Topological order in quantum dimer models

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Outline

- What is quantum dimer model?
- A toy model on the kagome lattice:
  - exact mapping to Kitaev’s toric code
  - dimer liquid with $Z_2$ topological order

- Liquid phase of the triangular lattice quantum dimer model (RK point):
  - 4 degenerate ground-state on a torus
  - Numerics showing that these state are locally undistinguishable
  - Computation of the topological entanglement entropy:
    - Kitaev-Preskill and Levin-Wen constructions
    - Modified construction to extract the topological entanglement entropy
      $\Rightarrow \gamma = 1.00(2) \log(2)$
What is a quantum dimer model?
Quantum dimer models

- Basis states = fully packed dimer coverings of the lattice

- Introduce some simple dynamics

\[ H = -t \sum \left[ \begin{array}{c} ++ \\ \end{array} \right] + H.c + V \sum \left[ \begin{array}{c} + - \\ \end{array} \right] \]

- Examples of phase diagrams

Valence-bond crystals (VBC) Staggered crystal

\[
\begin{align*}
\text{Critical point} & \quad V/J \\
1 & \quad 1
\end{align*}
\]

An exactly solvable quantum dimer model on the kagome lattice
Dimer coverings of the *kagome* lattice

On a lattice made of corner-sharing triangles (such as kagome), dimer coverings are easily represented with arrows:

There is a one-to-one correspondence between
i) configurations of arrows with $N_{\text{in}} = \text{even}$ everywhere and
ii) hard-core dimer coverings

Constraint
Number of incoming arrows must be *even* on every triangle $N_{\text{in}} = 0$ or 2
Exactly solvable dimer (toy) model

\[ \tau_i^z = \text{Flips the arrow } i \]

\[ \prod_{\text{closed loop}} \tau_i^z = \text{Physical dimer move (preserves the constraints)} \]

\[ h = \left( \prod \tau_i^z \right)^2 = 1 \]

\[ \prod \tau_i^z , \prod \tau_j^z \right\} = 0 \quad \forall h, h' \]

GM, Serban & Pasquier, PRL 2002

32 different loops when applying the hexagon symmetries

Hamiltonian

\[ H = -\sum_h \prod_{i=1}^{6} \tau_i^z \quad \text{dimer hoping} (= Z_2 \text{ flux}) \]
## Solvable kagome dimer model and Kitaev’s toric code

### Dimer model

- **Dimer kinetic energy**
  \[
  H = - \sum_h \left( \prod_{i=1}^{6} \tau_i^z \right)
  \]

- **Hard core constraint**
  \[
  \tau_1^x \tau_2^x \tau_3^x = 1
  \]
  \[
  \tau_i^x = \begin{cases} 
  +1 & \text{Arrow with same direction as in the reference config.} \\
  -1 & \text{Arrow in opposite direction}
  \end{cases}
  \]

- **Hamiltonian**
  \[
  H = - \sum_h \left( \prod_{i=1}^{6} \tau_i^z \right)
  \]

- **Ground-state**
  \[
  |g.s\rangle = \sum_{c \in \{\text{dimer coverings}\}} |c\rangle
  \]

### Kitaev’s toric code

- **Magnetic energy**
  \[
  B_p = \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x
  \]

- **Electric energy**
  \[
  A_s = \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x
  \]

- **Hamiltonian**
  \[
  H = - \sum_{s} A_s - \sum_{p} B_p
  \]

- **Ground-state**
  \[
  |g.s\rangle = \sum_{c \in \{\text{config. satisfying } A_s=1 \forall s\}} |c\rangle
  \]
The simplest $\mathbb{Z}_2$ dimer liquid – short-range RVB phys.

- Topological degeneracy
- No broken symmetry
  - Dimer-dimer correlations are \textit{strictly} zero beyond 2 lattice spacings
  - The degenerate ground-states are locally undistinguishable \Rightarrow No local order parameter, no (conventional, hidden) order of any kind. [Furukawa, GM, Oshikawa PRL 2006]
- Gapped excitations
  - All wave-functions are known explicitly
  - Exact mapping to an $\mathbb{Z}_2$ gauge theory
  - Excitations = pairs of (static and non-interacting) $\mathbb{Z}_2$ vortices (visons)
- Deconfined fractional excitations (monomers)
  - doped version of the model: GM, Pasquier, Mila & Lhuillier \textit{PRB 2005}
- Perturbation theory \Rightarrow vortex dispersion relation, PSG, etc.
Topological degeneracy in the $\mathbb{Z}_2$ liquid phase of the triangular lattice quantum dimer model
Dimer liquid phase in the triangular QDM

- Quantum dimer model [Moessner & Sondhi, 2001]

\[ H = -J \sum | \begin{array}{c} \text{dimer} \end{array} | + V \sum | \begin{array}{c} \text{dimer} \end{array} | + V \sum | \begin{array}{c} \text{dimer} \end{array} | \]

- Liquid phase:
  - Short-ranged dimer-dimer correlations (but non-zero correlation length)
  - Gapped excitations
  - The ground-state degeneracy depends on the topology.

- Phase diagram

First simple model with a short-ranged RVB liquid.
At $J=V=1$, the Hamiltonian can be written as

$$H = 2 \sum_r |\psi_r \rangle \langle \psi_r| = \text{Sum of projectors}$$

$$|\psi_r \rangle = \frac{1}{\sqrt{2}} \left( |\begin{array}{c} r \\ \end{array}\rangle - |\begin{array}{c} L \\ \end{array}\rangle \right)$$

$$|0\rangle = \sum_c |c\rangle \quad \langle \Psi_r |c\rangle = \begin{cases} \frac{1}{\sqrt{2}} |c_{\downarrow r}\rangle & \text{if } |c\rangle = |\cdots \begin{array}{c} L \\ \end{array} \cdots\rangle \\ -\frac{1}{\sqrt{2}} |c_{\downarrow r}\rangle & \text{if } |c\rangle = |\cdots \begin{array}{c} r \\ \end{array} \cdots\rangle \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \Psi_r |0\rangle = 0$$

$$\Rightarrow H|0\rangle = 0 \Rightarrow \text{a ground-state}$$
Rokhsar-Kivelson point - triangular QDM

- **Topological sectors**

- 4 zero-energy eigenstates (ground-states)
- Finite correlation length
- Excited states not exactly known

\[
\begin{align*}
|1\rangle &= \sum_{c\in(++)} |c\rangle \\
|2\rangle &= \sum_{c\in(+-)} |c\rangle \\
|3\rangle &= \sum_{c\in(--)} |c\rangle \\
|4\rangle &= \sum_{c\in(-+)} |c\rangle \\
H|\psi\rangle &= 0
\end{align*}
\]

How can we prove that this degeneracy is due to some topological order, and is not due to some (hidden/complicated) conventional order?
Measuring the *local* difference between quantum states

Furukawa, GM, Oshikawa *PRL 2006*

- Measure how much two states are « different »:

\[
\text{diff}\left( |A\rangle, |B\rangle, \Omega \right) = \text{Max}_{\hat{O} \text{ defined on } \Omega} \quad \left| \langle A | \hat{O} | A \rangle - \langle B | \hat{O} | B \rangle \right|
\]

\[
= \ldots
\]

\[
= \sum |\lambda_n| \quad \text{where} \quad \rho_A^{\Omega} - \rho_B^{\Omega} = \sum_{\text{eigenvalue}} \lambda_n |n\rangle\langle n|
\]

- Generalize to more than two states:

\[
|1\rangle, |2\rangle, \ldots, |d\rangle \quad \text{degenerate ground-states}
\]

\[
D_\Omega\left( |1\rangle, |2\rangle, \ldots, |d\rangle \right) = \text{Max}_{|A\rangle=\sum \alpha_i |i\rangle} \text{diff} \left( |A\rangle, |\text{ref}\rangle, \Omega \right) \quad \text{with} \quad |\text{ref}\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle
\]

- \(D_\Omega \rightarrow 0 \): topological degeneracy
- \(D_\Omega = O(1) \): conventional spontaneous symmetry breaking

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TQC 2007 - Hamilton Mathematics Institute, September 2007

G. Misguich
« Local difference » between the degenerate ground-states

Ground-states of the triangular-lattice quantum dimer model at J=V=1 (RK point)
Scaling of the « local difference » $D_\Omega$

Ground-states of the triangular-lattice quantum dimer model at $J=V=1$ (RK point)

\[
D_\Omega = c \exp(aR - bN^{1/2})
\]

- $a = 2.00399$
- $b = 0.92821$
- $c = 2.23221$

⇒ The 4 ground-states cannot be distinguished by *any local observable* in the thermodynamic limit
Topological entanglement entropy in the liquid phase of the triangular lattice quantum dimer model
### Entanglement Entropy

**Reduced density matrix**

\[ \rho = \text{Tr}_{\Omega} \left[ \psi \langle \psi \rangle \right] \]

**Entanglement entropy**

\[ S = -\text{Tr} \left[ \rho \log \rho \right] \]

**Idea:**

The entanglement entropy of \( S \) has contributions

i) coming from **local correlations** in the vicinity of the **boundary**, plus

ii) possible **non-local** contributions.

In a topologically ordered wave-function, this non-local contribution "\( S_{\text{topo}} \)" may be universal (and equal to the **total quantum dimension**).

Entanglement entropy – *kagome* RK wave-function

\[ |\psi\rangle = \text{Equal amplitude superposition of all dimer coverings (on the kagome lattice)} \]

\[ |\psi\rangle = \frac{1}{\sqrt{N_{\text{tot}}}} \sum_{c \in \{\text{arrows config.}\}} |c\rangle \]

\[ S = \sum_{b \in \{\text{arrow config. at the boundary}\}} \log[p_b] \quad , \quad p_b = \frac{N_\Omega(b)N_{\overline{\Omega}}(b)}{N_{\text{tot}}} = \begin{cases} 2^{1-L} & \text{if } b \text{ has an even # of incoming arrows} \\ 0 & \text{otherwise} \end{cases} \]

\[ S = L \log(2) \quad \text{Boundary law (local correlations/constraints)} \]

\[ -\log(2) \quad \text{Non-local contribution (global parity constraint)} \]
Entanglement entropy - *triangular* RK wave-function

We observe a "boundary law", as expected in a gapped system with short-range correlations.

\[ S(R) \approx \alpha R + \gamma \]

But ambiguity \( \sim O(1) \) on \( R \) \( (R_{\text{min}} \neq R_{\text{max}}) \)

\( \Rightarrow \) how to extract \( \gamma \) ?
Topological entanglement entropy

\[ S(R) \approx \alpha R + \gamma \]

- Two ways to get rid of the local contributions to measure \( \gamma \):


\[
2S_{\text{topo}} = S_{ABCD} - S_{ABD} - S_{ABC} + S_{AB}
\]


\[
S_{\text{topo}} = S_{ABC} - (S_A + S_B + S_C)
\]
Topological entanglement entropy - *triangular* RK wave-function

- Levin and X.-G. Wen

![Diagram](a) S00

![Diagram](b) Levin-Wen, S30

Kitaev and Preskill

![Diagram](a) S00
Kitaev-Preskill construction – triangular RK wave-function

![Images of S00, S30, T30, and T90 wave-functions]

### Table: S00 and T30 Cases

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<tr>
<th>Radius $R$</th>
<th>$N=52$</th>
<th>$N=64$</th>
<th>Radius $R$</th>
<th>$N=52$</th>
<th>$N=64$</th>
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<td></td>
<td>0.9898</td>
<td></td>
</tr>
</tbody>
</table>
Thin-strip entanglement – kagome RK wave-function

Dimer covering on the kagome lattice ↔ arrow configurations
↔ non-intersecting loops on the honeycomb lattice

Equal-amplitude ground-state (config. can have an odd or even winding number):

$$|RK\rangle = (-) + (+) + \ldots$$

$$S_{\Omega}^{RK} = L_x \log(2)$$

Fixed-parity-sector (say even) ground-state

$$|+\rangle = (+) + (+) + \ldots$$

$$S_{\Omega}^{+} = L_x \log(2) - \log(2)$$

[Hamma et al. PRA 2005]
Topological entanglement entropy - *triangular* RK wave-function

- Use a « thin strip » winding around the system ⇒ obtain $\gamma = -\log(2)$ with 0.2 % accuracy.

![Diagram with labels](image-url)

\[
\text{Zigzag area, RK point}
\]

\[
\begin{align*}
S &= 0.827(1)l_x - 0.022(10) \\
S &= 0.828(2)l_x - 0.726(11)
\end{align*}
\]

\[-\ln 2 = -0.6931\]
Thin-strip construction – triangular RK wave-function

\begin{table}
\centering
\begin{tabular}{lccccc}
\hline
\multicolumn{6}{c}{\((S[RK] - S[RK; p])/\ln 2\)} \\
\hline
\(l_x(=l_y)\) & ++ & -+ & +- & -- \\
\hline
4 & 1.0024* & 0.8051 & 1.4910 & 0.8051 \\
6 & 1.0315 & 0.9248 & 1.0315 & 1.0212* \\
8 & 0.9944* & 1.0022 & 1.0017 & 1.0022 \\
10 & 0.9981 & 1.0028 & 0.9981 & 1.0011* \\
\hline
\end{tabular}
\end{table}
Summary

- Quantum dimer model version of Kitaev toric code:
  - solvable toy model
  - explicit connection between $Z_2$ topological order and short-range RVB physics

- Triangular-lattice quantum dimer model:
  - Numerical investigation of the “undistinguishable character” of the different ground-states.
  - One of the first numerical investigation of the “topological entanglement entropy” introduced by Levin-Wen & Kitaev-Preskill (2006).
  - Thin strip construction.
  - Confirmation of $\gamma = -\log(2)$. 

\[\Omega\]