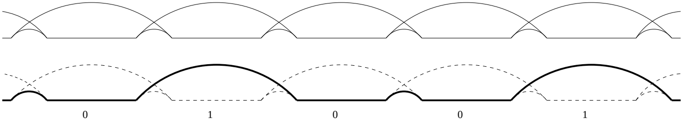
$$\Big(P\big[H(A \cup \{y\}) > L \big| H(A) \geq L\big]\Big)^{un^{1-\gamma}}$$ gives a sub-critical branching.

These conditions hold in all but $(c \log\log n)$ steps of the algorithm.

Super-critical regime: The sprinkling

We introduce the so-called 'long-range bridges'.

- Let (Y_i) be i.i.d. random walks of length L $(= n^\gamma)$.
- Denote by a_i and b_i the start and end points of Y_i.
- Connect all b_i with a_{i+j} (for $j \leq \log(n)$), with a bridge of length $\ell = \log^2(n)$.



- Kill some of the segments Y_i independently with probability $n^{-\gamma}$.
- Extract a random walk path in the remaining set (it will have law close to P).

Merging 'intermediate components'

- Consider two sets A and B with volume at least cn, obtained by collecting components of size n^δ.

$$\#\{\text{choices for } A \text{ and } B\} \leq 2^{n^{1-\delta}}$$

- By the isoperimetric inequality, there are of order n links from A to B in G_n.
- After the sprinkling, several of these links will become vacant and A and B will be joined.

$$\text{The probability that this fails} \leq c \exp\{-c' n^{1-c\gamma}\}$$

- Choose γ small.
- If all choices of A and B get joined in the end, we obtain a giant component.

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Thanks!