Heat kernel estimates for random walks on random media at criticality

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1 Introduction

Motivation

Analyze "anomalous" random walks or diffusions on disordered media

Math. Physicists' work

Survey: Ben-Avraham and S. Havlin ('00)

Detailed study of heat conduction and wave transmission

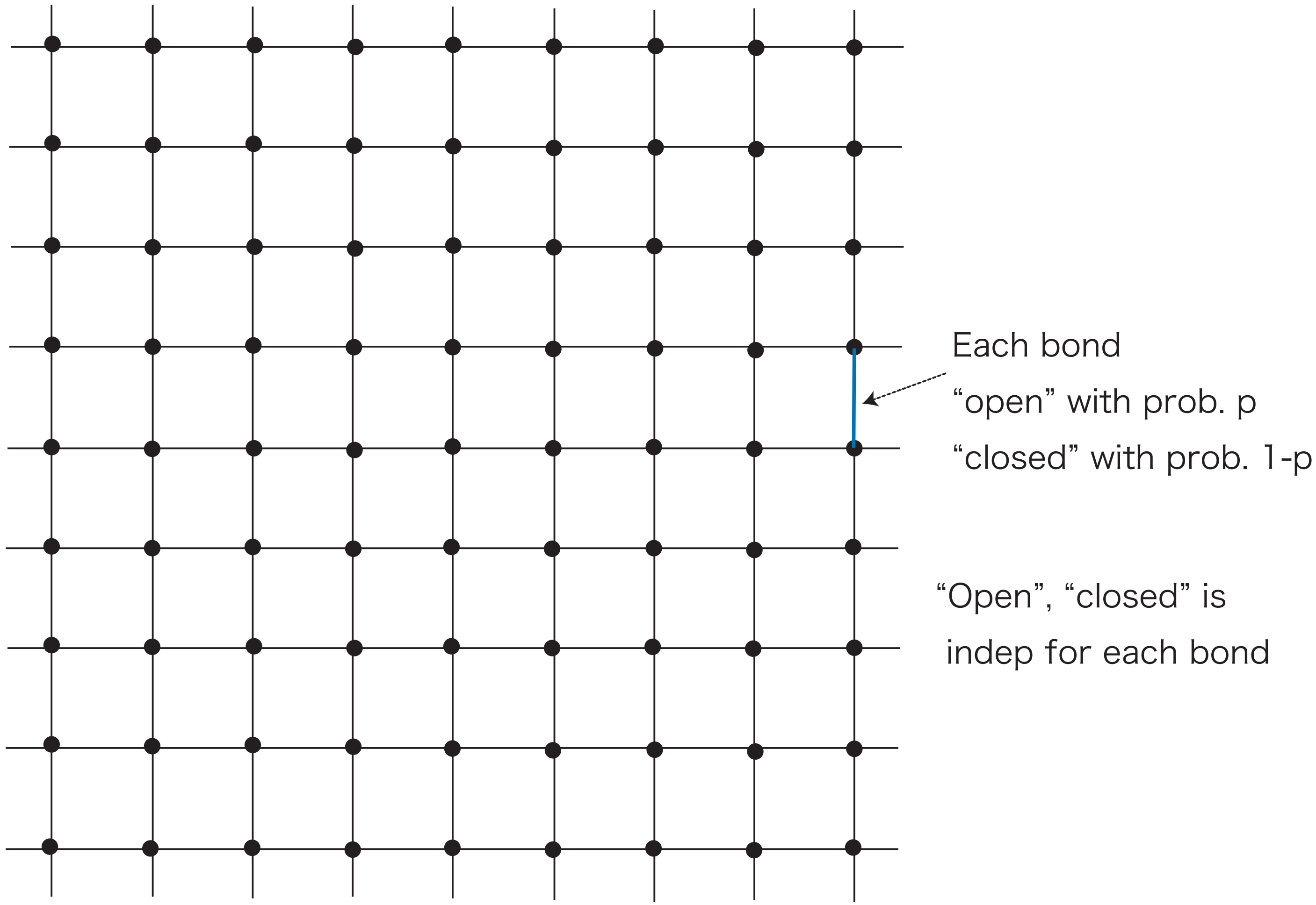
- Complicated network $\Rightarrow$ Random walk on "ideal" fractals

  Rammal-Toulose ('83) etc.

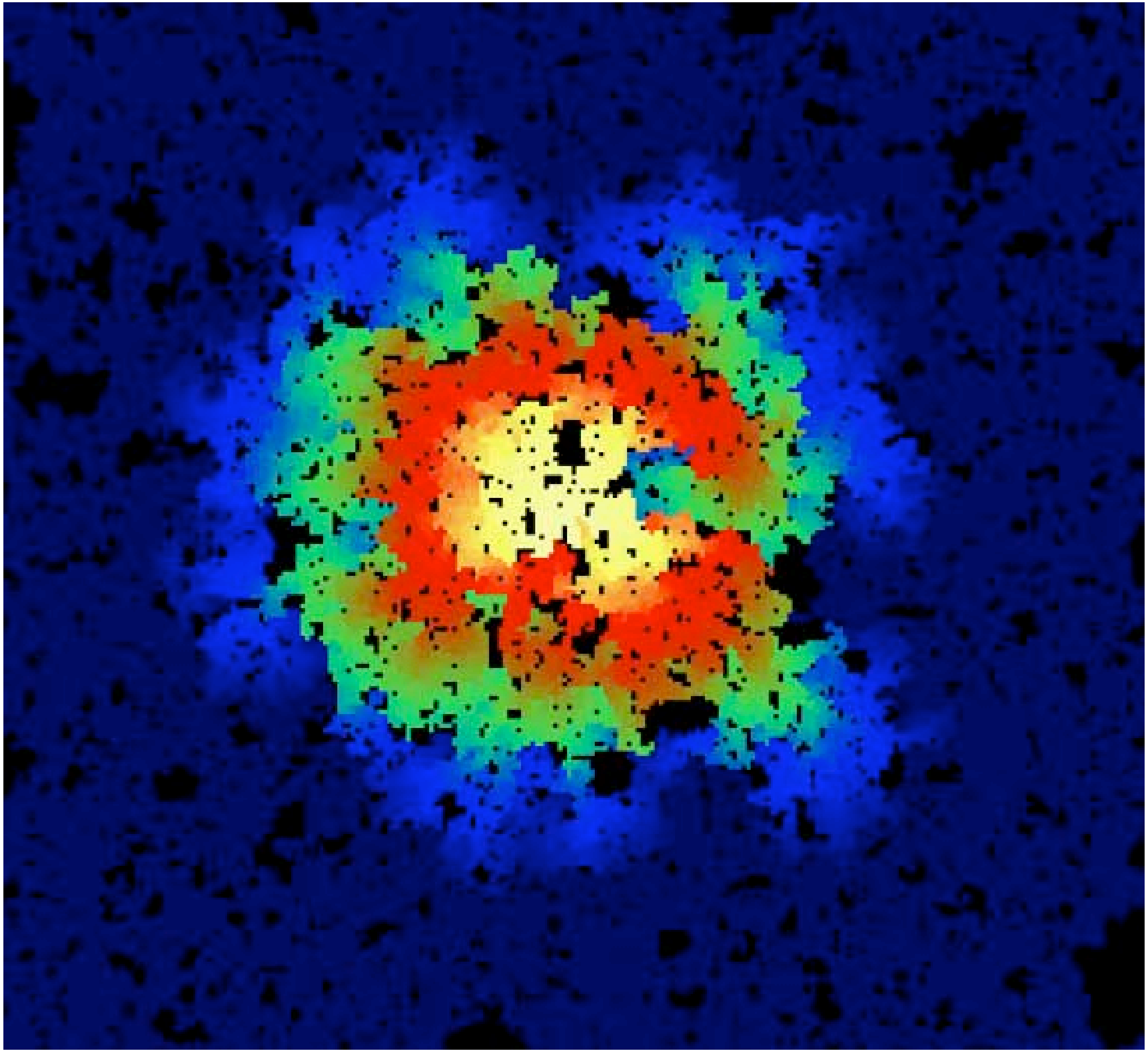
- Random models at critical probability (Percolation cluster etc.)

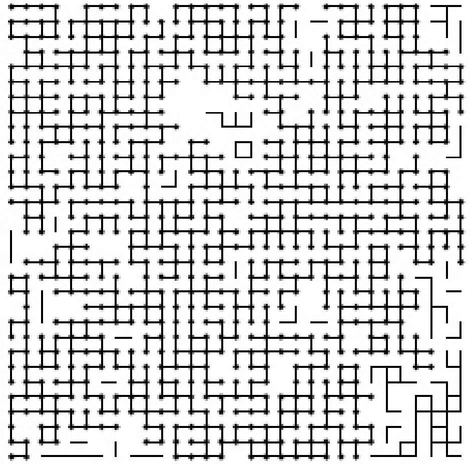
  De Gennes ('76) "the ant in the labyrinth"

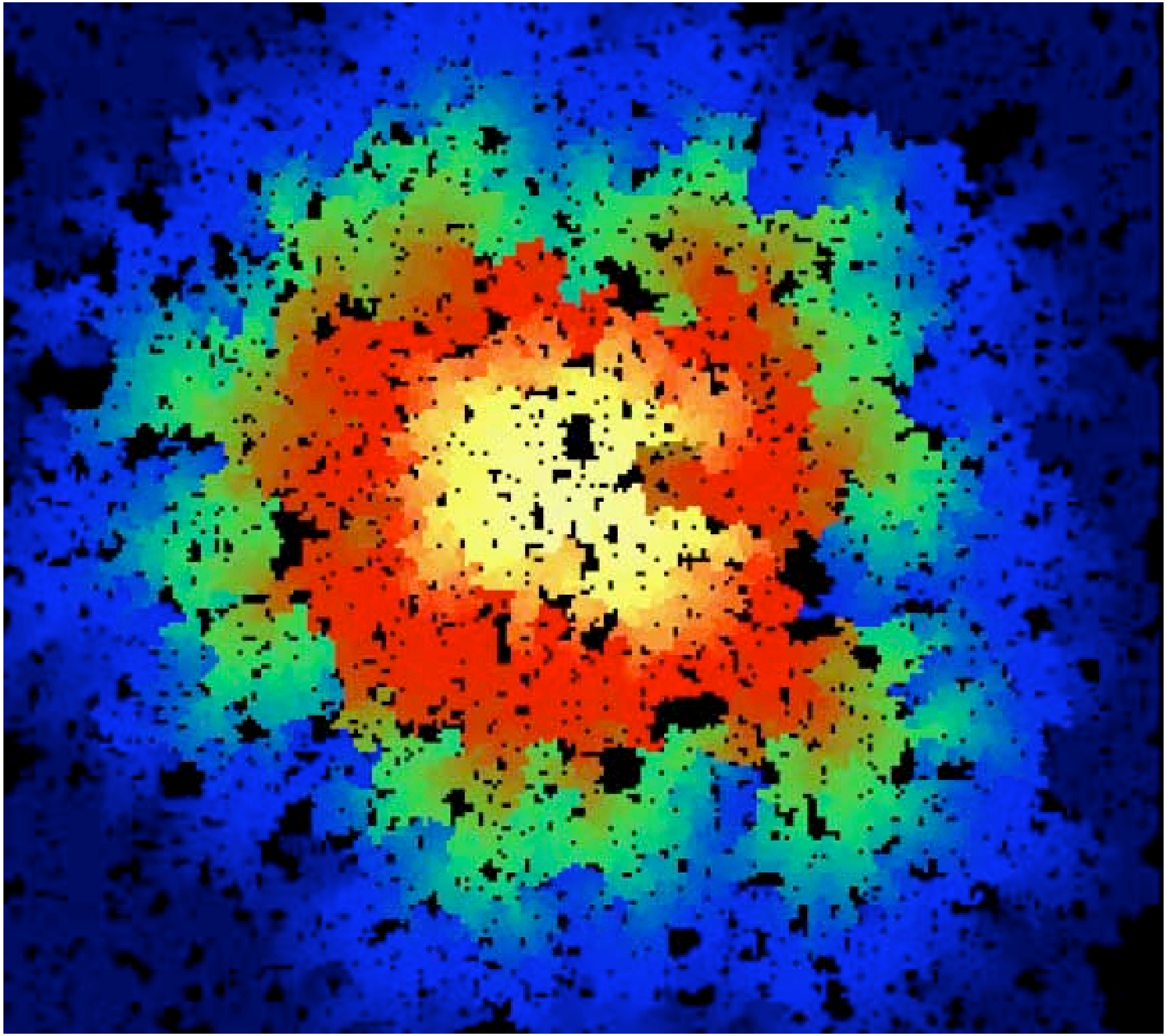
Bond percolation on $\mathbb{Z}^d \, (d \geq 2)$



$\exists p_c \in (0,1)$ s.t. $\exists 1 \infty$-cluster for $p > p_c$, no ∞-cluster for $p < p_c$.







'Anomalous' behaviour of the random walk at critical probability.

Let $p_n^\omega(x,y) := P_\omega^x(X_n = y)/\mu_y$ and

$d_s = -2 \lim_{n\to\infty} \log p_{2n}^\omega(x,x)/\log n$.

Alexander-Orbach conjecture (J. Phys. Lett., '82)

$d \geq 2 \Rightarrow$ $d_s = 4/3$ (NOT d).

(It is now believed that this is false for small d.)

Plan of the talk

- Random walk on random disordered media

  (i) Percolation cluster at criticality

  (ii) Percolation cluster for diamond lattice at criticality

  (iii) Random walk trace

  (iv) Erdös-Rényi random graph at critical window

2 Volume + Resistance $\Rightarrow$ HK estimates

$(\mathcal{G}(\omega), \omega \in \Omega)$: random graph on $(\Omega, \mathcal{F}, \mathbb{P})$, $\{Y_n\}$: simple RW on $\mathcal{G}$. For $\lambda \geq 1$, let

$$J(\lambda) := \{R \geq 1 : \frac{R^D}{\lambda} \leq \mu(B_R) \leq \lambda R^D, R_{\text{eff}}(0, B_R^c) \geq \frac{R^\alpha}{\lambda}, R_{\text{eff}}(0, y) \leq \lambda R^\alpha, \forall y \in B_R\},$$

for $D \geq 1, 0 < \alpha \leq 1$ where $B_r := B(0, r)$.

Theorem 2.1 (Barlow-Járai-K-Slade '08, K-Misumi '08)

If $q_0, c_1 > 0$ *s.t.* $\mathbb{P}(R \notin J(\lambda)) \leq c_1 \lambda^{-q_0}$, *for all* $R \geq 1$, *then* $\exists a_1, a_2 \geq 0$ *s.t.*

(i) $$(\log n)^{-a_1} n^{-\frac{D}{D+\alpha}} \leq p_{2n}^{\omega}(x, x) \leq (\log n)^{a_1} n^{-\frac{D}{D+\alpha}} \qquad \textit{for large } n, \quad \mathbb{P} - a.s.$$

Especially, $d_s(\mathcal{G}(\omega)) = \frac{2D}{D+\alpha}$, $\mathbb{P}$*-a.s.* ω , *and the RW is recurrent.*

(ii) $$(\log n)^{-a_2} n^{\frac{1}{D+\alpha}} \leq \max_{0 \leq k \leq n} d(0, Y_k) \leq (\log n)^{a_2} n^{\frac{1}{D+\alpha}}, \qquad \textit{for large } R, \quad P_{\omega}^x - a.s.$$

So, $d_f = D, d_w = D + \alpha, \ d_s/2 = d_f/d_w$.

3 (i) Percolation cluster at criticality

Consider the following models:

(I) (Spread-out) oriented percolation with $d > 6$

(II) Percolation with d large (say $d \geq 19$)

Let $\mathcal{C}(0)$ be the set of vertices connected to 0 by open bonds (random media!)

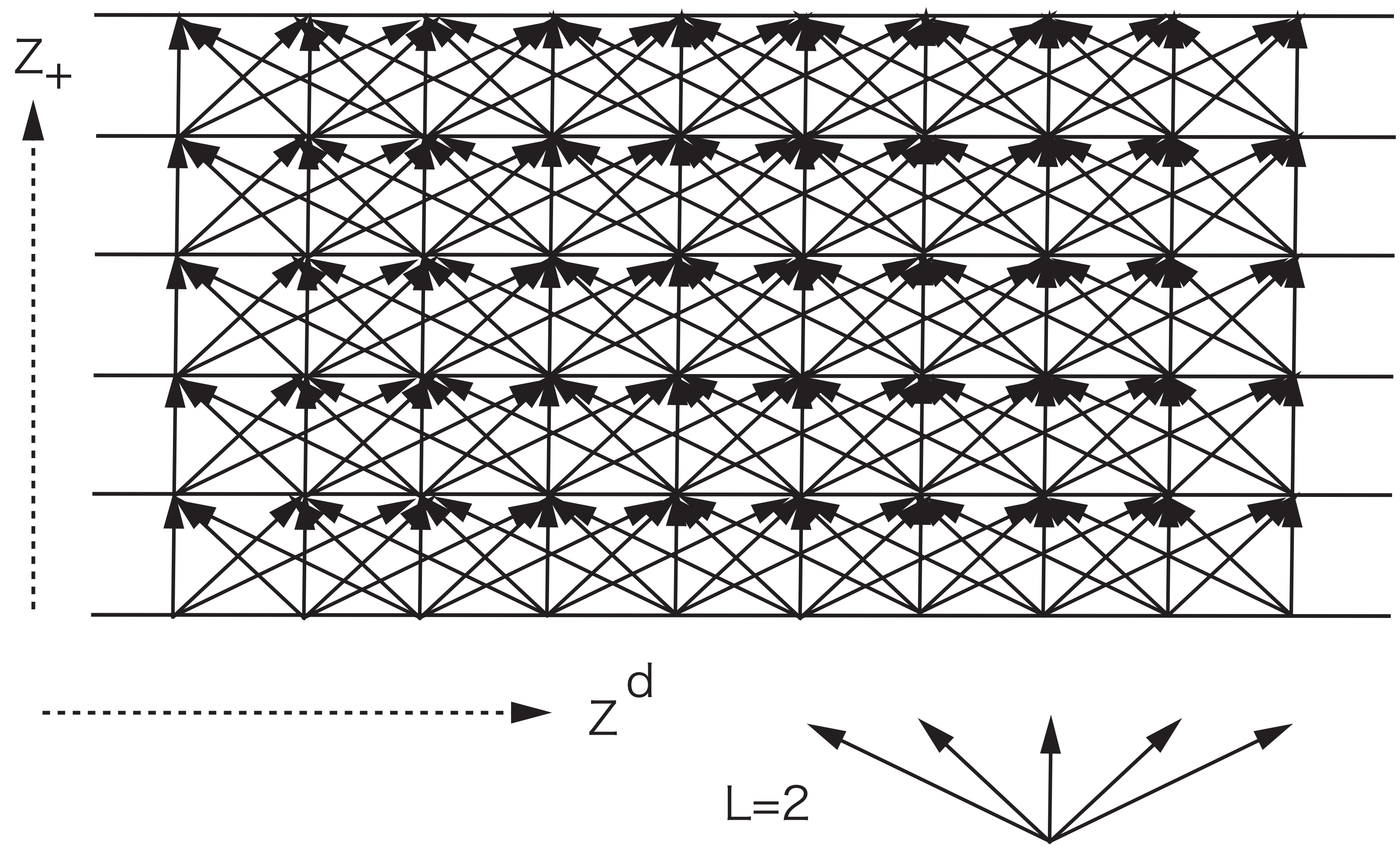
$\exists p_c = p_c(d) \in (0,1)$ s.t. $p > p_c \Rightarrow \exists$ infinite cluster, $p \leq p_c \Rightarrow$ no infinite cluster

So, at $p = p_c$, $\mathcal{C}(0)$ is a finite cluster with prob. 1!

$\Rightarrow$ Consider incipient infinite cluster (IIC). (I.e. at the critical prob., conditioned on $\sharp\mathcal{C}(0) = \infty$.) Existence of the IIC is known for the above models.

(OP: van der Hofstad-den Hollander-Slade '02, P: van der Hofstad-Járai '04)

$(\mathcal{G}(\omega), \omega \in \Omega)$: IIC, $(\Omega, \mathcal{F}, \mathbb{P})$: prob. space for the randomness of the space



Z_+
Z^d
L=2

For each $\mathcal{G} = \mathcal{G}(\omega)$, let $\{Y_n\}$ be a simple RW on $\mathcal{G}$.

P_ω^x: law of $\{Y_n\}$ starting at $x \in \mathcal{G}(\omega)$, $\quad p_n^\omega(x,y) := P_\omega^x(Y_n = y)/\mu_y$.

Theorem 3.1 (Barlow-Járai-K-Slade '08, Kozma-Nachmias '09)

For models I and II, $\exists a_1, a_2 \geq 0$ s.t. the following hold.

$$(i) \qquad (\log n)^{-a_1} n^{-2/3} \leq p_{2n}^\omega(x,x) \leq (\log n)^{a_1} n^{-2/3}, \qquad \textit{for large } n, \quad \mathbb{P}-a.s.$$

Especially, $d_s(G(\omega)) = \frac{4}{3}$, $\mathbb{P}$–a.s. ω (solves the A-O conj.), and the RW is recurrent.

$$(ii) \qquad (\log n)^{-a_2} n^{1/3} \leq \max_{0 \leq k \leq n} d(0, Y_k) \leq (\log n)^{a_2} n^{1/3}, \qquad \textit{for large } n, \quad P_\omega^x - a.s.$$

$$(iii) \qquad c_3 n^{-2/3} \leq \mathbb{E}(p_{2n}^\cdot(0,0)) \leq c_4 n^{-2/3}, \quad \forall R, n \geq 1.$$

⨀ Theorem 2.1 applies with $D = 2, \alpha = 1$. (Need probabilistic estimates.)

Remark: (i) For $p > p_c$ (at least for model II),

(a) (HK(2)) [Gaussian heat kernel estimates] holds $\mathbb{P}$-a.s. for large t (Barlow '04)

(b) $n^{-1}Y^{\omega}_{n^2 t} \to B_{\sigma t}$ $\mathbb{P}$-a.s. ω for some $\sigma > 0$ (Quenched invariance principle)

(Sidoravicius-Sznitman '04, Berger-Biskup '07, Mathieu-Piatnitski. '07)

(ii) A-O conjecture holds for $d > 6$ for model I and d large for model II.

Critical dimension is believed to be $d = 6$ for model II.

Numerical simulations suggest that A-O conjecture is false for $d \leq 5$ (model II).

$$d = 5 \Rightarrow d_s = 1.34 \pm 0.02, \quad d = 4 \Rightarrow d_s = 1.30 \pm 0.04$$

$$d = 3 \Rightarrow d_s = 1.32 \pm 0.01, \quad d = 2 \Rightarrow d_s = 1.318 \pm 0.001$$

Remark 2: For trees, the following results are known.

(1) Critical percolation on regular trees (Kesten '86, Barlow-K '06)

Theorem 2.1 holds with $D = 2, \alpha = 1$.

(2) Critical invasion percolation on regular trees (Angel-Goodman-Hollander-Slade '08)

Theorem 2.1 holds with $D = 2, \alpha = 1$.

(2) Critical G-W tree with ∞-variance offspring distri. (Kesten '86, Croydon-K '08)

$\{Z_n\}_{n\geq 0}$: critical G-W proc. $\mathbb{E}[Z_1] = 1$, $\mathbb{P}(Z_1 = 1) \neq 1$.

$\mathbb{E}[s^{Z_1}] = s + (1-s)^\beta L(1-s),\ \forall s < 1$, where $\beta \in (1, 2]$ and $L(x)$ is slowly varying

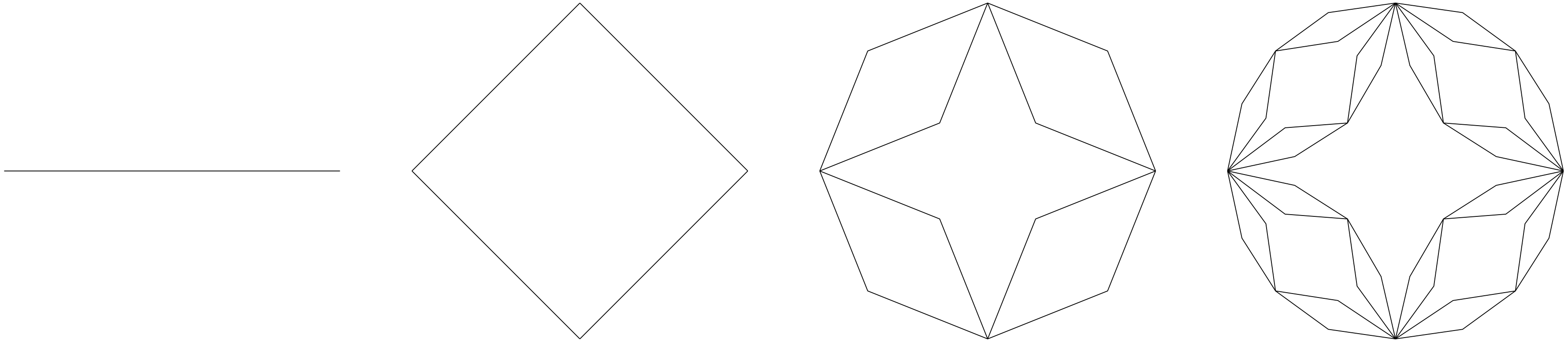
$\Rightarrow$ Theorem 2.1 holds with $D = \beta/(\beta - 1), \alpha = 1$.

Theorem 3.2 (Oscillations)

$\underline{\beta \in (1,2)}$ $\exists \varepsilon_1 > 0\ s.t.\ \ \liminf_{n\to\infty} n^{\frac{\beta}{2\beta-1}} (\log n)^{\varepsilon_1} p^\omega_{2n}(0,0) = 0,\ \ \mathbb{P} - a.e.\ \omega.$

$\underline{\beta = 2}$ $\exists \varepsilon_2 > 0\ s.t.\ \ \liminf_{n\to\infty} n^{\frac{2}{3}} (\log\log n)^{\varepsilon_2} p^\omega_{2n}(0,0) = 0,\ \ \mathbb{P} - a.e.\ \omega.$

(ii) Percolation cluster for diamond lattice at criticality (Hambly-K '08)



At each step, replace each edge by a parallelogram (diamond).

Let V_n be a set of vertices at the n-step, E_n a set of edges at the n-step.

$D_n := (V_n, E_n)$, $V_0 = \{0, 1\}$, $\cup_{m\geq 0} V_m$ is dense in K.

K (Scaling limit of the) Diamond hierarchical lattice Let $I = \{1, 2, , 3, 4\}$.

K is invariant under a family of contraction maps $\{\psi_i\}_{i=1}^4$: $K = \cup_{i\in I}\psi_i(K)$.

Diffusion on the diamond hierarchical lattice

$$\mathcal{E}_0(f,g) := \frac{1}{2}(f(0)-f(1))(g(0)-g(1)), \quad \mathcal{E}_n(f,g) = \sum_{i=1}^{4} \mathcal{E}_{n-1}(f\circ\psi_i, g\circ\psi_i).$$

Let $\quad \mathcal{E}(f,f) = \lim_{n\to\infty} \mathcal{E}_n(f,f), \ \forall f \in \mathcal{F}^* := \{f : \cup_{m\geq 0} V_m \to \mathbb{R} \mid \sup_n \mathcal{E}_n(f,f) < \infty\}.$

μ: Hdff meas. on K, $\mu(\psi_w(K)) = 4^{-|w|}$. **Note** μ does NOT satisfy volume doubling.

Theorem 4.1 *1)* $\exists \iota_\mu : \mathcal{F}^* \subset L^2(K,\mu)$ *compact imbedding. Let* $\mathcal{F} := \iota_\mu(\mathcal{F}^*)$.

Then $(\mathcal{E},\mathcal{F})$ *is a local reg. Dirichlet form on* $L^2(K,\mu)$.

2) $\exists p_t(\cdot,\cdot)$ *jointly cont. heat kernel that enjoys the following estimates:*

a) $0 < p_t(x,y) \leq \frac{c_1}{t}\exp\big(-c_2\frac{d(x,y)^2}{t}\big), \ p_t(x,x) \geq \frac{c_3}{\mu(B(x,c_4\sqrt{t}))} \quad \forall x,y \in K, \forall t \in (0,1),$

b) $\qquad c_1 t^{-1}|\log t|^{-a} \leq p_t(x,x) \leq c_2 t^{-1} \qquad$ *for* μ*-a.e.* $x \in K, \ \forall t < \exists T(x),$

c) $\qquad c_1 t^{-1/2} \leq p_t(0,0) \leq c_2 t^{-1/2} \qquad \forall t < 1.$

Percolation on K

For $p \in (0,1)$, construct D_n^p by retaining each edge in E_n indep. with prob. p.

From D_n^p, one can induce percolation on D_{n-1} by regarding that each edge is connected iff it is connected on the n-th level.

$\Rightarrow$ The induced percolation, $D_{n,1}$ is equal in law to $D_{n-1}^{f(p)}$, where

$$f(p) = p^4 + 4p^3(1-p) + 2p^2(1-p)^2 = 2p^2 - p^4$$

f has 3 fixed points in $[0,1]$; $0, 1$ (attractive) and $p_c = (\sqrt{5}-1)/2$ (repulsive)

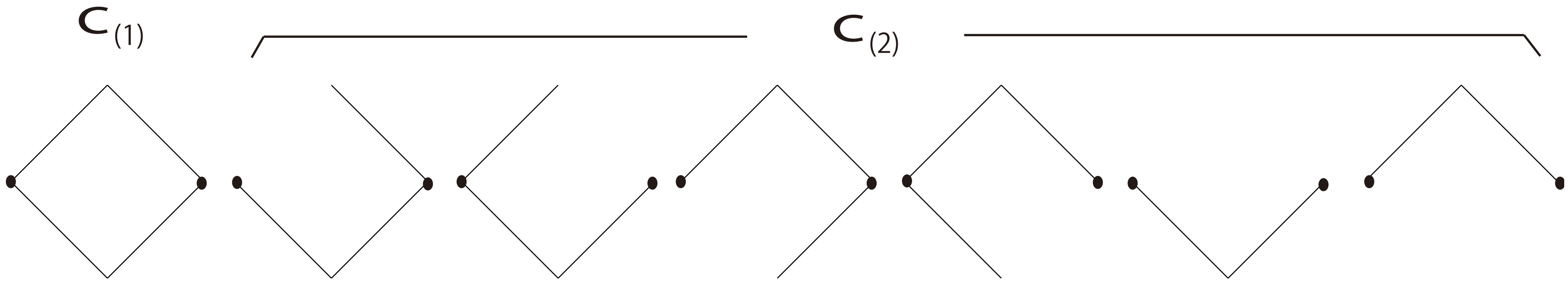
Lemma 4.2 *If $p > p_c$, then P(0 and 1 are connected in D_n) $\to 1$ as $n \to \infty$.*

If $p = p_c$, then P(0 and 1 are connected in D_n) $= p_c$ for all $n \geq 0$.

If $p < p_c$, then P(0 and 1 are connected in D_n) $\to 0$ as $n \to \infty$.

Let $\mathcal{C} = \mathcal{C}(\omega)$ be the crit. perco. cluster under $P^{p_c}(\,\cdot\,|\{0\}$ and $\{1\}$ are connected.))

Diffusion on the scaling limits of critical percolation clusters in K



Let $T := \cup_{i=1}^{\infty} I^i \cup \{\emptyset\}$ and $\mathcal{S} := \{c_{(1)}, c_{(2)}, d\}$. (Recall $I = \{1, 2, 3, 4\}$.)

$\Omega :=\ T \otimes \mathcal{S}$ probability space of labelled trees. So $\omega \in \Omega \Rightarrow \omega = \{(\mathbf{i}, u_{\mathbf{i}})\}_{\mathbf{i} \in T}$.

Define the resistance scale factors by $\rho_{u_{\mathbf{i}}} = 1$ if $u_{\mathbf{i}} = c_{(1)}$, and 2 otherwise.

Then $\exists \mu^{\omega}$: Borel meas. naturally defined on $\mathcal{C}(\omega)$ by using $\{\rho_{u_{\mathbf{i}}}\}_{\mathbf{i}}$.

Recall $\mathcal{E}_0(f, g) := \frac{1}{2}(f(0) - f(1))(g(0) - g(1))$. For $\omega = \{(\mathbf{i}, u_{\mathbf{i}})\}_{\mathbf{i} \in T}$, set

$$\mathcal{E}_1^{(\omega)}(f, g) = \sum_{i : u_i \in \{c_{(1)}, c_{(2)}\}} \mathcal{E}_0(f \circ \psi_i, g \circ \psi_i) \rho_{u_{\emptyset}}.$$

We now repeat this construction by setting

$$\mathcal{E}_n^{(\omega)}(f,g) = \sum_{i=1}^{4} \mathcal{E}_{n-1}^{(\sigma_i\omega)}(f\circ\psi_i, g\circ\psi_i)\rho_{u_\emptyset}.$$

Let $\quad \mathcal{E}^{(\omega)}(f,f) = \lim_n \mathcal{E}_n^{(\omega)}(f,f), \quad \forall f \in \mathcal{F}^{(\omega)} = \{f : \sup_n \mathcal{E}^{(\omega)}(f,f) < \infty\}.$

Theorem 4.3 *1)* $(\mathcal{E}^{(\omega)}, \mathcal{F}^{\omega})$ *is a local reg. D-form on* $L^2(\mathcal{C}(\omega), \mu^{\omega})$ $\forall \omega \in \Omega$ *s.t.*

$$\mathcal{E}^{(\omega)}(f,g) = \sum_{i=1}^{4} \mathcal{E}^{(\sigma_i\omega)}(f\circ\psi_i, g\circ\psi_i)\rho_{u_\emptyset} \qquad \forall f,g \in \mathcal{F}^{\omega}.$$

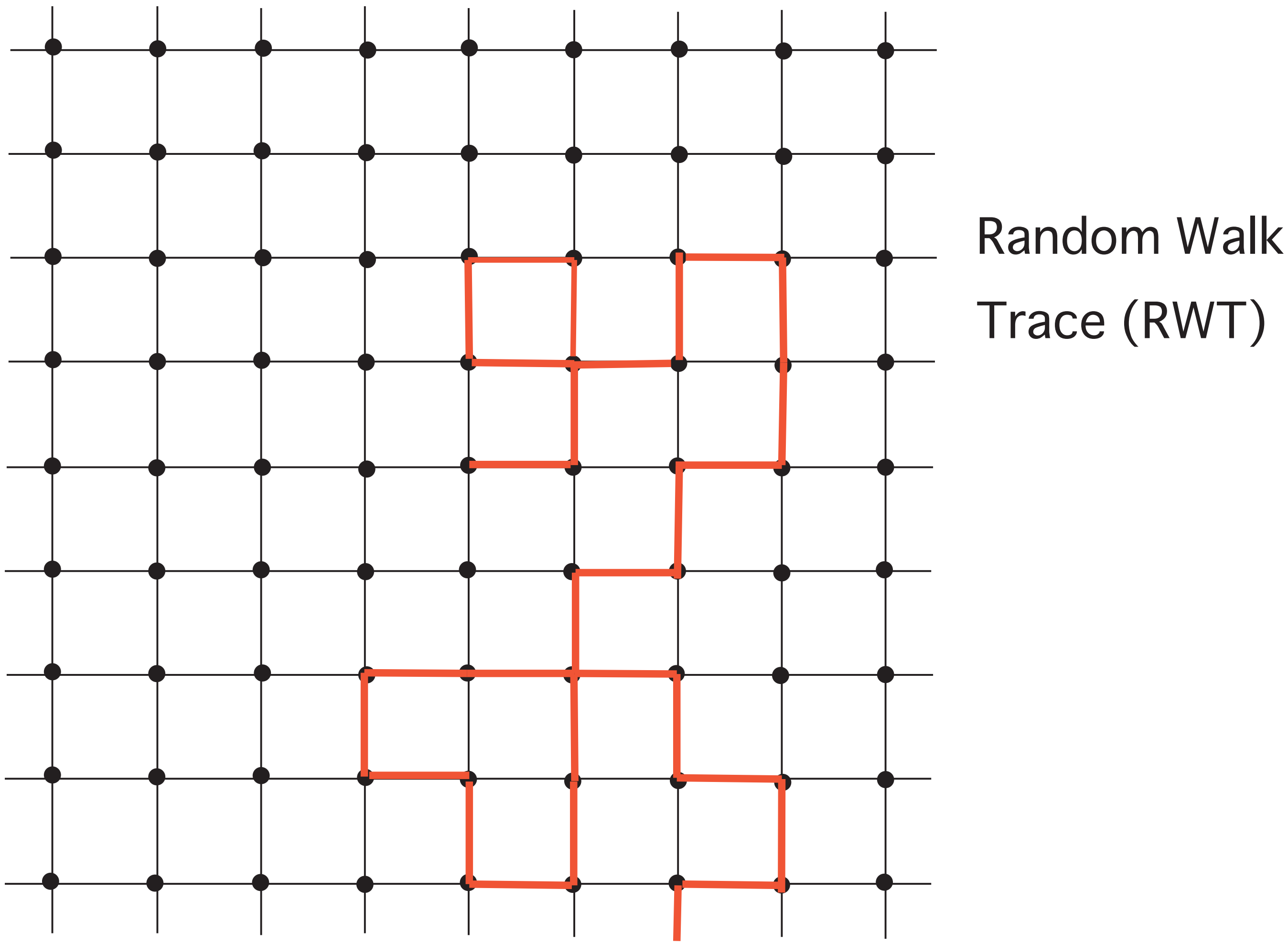
2) For $\mathbb{P}-a.s.\omega$, $\exists q_t^{\omega}(\cdot,\cdot)$ *jointly cont. heat kernel s.t.*

a) $c_1 t^{-\theta/(\theta+1)}|\log\log t|^{-b_1} \le q_t^{\omega}(x,x) \le c_2 t^{-\theta/(\theta+1)}|\log\log t|^{b_2}$, μ*-a.e.* $x \in \mathcal{C}(\omega)$, $\forall t < 1$,

b) $\quad c_3 t^{-(\theta-\nu)/(\theta-\nu+1)} \le q_t^{\omega}(0,0) \le c_2 t^{-(\theta-\nu)/(\theta-\nu+1)} \quad \forall t < 1$,

where $\quad \theta = 5.2654\ldots, \nu = 1.3384\ldots; \quad \frac{\theta}{\theta+1} = 0.8404\ldots, \quad \frac{\theta-\nu}{\theta-\nu+1} = 0.7970\ldots$

5 (iii) Random walk trace



Let $\mathcal{G} = \mathcal{G}(\omega)$ be the trace of RW on $\mathbb{Z}^d$ starting at 0.

For each $\mathcal{G}$, let $\{Y_n\}$ be a simple RW on $\mathcal{G}$ starting at 0.

Let $d_{\mathcal{G}}(\cdot, \cdot)$ be the graph distance on $\mathcal{G}$.

Theorem 5.1 (Croydon '09) $d \geq 5$. *Let B_t be BM and $W_t^{(d)}$ be indep. d-dim. BM.*

(i) $\exists c_1, c_2 > 0$ *such that*

$$c_1 n^{-1/2} \leq p_{2n}^{\omega}(0,0) \leq c_2 n^{-1/2} \quad \textit{for large } n, \quad \mathbb{P} - a.s.$$

(ii) $\exists \sigma_1 = \sigma_1(d) > 0$ *such that*

$$\{n^{-1/2} d_{\mathcal{G}}(0, Y_{[tn]})\}_t \xrightarrow{weak} \{|B_{\sigma_1 t}|\}_t, \quad \mathbb{P} - a.s.$$

(iii) $\exists \sigma_2 = \sigma_2(d) > 0$ *such that*

$$\{n^{-1/4} Y_{[tn]}\}_t \xrightarrow{weak} \{W^{(d)}_{|B_{\sigma_2 t}|}\}_t, \quad \mathbb{P} - a.s.$$

Theorem 5.2 (Shiraishi '08, '09)

(i) Let $d = 4$. $\exists c_1, c_2 > 0$ *and a slowly varying function* ψ *such that*

$$c_1 n^{-\frac{1}{2}}(\psi(n))^{\frac{1}{2}} \le p^{\omega}_{2n}(0,0) \le c_2 n^{-\frac{1}{2}}(\psi(n))^{\frac{1}{2}} \quad \textit{for large } n, \quad \mathbb{P} - a.s.$$

Further, $\psi(n) \approx (\log n)^{-\frac{1}{2}}$, *that is*

$$\lim_{n\to\infty} \frac{\log \psi(n)}{\log \log n} = -\frac{1}{2}.$$

(ii) Let $d = 4$. *Then the following holds* $\mathbb{P} - a.s.$ ω*:*

$$n^{\frac{1}{4}}(\log n)^{\frac{1}{24}-\delta} \le \max_{1\le k\le n} |Y^{\omega}_k| \le n^{\frac{1}{4}}(\log n)^{\frac{13}{12}+\delta} \qquad \textit{for large } n, \quad P^0_{\omega} - a.s.$$

(iii) Let $d = 3$. $\exists c_3 > 0$ *such that*

$$p^{\omega}_{2n}(0,0) \le n^{-\frac{10}{19}}(\log n)^{a} \quad \textit{for large } n, \quad \mathbb{P} - a.s.$$

Critical dimension for RW on RWT is 4!

Proposition 5.3 (Burdzy-Lawler '90)

$$
\begin{aligned}
& E[R_{\mathcal{G}}(0,S_n)] \sim cn \quad \textit{for } d \geq 5 \\
c(\log n)^{-\frac{1}{2}} \lesssim \quad & \frac{1}{n} E[R_{\mathcal{G}}(0,S_n)] \lesssim c'(\log n)^{-\frac{1}{3}} \quad \textit{for } d = 4 \\
& cn^{\frac{1}{2}} \lesssim E[R_{\mathcal{G}}(0,S_n)] \lesssim c' n^{\frac{5}{6}} \quad \textit{for } d = 3,
\end{aligned}
$$

Let L_n be the number of cut points (for $S[0,n]$) up to time n.

Also, let A_n be the number of points for loop-erased RW (for $S[0,n]$). Then

$$
E[L_n] \leq E[R_{\mathcal{G}}(0,S_n)] \leq E[A_n].
$$

Proposition 5.4 (Shiraishi '09) *For* $d = 4$,

$$
\frac{1}{n} E[R_{\mathcal{G}}(0,S_n)] \simeq (\log n)^{-\frac{1}{2}}
$$

$\because$ Let $\{T_j\}$ be the sequence of cut times up to time n. Then RW trace near S_{T_j} and $S_{T_{j+1}}$ intersects typically when $T_{j+1} - T_j$ is large, i.e. $\exists$"long range intersection".

6 (iv) Erdös-Rényi random graph at critical window

$G(n,p)$: Erdös-Rényi random graph   I.e. $V_n := \{1, 2, \cdots, n\}$ labeled vertices

Each $\{i, j\}$ $(i, j \in V_n)$ is connected by a bond with prob. p.

$\mathcal{C}_1^n$: largest connected component

Phase transition at $p = 1/n$: $p \sim c/n$ with $c < 1 \Rightarrow \sharp\mathcal{C}_1^n = O(\log n)$

with $c > 1 \Rightarrow \sharp\mathcal{C}_1^n \asymp n$

with $c = 1 \Rightarrow \sharp\mathcal{C}_1^n \asymp n^{2/3}$

Finer scaling (critical window): $p = 1/n + \lambda n^{-4/3}$ for fixed $\lambda \in \mathbb{R}$

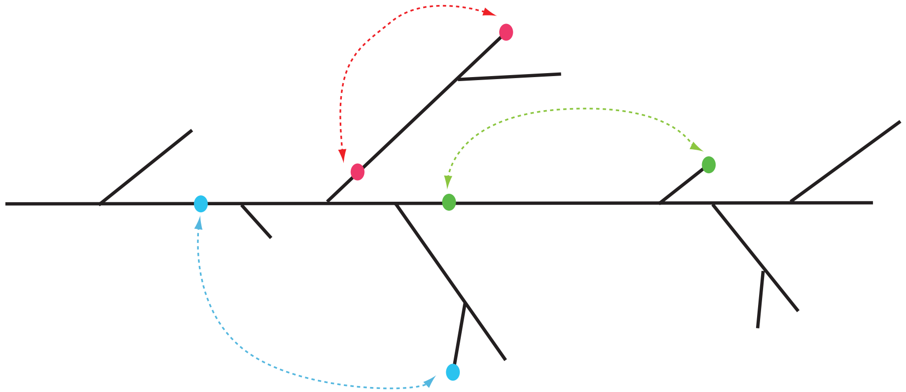
$\Rightarrow$ One can describe the asymptotics of $n^{-2/3}\sharp\mathcal{C}_1^n$ etc. (Aldous '97)

(Addario-Berry Broutin Goldschmidt '09)

$$n^{-1/3}\mathcal{C}_1^n \xrightarrow{d} \exists\mathcal{M} \qquad \text{(Gromov-Hausdorffsense),}$$

where $\mathcal{C}_1^n$ is considered as a rooted metric space.

Here $\mathcal{M}$ can be constructed from a (random) real tree by gluing a (random) finite number of points as in the following figure.



$Y_m^{\mathcal{C}_1^n}$: simple RW on $\mathcal{C}_1^n$. Heat kernel estimates on $\mathcal{C}_1^n$ (on-going work).

Theorem 6.1 (Croydon '09)

$\exists B_t^{\mathcal{M}}$: *Brownian motion on* $\mathcal{M}$ *and* $\exists p_t^{\mathcal{M}}(\cdot,\cdot)$ *its heat kernel s.t.*

$$\{n^{-1/3}Y_{[nt]}^{\mathcal{C}_1^n}\}_t \xrightarrow{weak} \{B_t^{\mathcal{M}}\}_t, \qquad \mathbb{P}-a.s.$$

$$c_1 t^{-2/3}(\log t^{-1})^{-c_2} \le p_t^{\mathcal{M}}(x,x) \le c_3 t^{-2/3}(\log t^{-1})^{-c_4}, \qquad \forall x \in \mathcal{M}, t \le T_0$$

(Cf. Tree case: Kesten '86, Barlow-K '06, Croydon '08, Croydon-K '08)