The loop-erased random walk and the uniform spanning tree on the four-dimensional discrete torus

by Jason Schweinsberg

University of California at San Diego

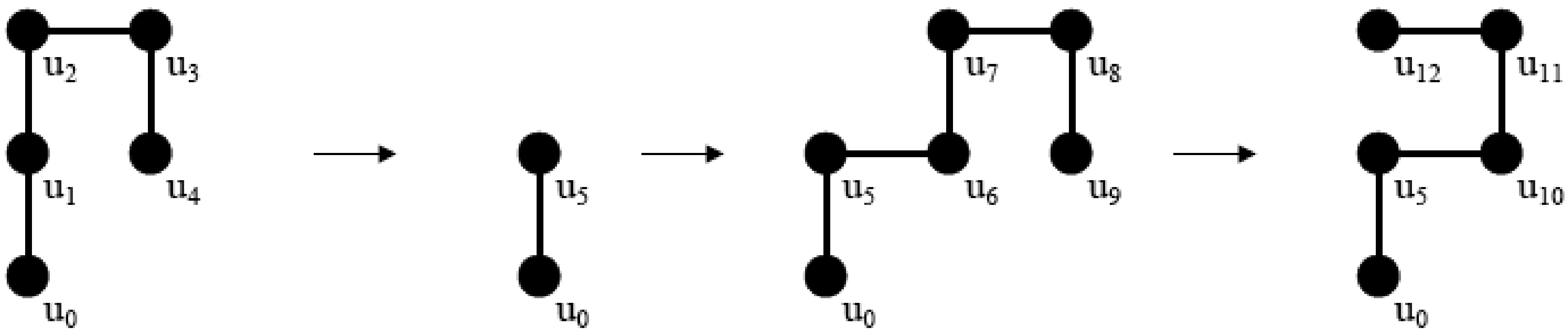
Outline of Talk

1. Loop-erased random walk
2. Uniform spanning tree
3. Continuum random tree
4. Main result
5. Outline of proof
6. Dynamics of loop-erased random walk

Loop-erased random walk

Let $\lambda = (u_0, u_1, \ldots, u_j)$ be a path in a graph $G = (V, E)$.

Define the loop-erasure $LE(\lambda)$ by erasing loops in the order in which they appear.



A random walk $(X_t)_{t=0}^{\infty}$ on a graph G is a V-valued Markov chain. At each step, moves to a randomly chosen neighboring vertex.

The loop-erased random walk (LERW) is the path $LE((X_t)_{t=0}^{\infty})$.

Well-defined when $(X_t)_{t=0}^{\infty}$ is transient.

Behavior of LERW on $\mathbb{Z}^d$:

- $d \geq 5$ (Lawler, 1980): All loops are short, length of the path $LE((X_t)_{t=0}^n)$ is $O(n)$, process converges to Brownian motion.

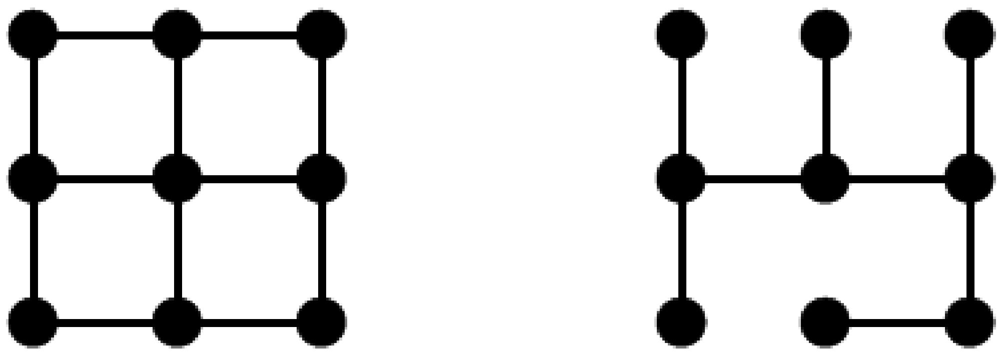
- $d = 4$ (Lawler, 1986, 1995): Length of the path $LE((X_t)_{t=0}^n)$ is $O(n/(\log n)^{1/3})$, process converges to Brownian motion.

- $d = 2$ (Kenyon, 2000; Lawler-Schramm-Werner, 2004): The length of $LE((X_t)_{t=0}^n)$ is $O(n^{5/8})$, process is conformally invariant and converges to SLE(2).

- $d = 3$ (Kozma, 2005): Scaling limit exists, invariant under dilations and rotations.

Uniform spanning tree

A spanning tree of a finite connected graph G is a connected subgraph of G containing every vertex and no cycles.



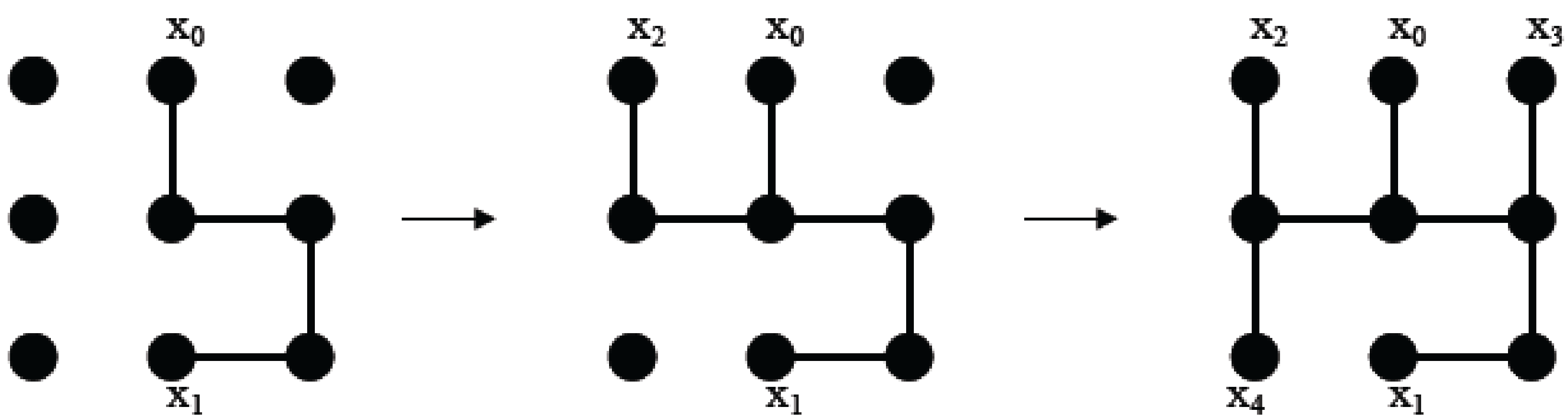
A uniform spanning tree (UST) is a spanning tree chosen uniformly at random.

Connections between UST and LERW

Theorem (Pemantle, 1991): The path from x to y in a UST has the same distribution as the LERW from x to y.

Wilson's Algorithm (Wilson, 1996): To construct UST of G,

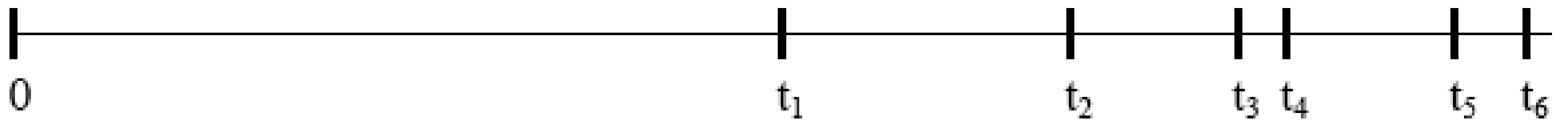
- Pick vertices x_0 and x_1, run LERW from x_0 to x_1 to get $\mathcal{T}_1$.
- Given $\mathcal{T}_k$, pick a vertex x_{k+1}, get $\mathcal{T}_{k+1}$ by adjoining to $\mathcal{T}_k$ an LERW from x_{k+1} to $\mathcal{T}_k$.
- Continue until all vertices are in the tree.



Can choose vertices in any order, can depend on current tree.

Continuum Random Tree (Aldous, 1991, 1993)

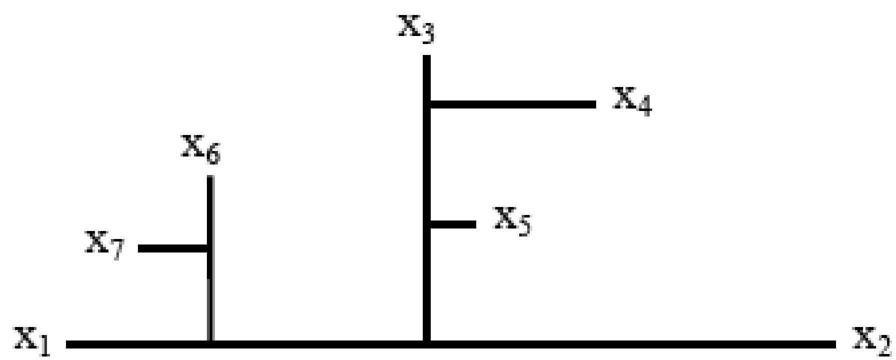
Consider Poisson process on $[0, \infty)$, intensity $r(t) = t$.



Begin with a segment of length t_1, call the endpoints x_1, x_2.

Attach segment of length $t_2 - t_1$ to a uniform point on the initial segment, label the endpoint x_3.

Continue, each segment orthogonal to all previous segments.



Limiting random metric space is continuum random tree (CRT).

Denote by μ_k the (exchangeable) distribution of $(d(x_i, x_j))_{1 \le i < j \le k}$.

View $x_1, \ldots, x_k$ as points picked from "mass measure" on CRT.

Scaling limits of UST on finite graphs

Theorem (Aldous, 1991): Consider the UST on the complete graph K_m with m vertices (equivalently, a uniform random tree on m labeled vertices). Let $y_1, \ldots, y_k$ be vertices chosen uniformly a random. Let $d(y_i, y_j)$ be the number of vertices on the path from y_i to y_j in the UST. Then

$$\left(\frac{d(y_i, y_j)}{\sqrt{m}}\right)_{1 \leq i < j \leq k} \to_d \mu_k.$$

Theorem (Peres-Revelle, 2004): Consider the UST on the torus $\mathbb{Z}_n^d$ for $d \geq 5$. Let $x_1, \ldots, x_k$ be vertices chosen uniformly at random. Then there is a constant β such that

$$\left(\frac{d(x_i, x_j)}{\beta n^{d/2}}\right)_{1 \leq i < j \leq k} \to_d \mu_k.$$

The CRT scaling limit also holds for UST on larger class of graphs including hypercubes $\mathbb{Z}_2^n$, expander graphs.

Corollary (Peres-Revelle, 2004): Let x and y be uniformly chosen from $\mathbb{Z}_n^d$, $d \geq 5$. Let $(X_t)_{t=0}^T$ be a random walk from x to y. Then

$$\lim_{n\to\infty} P(|LE((X_t)_{t=0}^T)| > \beta n^{d/2} z) = e^{-z^2/2}.$$

Limiting distribution called Rayleigh distribution.

Note: Benjamini-Kozma (2005) had proved that for $d \geq 5$ the length of LERW from x to y on $\mathbb{Z}_n^d$ is $O(n^{d/2})$. They conjectured the length in $\mathbb{Z}_n^4$ is $O(n^2(\log n)^{1/6})$.

Theorem (Schweinsberg, 2009): Consider the UST on $\mathbb{Z}_n^4$. Let $x_1, \ldots, x_k$ be vertices chosen uniformly at random. There is a sequence of constants $(\gamma_n)_{n=1}^\infty$ bounded away from 0 and ∞ such that

$$\left(\frac{d(x_i, x_j)}{\gamma_n n^2 (\log n)^{1/6}}\right)_{1\leq i<j\leq k} \to_d \mu_k.$$

In particular, if $(X_t)_{t=0}^T$ is a random walk from x_1 to x_2, then

$$\lim_{n\to\infty} P(|LE((X_t)_{t=0}^T)| > \gamma_n n^2 (\log n)^{1/6} z) = e^{-z^2/2}.$$

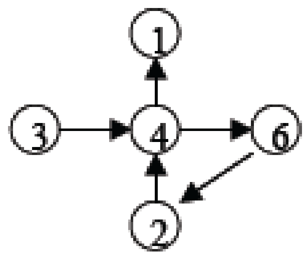
Coupling idea (Peres-Revelle, 2004)

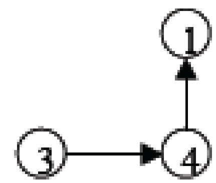
- Choose $y_1, \ldots, y_k$ uniformly from K_m.
- Choose $x_1, \ldots, x_k$ uniformly from $\mathbb{Z}_n^4$.
- Construct partial UST $\tilde{\mathcal{T}}_k$ on K_m using Wilson's algorithm, starting random walks from $y_1, \ldots, y_k$.
- Construct partial UST $\mathcal{T}_k$ on $\mathbb{Z}_n^4$ using Wilson's algorithm, starting random walks from $x_1, \ldots, x_k$.
- Segments in random walks of length $r = \lfloor n^2 (\log n)^{9/22} \rfloor$ on $\mathbb{Z}_n^4$ correspond to individual vertices in walks on K_m.
- Obtain tree $\mathcal{T}_k^*$ from $\mathcal{T}_k$ by collapsing random walk segments of length r into a single vertex.
- Find coupling such that $\tilde{\mathcal{T}}_k = \mathcal{T}_k^*$ with high probability.
- Deduce CRT limit for UST on $\mathbb{Z}_n^4$ from Aldous' result on K_m.

Coupling of random walks

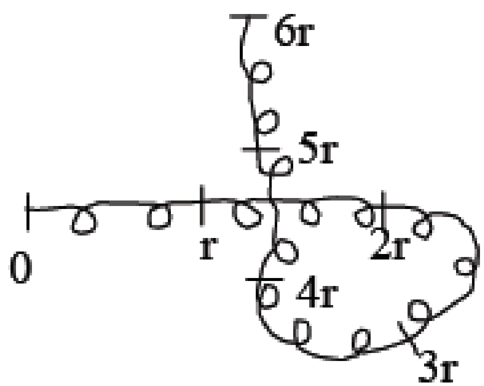
Random walk on $\mathbb{Z}_n^4$ makes short loops (occur within segment of length r) and long loops (occur on $\mathbb{Z}_n^4$ but not on $\mathbb{Z}^4$). Long loops correspond to loops of walk on K_m.

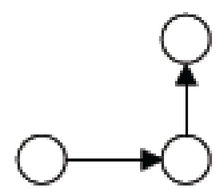
Example: random walk on K_6 begins $(3, 4, 6, 2, 4, 1)$.





Random walk $(X_t)_{t=0}^{6r-1}$ on $\mathbb{Z}_n^4$. Suppose that $X_s = X_t$ for some $s \in [r, 2r)$ and $t \in [4r, 5r)$.





Coupling k random walks gives coupling of $\tilde{\mathcal{T}}_k$ and $\mathcal{T}_k^*$.

Mixing times

Let $(X_t)_{t=0}^{\infty}$ be a random walk on $\mathbb{Z}_n^4$, modified to stay in its current state with probability 1/2. Assume $X_0 = 0$.

Markov chain with stationary distribution $\pi(x) = 1/n^4$ for all x.

$$\tau_n = \inf\left\{t : \max_{x\in\mathbb{Z}_n^4}\left|P(X_t = x) - \frac{1}{n^4}\right| \leq \frac{1}{2n^4}\right\}.$$

We have $C_1 n^2 \leq \tau_n \leq C_2 n^2$.

For $t \geq \tau_n$ and $x \in \mathbb{Z}_n^4$, we have exponentially fast convergence:

$$\left|P(X_t = x) - \frac{1}{n^4}\right| \leq \frac{2^{-\lfloor t/\tau_n \rfloor}}{n^4}.$$

A random walk $(X_t)_{t=0}^{\infty}$ gets close to the uniform distribution when t is $O(n^2)$, near the beginning of a segment of length r.

To find probability that two segments of length r intersect, assume they both start from uniform distribution.

Random walk intersections on $\mathbb{Z}^d$

Let $(V_t)_{t=0}^{\infty}$ and $(W_t)_{t=0}^{\infty}$ be independent random walks on $\mathbb{Z}^d$, started at the origin.

- If $d \leq 4$, almost surely the paths intersect infinitely often (infinitely many $x \in \mathbb{Z}^d$ such that $V_s = W_t = x$ for some s, t).
- If $d \geq 5$, almost surely the paths intersect only finitely often.

Let R_n be the cardinality of $\{s, t \in \{1, \ldots, n\} : V_s = W_t\}$.

- If $d \geq 5$, then $E[R_n] = O(1)$.
- If $d = 4$, then $E[R_n] = O(\log n)$.
- If $d \leq 3$, then $E[R_n] = O(n^{(4-d)/2})$.

Intersection probabilities for loop-erased segments

Let $(X_t)_{t=0}^{r-1}$ and $(Y_t)_{t=0}^{r-1}$ be random walks on $\mathbb{Z}_n^4$ started from the uniform distribution.

- $P(X_s = Y_t) = 1/n^4$ for all s, t.
- Expected number of intersections is r^2/n^4.
- If there is one intersection, there are $O(\log r)$ intersections.
- Probability of intersection is $O\left(\frac{r^2}{n^4 \log r}\right) = O((\log n)^{-2/11})$.

(Lyons-Peres-Schramm, 2003): Given two independent transient Markov chains with the same transition probabilities, loop-erasing one path reduces the probability that the paths intersect by at most a factor of 2^8.

Probability $LE((X_t)_{t=0}^{r-1})$ and $(Y_t)_{t=0}^{r-1}$ intersect is $O((\log n)^{-2/11})$.

Concentration of capacity

For $U \subset \mathbb{Z}_n^4$, let $\mathrm{Cap}_r(U) = P(Y_t \in U \text{ for some } t < r)$.

We have $E[\mathrm{Cap}_r(LE((X_t)_{t=0}^{r-1}))] = a_n(\log n)^{-2/11}$ for a sequence of constants $(a_n)_{n=1}^{\infty}$ bounded away from 0 and ∞.

Distribution of $\mathrm{Cap}_r(LE((X_t)_{t=0}^{r-1}))$ is highly concentrated around its mean (break walk into pieces, apply LLN to the probabilities of hitting individual pieces).

Let $m = \lfloor a_n^{-1}(\log n)^{2/11} \rfloor$. The next segment intersects each previous segment with probability approximately $1/m$.

Trees $\tilde{\mathcal{T}}_k$ and $\mathcal{T}_k^*$ coupled with high probability.

Length of loop-erased segments

Couple $(X_t)_{t=0}^{r-1}$ with walk $(Z_t)_{t=0}^{r-1}$ on $\mathbb{Z}^4$ so that $X_t = Z_t \pmod{n}$.

$$P(X_s = X_t \text{ and } Z_s \neq Z_t \text{ for some } s, t) \leq \frac{Cr^2}{n^4 \log r} = o(1).$$

Lengths of $LE((X_t)_{t=0}^{r-1})$ and $LE((Z_t)_{t=0}^{r-1})$ are $O\left(\frac{r}{(\log r)^{1/3}}\right)$.

We have $E[|LE((X_t)_{t=0}^{r-1})|] = b_n n^2 (\log n)^{5/66}$ for a sequence of constants $(b_n)_{n=1}^{\infty}$ bounded away from 0 and ∞. Distribution of length is concentrated around mean.

Approximate $d(x_i, x_j)$ by multiplying number of vertices on path in $\mathcal{T}_k^*$ by $b_n n^2 (\log n)^{5/66}$.

Distances $d(x_i, x_j)$ now coupled with $d(y_i, y_j)$ in K_m.

Adding a root vertex

For the first step of Wilson's algorithm, we need to run LERW from x to y, but it takes $O(n^4)$ steps of a random walk to hit y.

Walks on $\mathbb{Z}_n^4$ of length L intersect with probability $O\left(\frac{L^2}{n^4 \log L}\right)$.

Intersections first occur when $L = O(n^2(\log n)^{1/2})$.

Add root vertex ρ to $\mathbb{Z}_n^4$, connected to all vertices. Random walk goes to ρ after a geometric number of steps with mean $\beta n^2(\log n)^{1/2}$.

To apply Wilson's algorithm, first run LERW from x_1 to ρ, then start next walks at $x_2, \ldots, x_k$. This gives weighted spanning tree.

Removing edges leading to ρ gives spanning forest on $\mathbb{Z}_n^4$.

Stochastic domination

(Peres-Revelle, 2004): If $d'(x_i, x_j)$ denote distances in the spanning forest, total variation distance between $(d'(x_i, x_j))_{1 \le i < j \le k}$ and $(d(x_i, x_j))_{1 \le i < j \le k}$ is at most the probability that $x_1, \dots, x_k$ are in different tree components. Choose β large to reduce this below ϵ.

Add root to K_m, so the probability of going to the root in one step is same as probability that r-step walk on $\mathbb{Z}_n^4$ visits the root.

Where does the $1/6$ come from?

Need walks of length $O(n^2 (\log n)^{1/2})$ to get intersections.

After loop-erasure, length multiplied by $(\log n)^{-1/3}$.

Dynamics of LERW

Let $(X_t)_{t=0}^{\infty}$ be a random walk on $\mathbb{Z}_n^d$, $d \geq 4$.

Let $Y_t = |LE((X_s)_{s=0}^t)|$ be length of the loop-erasure at time t.

- Y_t increases linearly when there are no long loops.
- Long loops happen at rate proportional to Y_t.
- Long loops hit a uniform point on path.

Definition (Evans-Pitman-Winter, 2006): The Rayleigh process $(R(t), t \geq 0)$ is a $[0, \infty)$-valued Markov process such that:

- $R(t)$ increases linearly at unit speed between jumps.
- At time t, jump rate is $R(t-)$. At jump times, process gets multiplied by an independent Uniform$(0, 1)$ random variable.

Rayleigh distribution is stationary distribution.

Theorem (Schweinsberg, 2008): For some constants a_n and b_n, the processes $(b_n Y_{\lfloor a_n t \rfloor}, t \geq 0)$ converge to $(R(t), t \geq 0)$.

Note: This result was conjectured by Jim Pitman. Result for the complete graph was proved by Evans, Pitman, Winter (2006).