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Degeneracy of the ground-state of antiferromagnetic spin-1/2 Hamiltonians

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Abstract. In the first part of this paper, the extension of the Lieb-Schultz-Mattis theorem to dimensions *larger than one* is discussed. A counter example to the original formulation of Lieb-Schultz-Mattis and Affleck is exhibited and a more precise statement is formulated. The degeneracy of the ground-state in symmetry breaking phases with long-range order is analyzed. The second and third parts of the paper concern resonating valence-bond (RVB) spin liquids. In these phases the relationship between various authors approaches: Laughlin-Oshikawa, Sutherland, Rokhsar and Kivelson, Read and Chakraborty and the Lieb-Schultz-Mattis-Affleck proposal is studied. The deep physical relation between the degeneracy property and the absence of stiffness is explained and illustrated numerically. A new conjecture is formed concerning the absolute absence of sensitivity of the spin liquid ground-states to any twist of the boundary conditions (thermodynamic limit). In the third part of the paper the relations between the quantum numbers of the degenerate multiplets of the spin liquid phases are obtained exactly. Their relationship with a topological property of the wave functions of the low lying levels of this spin liquid phase is emphasized. In spite of the degeneracy of the ground-state, we explain why these phases cannot exhibit spontaneous symmetry breaking.

PACS. 71.10.Fd Lattice fermion models (Hubbard model, etc.) – 75.10.Jm Quantized spin models – 75.40.-s Critical-point effects, specific heats, short-range order – 75.50.Ee Antiferromagnetics

1 Introduction

The question of the degeneracy of the ground-state of spin-$\frac{1}{2}$ Hamiltonians has a long history. One part of this history goes back to the beginning of the sixties when Lieb, Schultz and Mattis proved that a spin-$\frac{1}{2}$ antiferromagnetic periodic chain of length L has a low energy excitation of order $\mathcal{O}(1/L)$ [1]. This theorem (called in the following LSMA) was then extended by Lieb and Affleck to *half-integer* spin but fails for integer ones [2]. It states that in one dimension (1D), SU(2) invariant Hamiltonians with half-integer spins in the unit cell, either have gapless excitations or degenerate ground-states in the thermodynamic limit. The next important contributions to this long debated question came at the end of the eighties when various authors [3–6] showed that the ground-state and first excitations of short-range Resonating Valence-Bond (RVB) spin liquids with one spin-$\frac{1}{2}$ per unit cell were 4-fold degenerate on a two dimensional (2D) torus (periodic boundary conditions). The connection between the two statements in 1D was promptly established [7]. In fact Lieb Schultz and Mattis had suggested in their original paper that their 1D theorem could probably be extended to higher space dimensions: to support this conjecture, they developed an heuristic argument, that has been refined later on by Affleck [8].

In this paper we examine the various arguments that have been formulated for a 2D spin systems at the light of a large number of numerical results.

In Section 2, we study the situations with long-range order (either magnetic Néel long-ranged order or Valence Bond Crystals) and we show that these systems indeed have a degenerate ground-state in the thermodynamic limit. We study two examples in which one aspect of the LSMA conjecture appears to be incorrect in 2D.

In Section 3, we go through the analysis of a recent argument by Oshikawa [10]. We present a numerical analysis illustrating the relative part of exact results and conjecture in Oshikawa's argument and we show numerically

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that for the multiple-spin exchange (MSE) spin liquid Oshikawa's assumption is verified and the ground-state is 4-fold degenerate.

In Section 4 we consider the general case of short-range RVB spin liquids (RVBSL). In such systems the degeneracy is intimately related to the existence of topological sectors in the Hilbert space of short-range dimer coverings. In such a framework we determine the relations between the quantum numbers of the degenerate multiplets both for even×odd samples and for even×even ones. These results are illustrated on the quantum hard-core dimer (QHCD) model on the triangular lattice [9]. We finally explain why in spin liquids spontaneous symmetry breaking is impossible in spite of the degeneracy of the ground-state.

In the last section (Sect. 5), we summarize the most salient results of the present work and conclude. Some technical points are presented in two appendices.

2 The LSMA conjecture for 2D systems

2.1 The LSMA theorem for chains and stripes of vanishing aspect ratio

The LSMA theorem applies to spectra of quantum antiferromagnetic chains with periodic boundary conditions, and half-integer spin in the unit cell. The proof of the theorem is based on the construction of a low lying excitation, which may be pictured as a slow twist of the exact ground-state. Precisely the excitation is determined by the action on the exact ground-state $|\psi_0\rangle$ of the unitary operator U defined as:

$$U(\phi) = \exp\left(\mathrm{i}\frac{\phi}{L_x} \sum_{j=0}^{L_y-1} \sum_{n=0}^{L_x-1} n S^z_{n,j} \right), \qquad (1)$$

where $S^z_{n,j}$ represents the z component of the spin operator at site (n, j), the chain being of length L_x and width L_y. The low energy variational excitation reads:

$$\left|\theta_{2\pi}^{\mathrm{LSMA}}\right\rangle = U(2\pi)\left|\psi_0\right\rangle \cdot \qquad (2)$$

We will design as a column of the sample all the spins swept by the unit cell subjected to the L_y translations along y. Let us call $\mathcal{T}_x$ the operator for one-step translation in the x direction: the Hamiltonian is supposed to be translationally invariant $[\mathcal{H}, \mathcal{T}_x] = 0$. It is easy to show that $U(2\pi)$ anti-commutes with $\mathcal{T}_x$ as soon as the number of spins in a column is an odd integer [1,8]. In these conditions, if the ground-state wave-vector is $\mathbf{k}_0$, the variational excitation $\left|\theta_{2\pi}^{\mathrm{LSMA}}\right\rangle$ has a wave-vector $\mathbf{k}_0 + (\pi, 0)$: it is thus orthogonal to the ground-state.

The energy of this variational state is:

$$\left\langle\theta_{2\pi}^{LSMA}\right| \mathcal{H}_0 \left|\theta_{2\pi}^{LSMA}\right\rangle = E_0 + \alpha L_x L_y \left[1 - cos\left(\frac{2\pi}{L_x}\right)\right], \qquad (3)$$

where $E_0 = \langle\psi_0| \mathcal{H}_0 |\psi_0\rangle$ is the ground-state energy and α a finite quantity of order $\mathcal{O}(1)$ measuring the average value in the ground-state of spin-spin correlation. The LSMA theorem concerns short range Hamiltonians, for the simple nearest-neighbor Heisenberg case α is $-2/3$ times the ground-state energy per bond.

Equation (3) expresses the LSMA theorems: if the system under consideration is a chain, $\left|\theta_{2\pi}^{LSMA}\right\rangle$ is orthogonal to the ground-state and collapses onto it as $\mathcal{O}(1/L_x)$ when the chain length L_x goes to infinity. The property remains true for a stripe with an odd number of rows and an odd number of spins $\frac{1}{2}$ in the unit cell as soon as the aspect ratio L_y/L_x goes to zero when the size of the sample goes to infinity: the energy of the excited state is then $\mathcal{O}(L_y/L_x)$ above the ground-state. On the other hand, if the thermodynamic limit is taken with a non vanishing aspect ratio, the gap between the ground-state and the variational excitation does not close in 2D and higher dimensions.

Faced to this alternative, Lieb *et al.* [1] expressed the conjecture that the degeneracy observed in the odd stripes with vanishing aspect ratio could be a true physical property of the 2D samples with aspect ratio equal to 1. They wrote: "Because the excitation energy of exact low-lying states should not depend on the shape of the entire lattice, there should be no energy gap for a lattice of $N \times N$ sites either". Elaborating on this consideration, Affleck [8] expressed essentially the same conjecture.

In the following sections, we want to precise these statements at the light of a numerical analysis of the exact spectra of various generic 2D quantum antiferromagnets. To discuss the validity of the above-mentioned conjecture for dimensions larger or equal to 2, we extract from it different proposals with increasing specificity:

- proposal A: The ground-state of an antiferromagnetic system with an odd number of spins $\frac{1}{2}$ in the unit cell is degenerate in the thermodynamic limit.
- proposal B: If $\mathbf{k}_0$ is the wave-vector of the ground-state there are at least 2 additional eigenstates with wave vectors $\mathbf{k}_0 + (\pi, 0)$ and $\mathbf{k}_0 + (0, \pi)$ that collapse onto the ground-state in the thermodynamic limit.
- proposal B$'$: In the thermodynamic limit the unit cell of the ground-state is enlarged by a factor 2^d, where d is the dimension of the lattice. This statement is a bit more precise that statement B: in 2D it implies the degeneracy of the ground-state with eigenstates of wave vectors: $\mathbf{k}_0 + (\pi, 0)$, $(0, \pi)$ and $\mathbf{k}_0 + (\pi, \pi)$.
- proposal C: A good variational estimate of the excited states of proposal B can be obtained by a twist of the ground-state (to be specified in Sect. 2.4).

2.2 Proposal A and spectra of different generic 2D quantum antiferromagnets

At $T = 0$ the ground-state of an antiferromagnetic system can be Néel ordered (the spin-spin correlations exhibit long-range order), a Valence-Bond-Crystal (VBC) (the dimer-dimer correlations show long-range order) or an RVBSL state with no long-range order [11].

In any of these cases and for various reasons the ground-state is degenerate in the thermodynamic limit.

In the first two-cases the symmetry breaking of the macroscopic ground-state implies a degeneracy of the ground-state in the thermodynamic limit.

i) In the case of Néel long-range order, there are both degenerate states (which form the true thermodynamic ground-state) and gapless excitations. The gapless excitations, the antiferromagnetic magnons, are the Goldstone mode associated to the broken continuous SU(2) symmetry. Whereas it is well known that the softest magnons scale as $\mathcal{O}(1/L)$, it is sometimes taken for granted that these collective "first excitations" are the first excited levels of the multi-particle spectra. This is indeed false: the "$T = 0$ Néel ground-state" (or "vacuum of excitations") is itself a linear superposition of $\sim N^{\alpha}$ eigenstates of the $SU(2)$ invariant Hamiltonian which collapse to the ground-state as $\mathcal{O}(1/N)$ (α is the number of sub-lattices of the Néel state, L is the linear size of the sample and N the number of sites of the sample): this is the tower of states invoked in P. W. Anderson 1952 famous paper [12–15].

ii) In the case of a discrete broken symmetry, as for example the space symmetry breaking associated to long-range dimer or plaquette order (VBC) [16–22], or in the case of T-symmetry breaking associated to long-range order in chirality, the vacuum of excitations is also degenerate in the thermodynamic limit. The spectrum of a finite-size sample shows a quasi-degeneracy of the ground-state. The number of quasi degenerate states is finite and a function of the symmetry of the order parameter, it is independent of the sample size. The collapse of these quasi-degenerate levels on the ground-state is supposed to be exponential with the size of the lattice.

iii) In the last case of an RVB state with no long-range order we have found two generic situations:

α) A spin liquid with a gap to all excitations with up to now two examples: a phase of the multiple-spin exchange Hamiltonian on the triangular lattice [23], and a phase of the frustrated $J_1 - J_2$ model on the honeycomb lattice [21] (we shall call this phase a type I spin liquid). Finite size studies of these cases point to a degenerate ground-state on the triangular lattice and a unique one on the honeycomb lattice. As explained in Section 4, this difference comes from the fact that the triangular lattice has one spin $\frac{1}{2}$ per unit cell and the honeycomb lattice has 2.

β) A spin liquid with a spin gap filled with a continuum of singlets on the kagomé lattice [24], and on the triangular lattice [25]. In the last two cases the continuum of singlets appears to be adjacent to the ground-state (type II spin liquid).

In any of these cases Proposal A appears to be true, and the restriction to problems with half-odd integer spins in the unit cell necessary (as can be seen on the example of the $J_1 - J_2$ model on the honeycomb lattice [21]).

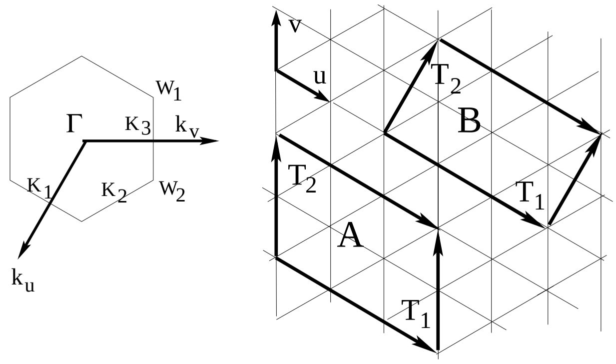

Fig. 1. Brillouin zone of the triangular lattice: wave-vectors $\mathbf{K_1} = (\pi, 0)$ frequently invoked in the text are shown in the Brillouin zone. When the C_{6v} symmetry of the sample is preserved, eigenstates with wave-vectors $\mathbf{K_1}$, $\mathbf{K_2}$, $\mathbf{K_3}$ are degenerate. The two kinds of samples used in Section 2.3 are designed as A, and B.

2.3 Proposal B and B': a thorough study of one counter-example

Proposal B$'$ is valid when the macroscopic ground-state unit cell is multiplied by a factor 2^d. This is for example the case of the 4-spin $S = 0$ plaquette order of the $J_1 - J_2$ model on the square lattice for $J_2/J_1 \sim 0.5$ [1]. Proposal B' is not valid in the ground-state of the J_2/J_1 model for $J_2/J_1 \gg 0.5$. In that case the ground-state is a two-sublattice columnar antiferromagnet which spontaneously breaks the translation and $\pi/2$ rotation symmetries. It is easy to check that proposal B' is not satisfied in that case since the spatial symmetry breaking in that phase is incompatible with momentum $\mathbf{k} = (\pi, \pi)$. The four-fold ground-state degeneracy is obtained by a combination of *two* states with zero momentum, one with $\mathbf{k} = (0, \pi)$ and one with $(\pi, 0)$.

We will now show that even the weaker proposal B can be false in true 2D spin systems and that, contrary to LSMA assumption, a 2D sample with vanishing aspect ratio can have a mathematical spectrum different from that of the true 2D samples. To illustrate this assumption we have done a thorough numerical study of the exact spectra of the Heisenberg model on the triangular lattice for various shapes and sizes of the samples.

The Heisenberg Hamiltonian with nearest neighbor interactions reads:

$$\mathcal{H}_0 = 2 \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j. \tag{4}$$

The Bravais lattice is defined by the two unit vectors $\mathbf{u}$ and $\mathbf{v}$. The reciprocal lattice is hexagonal: the lattice and Brillouin zone are displayed in Figure 1. The sample is defined by two vectors:

$$\mathbf{T}_{1(2)} = l_{1(2)}\mathbf{u} + m_{1(2)}\mathbf{v}. \tag{5}$$

[1] The existence of a VBC phase for such a coupling is rather well documented, but the exact nature of the order is still under debate [16,26,17,27].

Periodic boundary conditions read:

$$\mathbf{S}(\mathbf{R}_i + \mathbf{T}_j) = \mathbf{S}(\mathbf{R}_i). \quad (6)$$

For a given sample the wave-vectors compatible with the boundary conditions depend on $\mathbf{T}_1, \mathbf{T}_2$.

Previous studies on samples with aspect ratio equal to 1 [15] have established that this model has a three sublattice Néel order. The Anderson tower of states collapsing to the absolute ground-state as $1/N$ encompass eigenstates with wave-vectors: $\mathbf{k} = \mathbf{0}$ and $\mathbf{k} = \pm(2\pi/3, -2\pi/3)$. The low-lying magnons have wave vectors in the neighborhood of the points Γ and W_i of the Brillouin zone of the system (see Fig. 1). Due to the overall consistency of the picture, we suspected that eigenstates with wave-vectors $(0, \pi)$ or $(\pi, 0)$ (points $\mathbf{K_i}$ of Fig. 1) appear only in components of energetic magnons.

In this work we report a thorough numerical study of the behavior of the first $\mathbf{k} = \mathbf{k}_0 + (\pi, 0)$ exact eigenstate of this model as a function of the aspect ratio and size of the samples ($\mathbf{k}_0$ is always the wave vector of the absolute ground-state for the sample under consideration).

We have studied samples with various values of (L_x, L_y) (even, odd), (even, even) accommodating eigenstates with wave-vector $(\pi, 0)$. For a given aspect ratio two kinds of samples have been studied:

- The simplest one has its sides parallel to the primitive vectors of the lattice: $\mathbf{T}_1 = L_x\mathbf{u}, \mathbf{T}_2 = L_y\mathbf{v}$. If L_x or $L_y = 1, 2 \bmod 3$, PBC on such samples frustrate the natural Néel order of the lattice: the ground-state is usually not a $\mathbf{k} = \mathbf{0}$ state and the spectrum is not typical of Néel order. We nevertheless study these samples which are perfectly reasonable samples with respect to the LSMA conjecture (A sample of Fig. 1).
- If one dimension only is not multiple of 3, one may use a different shape, with the same aspect ratio, where PBC do not frustrate Néel order (see B sample of Fig. 1 and double starred samples in Tab. 1). On these non frustrating samples (NFS) the absolute ground-state is a $\mathbf{k} = \mathbf{0}$ state and the spectrum has the typical structure of a Néel ordered system. On such samples one can also build the variational state described by equation (2). It has a wave-vector $(\pi, 0)$, and its gap to the absolute ground-state is $3\alpha L_y/L_x$ (to be compared with equation (3) for an A sample).

This study leads to the following observations (see Tab. 1):

- For a given size the spin gap Δ_s has a weak dependence with the sample shape (it is especially weak on NFS). It decreases with the sample size as expected for a Néel ordered system (see Fig. 2):

$$\Delta_s = \frac{1}{\chi N}\left(1 - \beta\frac{c}{\rho\sqrt{N}}\right) + \mathcal{O}\left(\frac{1}{N^2}\right) \quad (7)$$

where χ is the homogeneous spin susceptibility, c the spin-wave velocity, ρ the spin stiffness and β is a number of order one. The first triplet excited states of NFS have wave-vectors $\mathbf{k} = \pm(2\pi/3, -2\pi/3)$ and $\mathbf{k} = \mathbf{0}$.

Table 1. Gaps of various excited states (from the absolute ground-state): Δ_s, spin gap; Δ_0, gap between the absolute ground-state (with wave vector $\mathbf{k_0}$) and the first excited state with wave vector $\mathbf{k_0} + (\pi, 0)$ in the $S = 0$ sector; Δ_1, gap between the absolute ground-state and the first state with wave vector $\mathbf{k}_0 + (\pi, 0)$ in the $S = 1$ sector; δ, is the ratio of Δ_0 to the aspect ratio of the sample L_y/L_x. In the last column we indicate if $|\mathbf{k}_0 + (\pi, 0), S = 0\rangle$ is the first excited state of the spectrum. The lines with one or two (*) indicate A or B samples with PBC compatible with 3-sublattice Néel long-range order (See Fig. 1 for the double-starred samples).

N	L_x	L_y	Δ_s	Δ_0	Δ_1	δ	
12*	$\sqrt{12}$	$\sqrt{12}$	1.726	2.471	2.836	2.47	no
18*	6	3	1.131	1.483	2.882	2.96	no
24	8	3	0.942	0.474	1.486	1.26	yes
24	6	4	1.193	1.424	1.697	2.13	no
24**	8	3	0.926	1.034	2.073	2.75	no
24**	6	4	0.922	1.061	2.175	1.59	no
30	10	3	0.799	0.344	1.183	1.15	yes
30	6	5	0.825	0.762	1.652	0.91	yes
30**	10	3	0.782	0.732	1.618	2.43	yes
30**	6	5	0.776	1.067	2.003	1.28	no
36*	12	3	0.692	0.548	1.324	2.19	yes
36*	6	6	0.688	1.359	1.785	1.36	no

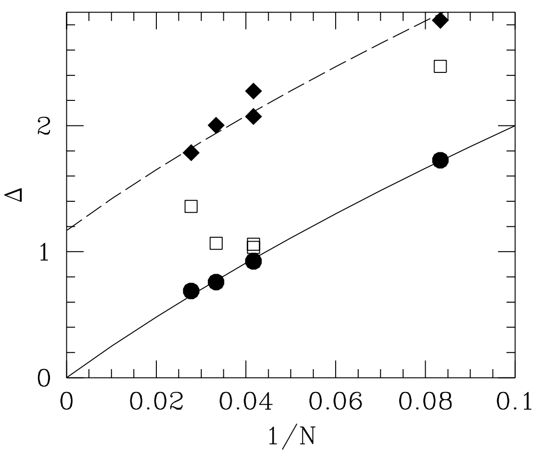

Fig. 2. Finite-size scaling of various gaps to excited states. Black circles: Δ_s gap to the first $S = 1$ state (it has wave-vector $W_{1(2)}$ in the Brillouin zone of Fig. 1). Open squares: Δ_0 gap to the first $|(\pi, 0), S = 0\rangle$ state (points K_i in the Brillouin zone). Black diamonds: Δ_1 gap to the first $|(\pi, 0), S = 1\rangle$ state.

- The gap Δ_0 to the first excited state with total spin $S = 0$ and wave-vector $\mathbf{k}_0 + (\pi, 0)$ behaves differently. (This state is the first exact state with the same wave-vector as $\left|\theta_{2\pi}^{LSMA}\right\rangle$).
- Δ_0 is small for small aspect ratio and for a given size is very sensitive to the aspect ratio and to the boundary conditions. For a small aspect ratio this state is indeed the first excited state of the exact spectrum (last column of Tab. 1).

- For a given size and aspect ratio, Δ_0 increases when going from frustrating periodic boundary conditions, to boundary conditions compatible with the three sublattice Néel order (except for the 6×4 sample, where Δ_0 is always large).
- For a given size, Δ_0 increases with the sample aspect ratio: whatever the size of the sample this gap is always of the same order of magnitude as the aspect ratio (6th column of Tab. 1).
- Let us consider now the two samples with 6×5 and 6×6 sites which do not frustrate Néel order (double starred and starred lines of Tab. 1). When going from the $N = 30$ spectrum to $N = 36$ all gaps decrease, which seems natural, except the Δ_0 gap which increases noticeably. This is a second indication of the strong dependence of the position of this specific level on the sample's aspect ratio.
- We finally looked at the size and shape effects on the gap Δ_1 to the first state with wave-vector $(\pi, 0)$ and total spin $S = 1$: $|(\pi,0), S = 1\rangle$ (Let us recall that $\left|\theta_{2\pi}^{LSMA}\right\rangle$ has not a fixed total spin). For any size and shape $\Delta_1 > \Delta_0$. Δ_1 data corresponding to NFS with the largest aspect ratios obey the same finite size scaling law as the Δ_s spin gap (Fig. 2): this is exactly what is expected for a $(\pi, 0)$ magnon excitation (the magnon excitation could be written as a twist of a symmetry-breaking ground-state, which itself is a coherent superposition of the low lying levels of the Anderson tower of states). Extrapolation to the thermodynamic limit gives a energy of 1.17 (the linear spin wave theory gives the value 2, but all the ED data point to a softening of the spin-wave spectrum by quantum fluctuations[2]).
- It is difficult to assign a value to the thermodynamic limit of the gap Δ_0 to the first $|(\pi,0), S = 0\rangle$ state for NFS with the largest aspect ratio (see Fig. 2) (two samples only, 12 and 36 have aspect ratio equal to 1): however examination of all the data pleads in favor of a non zero gap. If the semi-classical picture is valid, the state $|(\pi,0), S = 0\rangle$ belongs to the tower of states of the one-magnon excitation of wave vector $(\pi, 0)$, and in the thermodynamic limit

$$\lim_{N\to\infty} (\Delta_0) = \lim_{N\to\infty} (\Delta_1)\,. \tag{8}$$

The present data are too scarce to sustain this last assertion but they do not go against it.

This study suggests that:

i) The thermodynamic limit: $L_x, L_y \to \infty$ with $L_y/L_x \to 0$ is special, it could depend on the number of rows in the sample (see sample 6×4) and is different from the limit: $L_x, L_y \to \infty$ with $L_y/L_x = \mathcal{O}(1)$.

ii) The first state $|(\pi,0), S = 0\rangle$ which collapses to the ground-state in the quasi one-dimensional system is not a "natural" excitation of the 2D system: the natural excitations are the magnons and the magnon with the wave-vector $(\pi, 0)$ is not a soft mode as soon as the period of the symmetry-breaking ground-state is not even (as in the case above, where the period is 3).

iii) The nearest neighbor Heisenberg model on the triangular lattice appears to be a serious counter-example to proposal B and B$'$ for a true 2D system. We suspect that the same model on the square lattice would equally be a counter-example to these proposals. We have not done a full size and shape analysis of this case as we have done for the triangular lattice. But we have verified that for samples with non zero aspect ratios the Anderson tower of states encompasses eigenstates with wave-vectors $\mathbf{k} = (0, 0)$ and $\mathbf{k} = (\pi, \pi)$ as expected, and the $\mathbf{k} = (\pi, 0)$ and $\mathbf{k} = (0, \pi)$ are components of energetic magnons [3].

2.4 Proposal C

Equation (3) implies that in a true 2D system it is in general impossible to build a low-lying excitation using the LSMA twist operator $U(2\pi)$. In reference [8], Affleck discussed the general possibility to build a one-dimensional excitation in a two-dimensional medium. We do not know how to do this in a general case, and as explained in the previous section, we suspect that this quest is in general doomed to failure.

However, in the specific case of spin liquids, it could be possible to build such an excitation, inasmuch as the stiffness of a 2D spin liquid vanishes when the size of the sample goes to infinity.

In an ordered system, the stiffness ρ, which measures the response of the sample $L_x \times L_y$ to a twist ϕ of the boundary condition in the x direction, can be defined as

$$E(\phi) = E(0) + \frac{N}{2}\rho\left(\frac{\phi}{L_x}\right)^2 + \mathcal{O}(\phi^4) \tag{9}$$

with $E(\phi)$ the total energy of the sample. Far from the quantum critical point the stiffness is usually of the order of magnitude of the coupling constant ($\sim 1/9$ for the Heisenberg model on the triangular lattice).

Let us now consider the following one-dimensional perturbation of the ground-state with PBC: a constraint on line number one is forced with the twist operator

$$u^1(2\pi) = \exp\left(\mathrm{i}\frac{2\pi}{L_x}\sum_{n=0}^{L_x-1} n S^z_{n,1}\right) \tag{10}$$

[2] This softening due to quantum fluctuations always noticeable on the stiffness and spin-wave velocity, seems even larger on the maxima of the dispersion curves, as it has also be noticed on the square lattice [28].

[3] The first excited state of the spectrum is an $S = 1$ $\mathbf{k} = (\pi, \pi)$ eigenstate: its gap to the ground-state goes to zero linearly with the size of the sample; its value is 0.630 (resp. 0.575) in the $N = 32$ (resp. $N = 36$) sample. The first eigenstate with a wave-vector $\mathbf{k} = (\pi, 0)$ has a gap to the ground-state of 4.885 (resp. 4.850) in these two samples: it is triplet ($S = 1$). The first eigenstate with $S = 0$ and $\mathbf{k} = (\pi, 0)$ has a huge gap to the ground-state of 5.195 (resp. 5.075).

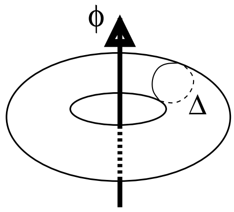

Fig. 3. 2-torus with one cut Δ.

and the system is supposed to relax around this default at fixed momentum (Eq. (10) insures that the variational state has momentum $(\pi, 0)$).

In an ordered system with stiffness ρ the variational energy of such an excitation could be approximated as:

$$E \sim E_0 + \frac{1}{2}\, \beta\rho \frac{L_y}{L_x}\, (\pi)^2\,, \tag{11}$$

with β a number of order one. In a true 2D ordered system the correction to the ground-state energy can never go to zero for a non-zero aspect ratio. On the contrary, in a spin liquid sample, with linear dimensions larger than the correlation length, we conjecture that the second term of the above estimate goes to zero with the stiffness.

This heuristic argument is in agreement with the behavior of the exact spectra described in the next sections. The key ingredient to discriminate between the two cases is the presence of a finite (or vanishing) stiffness: a fact intimately related to the central hypothesis of Oshikawa that we will discuss in the following section.

3 Oshikawa's argument

In a recent paper [10] Oshikawa discussed the consequences of the presence of a gap in a quantum many-particle system with conserved particle number on a periodic lattice in arbitrary dimensions. Inspired by Laughlin's topological arguments for the quantum Hall effect, Oshikawa argues that a finite excitation gap is possible only when the particle number per unit cell of the ground-state is an integer. The above-mentioned formulation translated for a spin system reads: "A spin system can have a gap only if the spin in the ground-state unit cell is integer" and Oshikawa statement for a spin-$\frac{1}{2}$ system is "a spin-$\frac{1}{2}$ system can have a gap only if, the translational symmetry is spontaneously broken": the ground-state unit cell is thus adequately enlarged to be consistent with the previous statement.

To support his statement Oshikawa developed an argument à la Laughlin involving the adiabatic transform of the ground-state under the action of a fictitious magnetic flux piercing the ring of the sample submitted to periodic boundary conditions (Fig. 3). In this section we illustrate this adiabatic transform on a spin system. This transform is here realized in the spin language by twisting the boundary conditions (Sect. 3.1). We then show the general mathematical properties of the spectrum of *any* spin Hamiltonian under such an adiabatic transform (Sect. 3.2 and Appendix A).

Oshikawa's result is based on a crucial hypothesis that the gap from the ground-state manifold to excited state *does not close during the adiabatic transform.* In Section 3.3 we illustrate this assumption on exact spectra of the MSE Hamiltonian, in a range of parameter where this model exhibits a RVBSL phase in the thermodynamic limit (see Appendix B for the precise definition of the model and the range of parameters where it exhibits a spin liquid phase). We postpone the discussion of the second issue of Oshikawa's paper, *i.e.* the discussion of spontaneous symmetry breaking to the next section.

3.1 Adiabatic transform of a spin-system underthe twist of the boundary conditions

If the spin-$\frac{1}{2}$ degrees of freedom are represented by hard-core bosons (Holstein-Primakoff), coupling these bosons to a fictitious flux is completely equivalent, in the spin language, to a *twist* of the boundary conditions. Generalized twisted boundary conditions are defined by the choice of the twist axis (here the z axis in the original $\mathcal{B}_0$ spin frame) and a set of 2 angles ϕ_j as:

$$\mathbf{S}(\mathbf{R}_i + \mathbf{T}_j) = \mathrm{e}^{\mathrm{i}\phi_j S^z(\mathbf{R}_i)} \mathbf{S}(\mathbf{R}_i) \mathrm{e}^{-\mathrm{i}\phi_j S_z(\mathbf{R}_i)} \tag{12}$$

(see Fig. 1 for the definitions of T_1 and T_2). For the sake of simplicity we will illustrate the general properties of this transform on the case of a twist ϕ in the $\mathbf{T}_1$ direction for the simplest $SU(2)$ invariant Hamiltonian: the nearest neighbor Heisenberg Hamiltonian (extension to a more complex and relevant Hamiltonian is just a question of notations). With periodic boundary conditions the Hamiltonian in the $\mathcal{B}_0$ spin frame reads:

$$\mathcal{H}_0 = \sum_{p=0}^{L_y-1} \sum_{n=0}^{L_x-1} \left(\mathbf{S}_{n,p} \cdot \mathbf{S}_{n+1[L_x],p} + \mathbf{S}_{n,p} \cdot \mathbf{S}_{n,p+1[L_y]}\right). \tag{13}$$

A twist ϕ in the $\mathbf{T}_1$ direction implies the calculation of the eigenstates of

$$\begin{aligned} \mathcal{H}_\phi = &\sum_{p=0}^{L_y-1} \left(\mathbf{S}_{L_x-1,p} \cdot \mathbf{S}^\phi_{0,p} + \mathbf{S}_{L_x-1,p} \cdot \mathbf{S}_{L_x-1,p+1[L_y]}\right) \\ &+ \sum_{p=0}^{L_y-1} \sum_{n=0}^{L_x-2} \left(\mathbf{S}_{n,p} \cdot \mathbf{S}_{n+1,p} + \mathbf{S}_{n,p} \cdot \mathbf{S}_{n,p+1[L_y]}\right) \\ = &\,\mathcal{H}_0 + \frac{1}{2} \sum_{p=0}^{L_y-1} \left((\mathrm{e}^{\mathrm{i}\phi} - 1) S^-_{L_x-1,p} S^+_{0,p} + \mathrm{h.c.}\right) \end{aligned} \tag{14}$$

where

$$\mathbf{S}^\phi_{0,p} = \mathrm{e}^{\mathrm{i}\phi S^z_{0,p}} \mathbf{S}_{0,p} \mathrm{e}^{-\mathrm{i}\phi S^z_{0,p}}. \tag{15}$$

The ϕ twist term in equation (14) is equivalent to the phase factor induced by a magnetic flux on the kinetic term of charged particles in Laughlin's transform. Under an adiabatic twist of the boundary conditions the spectrum of $\mathcal{H}_\phi$ (Eq. (14)) evolves periodically with a period 2π. But the eigenstates evolution could be more complicated. There is no guaranty that the many-body ground-state of $\mathcal{H}_{\phi=0}$ adiabatically transforms into the ground-state of $\mathcal{H}_{\phi=2\pi}$. As we will show below, this is generally not the case and the true periods of the eigenstates are in general multiples of 2π.

3.2 Twisted boundary conditions and translation invariance

To follow adiabatically an eigenstate of a many particle spectrum while twisting the boundary conditions could seem difficult because of level crossings during the twist. In fact there is an unique, unambiguous way to do it as these eigenstates belong to one-dimensional representations of a translation symmetry group. Diagonalization in such subspaces leads to non degenerate eigenstates which never cross. The translational invariance of the problem is hidden in its representation in the $\mathcal{B}_0$ frame but can be restored by a gauge transform.

Let us define $\mathcal{B}_\phi$, the new frame deduced from the original $\mathcal{B}_0$ by the spatially dependent twist described by the unitary operator:

$$U(\phi) = \exp\left(\mathrm{i}\frac{\phi}{L_x}\sum_{p=0}^{L_y-1}\sum_{n=0}^{L_x-1} n S^z_{n,p}\right). \tag{16}$$

In this new frame, the twisted Heisenberg Hamiltonian reads:

$$\tilde{\mathcal{H}}_\phi = U(\phi)\mathcal{H}_\phi U(\phi)^{-1}; \tag{17}$$

(in this equation and in the following, we put a tilde on the observables measured in the $\mathcal{B}_\phi$ frame. We will indicate by a superscript ϕ the eigenstates of $\tilde{\mathcal{H}}_\phi$). This unitary transform is chosen so that the boundary terms in equation (14) disappear:

$$\tilde{\mathcal{H}}_\phi = \mathcal{H}_0 + \frac{1}{2}\sum_{p=0}^{L_y-1}\sum_{n=0}^{L_x-1}\Big((\mathrm{e}^{\mathrm{i}\phi/L_x}-1)S^-_{n,p}S^+_{n+1[L_x],p} + \mathrm{h.c.}\Big). \tag{18}$$

$\tilde{\mathcal{H}}_\phi$ is translation invariant: $\left[\tilde{\mathcal{H}}_\phi, \mathcal{T}_{x(y)}\right] = 0$, where $\mathcal{T}_{x(y)}$ are the operators for one-step translations in the x (resp. y) directions. Its spectrum is indeed identical to the spectrum of $\mathcal{H}_\phi$. After this gauge transform, we can now define one-dimensional irreducible representations of the translation group labeled by their wave-vectors $\mathbf{k}$ in the $\mathcal{B}_\phi$ frame and adiabatically follow a $\mathbf{k}$ eigenstate of $\tilde{\mathcal{H}}_\phi$ in the successive $\mathcal{B}_\phi$ frames while increasing ϕ. This is what is done in Figures 4 and 5, where we see that the true period of the many-body ground-state $|\mathbf{k}=(0,0), S^z_{\mathrm{tot}}=0\rangle^\phi$ and of the first exact $|\mathbf{k}=(\pi,0), S^z_{\mathrm{tot}}=0\rangle^\phi$ state of the 8×3 sample (resp. of the 4×5 one) is 4π.

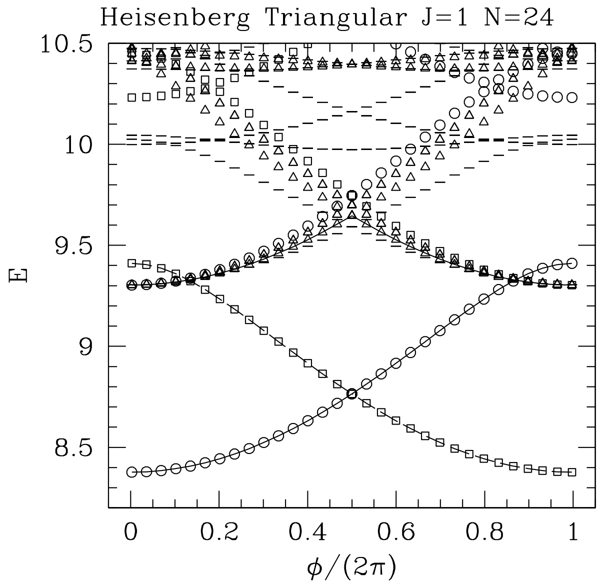

Fig. 4. Low-lying spectrum of the 8×3 sample of the Néel ordered Heisenberg model on the triangular lattice as a function of the twist ϕ at the x boundary of the sample. Circles are for $|S^z_{\mathrm{tot}}=0,\mathbf{k}=\mathbf{0}\rangle^\phi$ states and squares stand for states with $|S^z_{\mathrm{tot}}=0,\mathbf{K}_1\rangle^\phi$ (wave-vector $(\pi,0)$). Triangles represent the lowest $S^z_{\mathrm{tot}}=1$ state (whatever its wave vector). Continuous lines are drawn through the first $|S^z_{\mathrm{tot}}=0,\mathbf{k}=\mathbf{0}\rangle^\phi$ state, the first $|S^z_{\mathrm{tot}}=0,\mathbf{K}_1\rangle^\phi$ and the lowest $S^z_{\mathrm{tot}}=1$ state thus giving the spin gap. Due to the finite size, the $\mathbf{k}$ wave vector of this lowest $S^z_{\mathrm{tot}}=1$ state changes with the twist angles (explaining the discontinuities in the slope of the curve) and with the sample size.

This property is an exact property of the $S^z_{\mathrm{tot}}=0$ eigenstates on even$\times$odd samples. A straightforward algebra (computation of $\mathcal{T}_{x(y)}U(2\pi)\mathcal{T}^{-1}_{x(y)}$), shows that $U(2\pi)$ maps a state with wave-vector $\mathbf{k}$ and total spin S^z_{tot} onto a state with the same z-component of the total spin but with a wave vector $\mathbf{k}'$ given by [4]:

$$\begin{aligned} k'_x &= k_x + 2\pi\frac{S^z_{\mathrm{tot}}}{L_x} + \pi\,[L_y \bmod 2] \\ k'_y &= k_y. \end{aligned} \tag{19}$$

From these relations one immediately deduces that a 2π twist on an even$\times$odd sample maps the absolute ground-state $|\mathbf{k}=(0,0),S^z_{\mathrm{tot}}=0\rangle$ of $\mathcal{H}_0$ onto the first $|\mathbf{k}=(\pi,0),S^z_{\mathrm{tot}}=0\rangle$ eigenstate and reversely, *whatever*

[4] This relation has also been used to generalize to $S^z_{\mathrm{tot}}\neq 0$ the LSMA argument in order to discuss magnetization plateaus in one dimension [29].

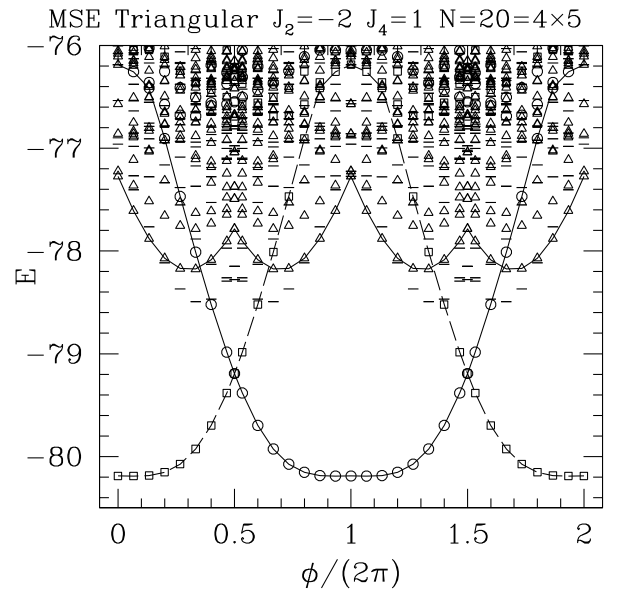

Fig. 5. Low-lying spectrum of the 4×5 sample of the multiple-spin exchange model (Eq. (55)) as a function of the twist ϕ at the x boundary of the sample. Same symbols as in Figure 4. We suspect that for this spin liquid, in the thermodynamic limit the wave vector of the first $S^z_{\rm tot}=1$ state is $(\pi/2,0)$. But in the finite small samples that are available we never have this wave vector (except in the 4×6 sample): this explains the discontinuities in the shape of the lowest $S^z_{\rm tot}=1$ state.

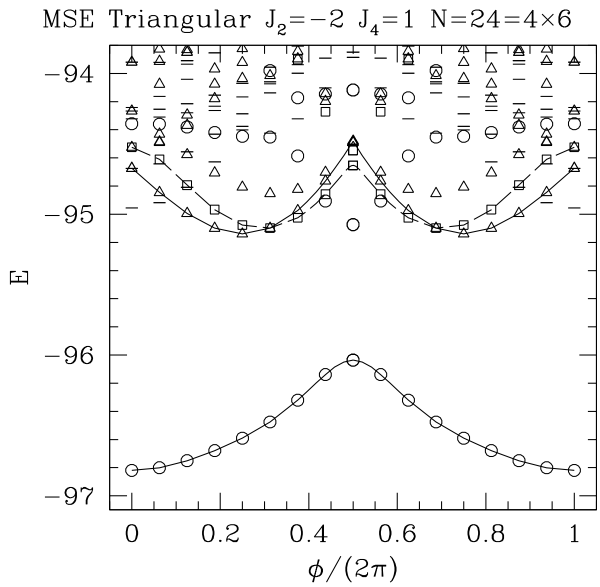

Fig. 6. Low-lying spectrum of the 4×6 sample of the multiple-spin exchange model (Eq. (55)) as a function of the twist ϕ at the x boundary of the sample. Same symbols as in Figure 4.

the nature of the Hamiltonian and the intrinsic properties of its ground-state. Figure 4 is a spectrum of a Néel ordered phase and Figures 5 and 8 of spin liquid phases.

More generally the $S^z_{\rm tot}=0$ eigenstates have periodicity 4π on even×odd samples (see Figs. 4, 5 and 8) and 2π on even×even ones (Figs. 6, 7) . Let us finally remark that all eigenstates of a given problem do not have the same periodicity: the periodicity depends both on the z component of the total spin of the eigenstate under consideration and on its wave-vector (Eqs. (19)). Additional information on the quantum numbers associated with the point-group symmetries can be found in Appendix A.

3.3 Robustness of the gap in presence of twisted boundary conditions in spin liquids

To establish the existence of a ground-state degeneracy in a gapped system, Oshikawa makes the following assumption: the ground-state of $\mathcal{H}$ can only be transformed in itself or into another degenerate ground-state. In other words, he assumes that the gap from the ground-state manifold to excited states does not close during the adiabatic evolution. Thanks to the results of the previous subsection, this hypothesis directly implies that in even×odd samples the $|S^z_{tot}=0,\mathbf{k}=(0,0)\rangle$ absolute ground-state is degenerate with the lowest $|S^z_{tot}=0,\mathbf{k}=(\pi,0)\rangle$ eigenstate. *If we take for granted that in the thermodynamic limit the spectra of even× odd and even× even samples are identical*, Oshikawa's assumption implies Proposal B'[5].

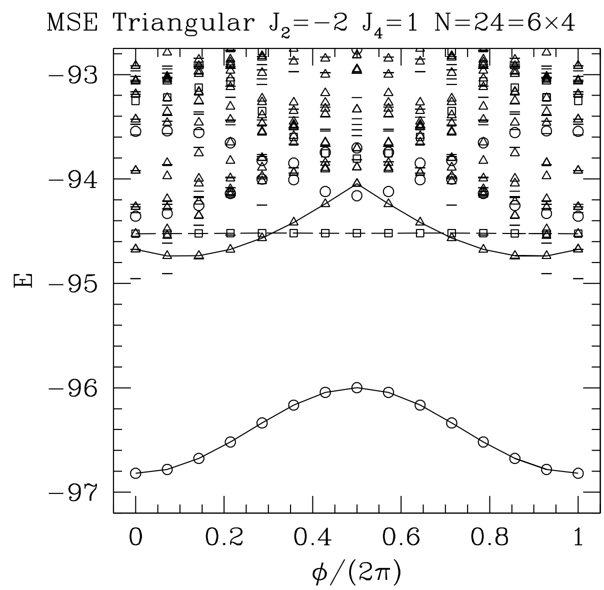

Fig. 7. Low-lying spectrum of the 6×4 sample of the multiple-spin exchange model (Eq. (55)) as a function of the twist ϕ at the x boundary of the sample. Same symbols as in Figure 4.

[5] In fact we will show below that the four-fold degeneracy of the RVBSL can sometimes be realized in even×odd and even×even samples with different irreducible representations of the space group.

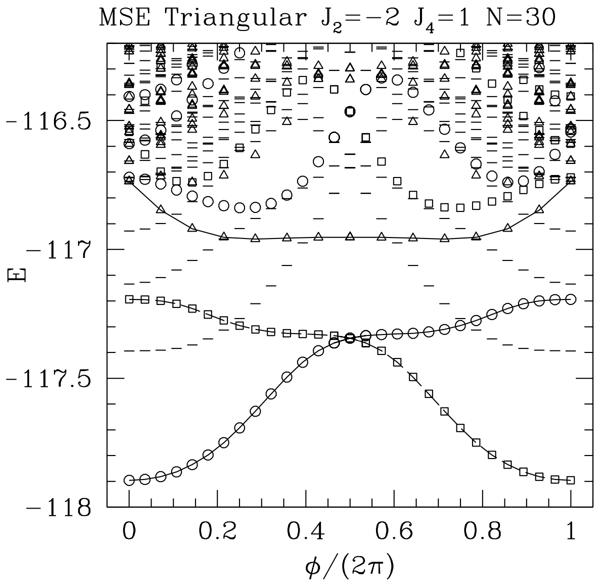

Fig. 8. Low-lying spectrum of the 6×5 sample of the multiple-spin exchange model (Eq. (55)) as a function of the twist ϕ at the x boundary of the sample. Same symbols as in Figure 4. On this small sample the lowest $S^z_{\rm tot}=1$ state is a $|S^z=1,\mathbf{k}=(2\pi/3,0)\rangle$. We suspect that in the thermodynamic limit the lowest $S^z=1$ eigenstate has a wave-vector belonging to the star of $(\pi/2,0)$ (these wave-vectors are not allowed in the present sample).

Table 2. Spin liquid spectra (MSE model defined in Appendix B). Δ_0 measures the gap between the absolute ground-state with wave vector $(0,0)$, and the first excited state with wave vector $(\pi,0)$ in the $S=0$ sector; $\Delta_0^{'}$ measures the increase in the absolute ground-state total energy in a π twist of the boundary conditions; Δ_S is the spin gap.

N	20	24	28	30	36
Δ_0	4.0027	2.2944	1.5683	0.7026	0.2527
$\Delta_0^{'}$	0.9967	0.8195	0.6958	0.5615	
Δ_S	2.9168	2.1452	1.2555	1.1607	0.9847

Numerical results for the RVBSL phase of the MSE model support these two hypotheses (see Appendix B for the precise description of the model and spectral results in Figs. 5, 6, 7, 8 and Tab. 2). The gap Δ_0 between the absolute ground-state $|S^z_{\rm tot}=0,\mathbf{k}=(0,0)\rangle$ and the first $|S^z_{\rm tot}=0,\mathbf{k}=(\pi,0)\rangle$ decreases steadily with the size of the sample with no difference between even$\times$odd and even$\times$even samples (see Tab. 2 and Fig. 9).

These results pull into light the central conjecture of Oshikawa, *i.e.* what Laughlin calls the robustness of the gap or the uncompressibility of the spin liquid and its relationship with our conjecture of the vanishing stiffness of the 2D spin liquids in the thermodynamic limit. Strictly speaking numerical calculations do not allow a consistent determination of the "stiffness" of a finite size sample in

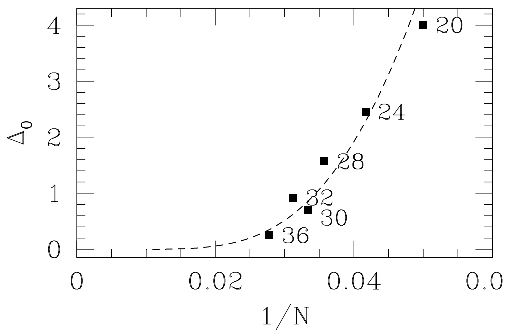

Fig. 9. Δ_0 gap of the first multiplet in the singlet sector *versus* $\sqrt{N}$: the line is an exponential fit $\propto \exp(-\sqrt{N}/\xi)$ ($\xi \sim 0.6$).

a spin liquid phase: depending on the shape of the sample and on the direction of the applied twist the response of the system can vary in a range going from half to one hundredth of the coupling constant (contrary to the case of a Néel ordered system where the stiffness is very well defined and totally independent on the direction of the twist). Most of the results obtained on the 28, 30, 32-sites samples are 10 to 100 times smaller than those of the Néel ordered phase. But the conjecture that can be formed from the above mentioned results is even stronger than the vanishing of the stiffness and amounts to *an absolute absence of sensitivity of the thermodynamic ground-state of a spin liquid to any twist of the boundary conditions* (*i.e.* all terms of the Taylor expansion of Eq. (9) are zero in the thermodynamic limit, not only the ϕ^2 term).

This conjecture is based on the following observations concerning spectra of the MSE model (see Tab. 2 and Figs 5, 6, 7, 8):

- in even$\times$odd samples the ground-state energy of the system under a twist oscillate between the energy of the absolute ground-state at zero twist and that of the first excited state $|S^z_{\rm tot}=0,\mathbf{k}=(\pi,0)\rangle$ reached for a 2π twist[6]. The amplitude of the oscillation measured by the energy gap Δ_0 between these two states decreases steadily with the system size.
- In even$\times$even samples the amplitude of oscillation is reduced: the maximum of energy of the ground-state is obtained for a π twist. The amplitude of this oscillation Δ_0' given by the difference in total energy between these two states is given in Table 2. As the previous gap, this amplitude decreases with the system size.
- The number of samples is too small to ascertain an exponential decrease to zero with system length, but this conjecture remains plausible (Fig. 9).

On physical grounds, the absence of any response to a twist is a rather natural property in an RVBSL since there such a phase offer no local order parameter this perturbation could couple to. In Section 4 we will come back to this

[6] In small frustrated samples the absolute minimum could be observed for a very small, non zero, twist. This peculiarity is no more present for the stablest symmetric samples of size larger or equal to 28.

question from the point of view of the short-range RVB picture of spin liquids.

3.4 Oshikawa's conjecture and valence bond crystals

The numerical work presented in the previous section concerns an RVBSL. We have not investigated the behavior of a VBC with twisted boundary conditions but it is clear that the stiffness should be zero. In principle non-linear effects could be different in VBC and RVBSL.

4 The short-range RVB picture of the ground-state multiplicity of a spin liquid

Because of the short-range and antiferromagnetic nature of magnetic correlations and spin-rotation invariance, short-range dimer configurations offer a natural description of low-energy degrees of freedom in spin liquids. Indeed, much insights into their physical properties come from the investigation of dimer models. The quantum hard core dimer (QHCD) model of Rokhsar and Kivelson [4,3,5] on the square lattice is the simplest dimer model but one has to consider the triangular lattice (Moessner and Sondhi [9]) to find a genuine liquid with a finite correlation length. The short-range dimer point of view also gave a significant insights on another different kind of spin liquid: the Kagome Heisenberg antiferromagnet (see Ref. [30] and references therein).

First (Sect. 4.1) we recall that the Hilbert space of short-ranged dimer coverings on a torus is made of 4 topological sectors which cannot be coupled by any local operator. We show that any wave function described in a short-range dimer basis is two-fold degenerate on large enough even×odd and can be two- or three-fold degenerate on even×even samples with appropriate point-group symmetries. This is illustrated on spectra of the QHCD model on the triangular lattice. Then (Sect. 4.2) we argue that for a spin liquid where all dimer correlations decay exponentially fast with distance this degeneracy is extended to 4 for large enough samples. This property is checked on the QHCD model and in MSE spectra.

After some remarks on the four-fold degeneracy in the liquid phase (Sect. 4.3) we explain in Section 4.4 why RVBSL, in spite of the degeneracy of their ground-states cannot manifest spontaneous symmetry breaking.

4.1 Decomposition of the space of short-range dimer coverings in topological subspaces and exact degeneracies

The space of nearest neighbor dimer coverings of a torus is a small fraction of the total $S = 0$ sector of the full Hilbert space which increases as α^N (with $\alpha = 1.1753, 2^{1/3}, 1.3385, 1.5351$ respectively on the hexagonal, kagomé, square and triangular lattices). These different coverings are not orthogonal, but for large enough sizes they form a set of linearly independent vectors: an exact demonstration exists for the coverings of the square lattice [31], numerical studies have shown that the property is true for the kagomé lattice [30], and in the triangular lattice for any size large enough to insure than any periodic image of a site is at a distance larger than 4 units (this work).

4.1.1 Definition of the topological sectors

Let us draw a cut Δ encircling the torus created by periodic boundary conditions (see Fig. 3). This hyper-surface of dimension $d-1$ cuts bonds of the lattice but there is no site sitting on it. The position of the cut is arbitrary. We may decide in order to follow closely our previous discussion to put it between spin $L-1$ and spin 0 of each row of the lattice. The family of nearest-neighbor dimer coverings can be decomposed into two subspaces $\mathcal{E}^{\pm}_{\Delta}$ depending on the parity Π_{Δ} of the number of dimers crossing the cut Δ [7]. By considering a set of d cuts $\Delta_{i=1,\ldots,d}$ encircling the torus in all possible directions one obtains 2^d families of dimer covering.

Any movement of dimers can be represented as a set of closed loops around which dimers are shifted in a cyclic way. A *local* operator will only generate contractible loops which will cross each cut a *even* number of times. The number of dimers crossing the cut can therefore only be changed by an even integer and the parities Π_{Δ_i} are unchanged.

This property remains true as long as one works in a subspace where the dimer lengths are smaller than the linear system size, that is when the topological sector are well defined (if a dimer length is half the linear of the system one cannot decide by which side it goes). On the other hand, we checked on the triangular (resp. Kagome) lattice that these 4 sectors are the only topological sectors: local 3- (resp 4-) dimer moves can be used to transform a configuration of a given sector into any other configuration of that sector.

We show that these subspaces are orthogonal in the thermodynamic limit [7]. The graph of the scalar product $\langle c^+|c^-\rangle$ of two dimer configurations belonging to different subspaces $\mathcal{E}^+$ and $\mathcal{E}^-$ has at least one long loop encircling the torus in the L_x direction. When L_x goes to infinity this contribution to the scalar product is smaller than $2^{-L_x/2}$. Consider two normalized vectors $|\Psi^+\rangle$ and $|\Psi^-\rangle$ belonging to two different sectors:

$$\left|\Psi^{\pm}\right\rangle = \sum_{c^{\pm}\in\mathcal{E}^{\pm}} \Psi^{\pm}(c^{\pm})\left|c^{\pm}\right\rangle \cdot \tag{20}$$

[7] Another equivalent definition is based on winding numbers of transition graphs with a reference configuration [4].

Because of the exponential number of dimer coverings in each subspace it is not obvious that $|\Psi^+\rangle$ and $|\Psi^-\rangle$ are orthogonal in the thermodynamic limit. It is indeed the case:

$$\left|\langle\Psi^+|\Psi^-\rangle\right| = \left|\sum_{c^+,c^-} {\Psi^+}^*(c^+)\Psi^-(c^-)\langle c^+|c^-\rangle\right| \quad (21)$$

$$\le 2^{-L/2}\sum_{c^+,c^-}\left|{\Psi^+}^*(c^+)\Psi^-(c^-)\right| \quad (22)$$

$$\le 2^{-L/2}\sqrt{\sum_{c^+}|\Psi^+(c^+)|^2\sum_{c^-}|\Psi^-(c^-)|^2} \quad (23)$$

$$\le 2^{-L/2}\|\Psi^+\|\cdot\|\Psi^-\| \quad (24)$$

where the norm in the last inequality refers to the diagonal scalar product $\langle c_1||c_2\rangle = \delta_{c_1,c_2}$. We now wish to show that these later bound is finite. In order to see that it is the case one can expand the usual scalar product $\langle c_1|c_2\rangle$ in powers of $x = 2^{-1/2}$ in the spirit of reference [4]. The zeroth order term is nothing but the diagonal scalar product:

$$\langle c_1||c_2\rangle = \delta_{c_1,c_2} + \mathcal{O}(x^4). \quad (25)$$

The convergence of this loop expansion implies that $\|\Psi\|$ is finite if $|\langle\Psi|\Psi\rangle = 1$ and therefore $\langle\Psi^+|\Psi^-\rangle = \mathcal{O}(2^{-L/2})$. From this we see that not only scalar products of *dimer configurations* in different topological sectors vanish exponentially but this is also true for any pair of *states*.

In the following, unless explicitly mentioned, we consider the 2D case for simplicity but most of the topological arguments about dimer covering immediately extend to higher dimensions.

4.1.2 Two-fold degeneracy in even×odd samples

In the special case of tori with an odd number of rows (and an odd number of spin-$\frac{1}{2}$ per crystallographic unit cell), one step translation along the x axis (called $\mathcal{T}_x$ in the following) maps $\mathcal{E}^+$ on $\mathcal{E}^-$ and reversely. Some point-group symmetry can also do this job. A π rotation about a lattice site nearby the cut (called $\mathcal{R}_\pi$ in the following) has the same effect. If the cut is chosen parallel to a symmetry axis of the sample, a reflection with respect to this axis (called Σ_y in the following) will equally map $\mathcal{E}^+$ on $\mathcal{E}^-$ and reversely.

All these symmetry operations isolate a single column C of lattice sites between Δ and its transform Δ'. In that case columns have an odd number of sites and an odd number of dimers must connect some sites inside C with sites outside C. Therefore Π_Δ differs from $\Pi_{\Delta'}$ and the two subspaces $\mathcal{E}^+$ and $\mathcal{E}^-$ are exchanged.

For a large enough system these two sectors are 1) orthogonal, 2) uncoupled by any local Hamiltonian and 3) exchanged by symmetry operations (even×odd). This is enough to insures that they have the same spectrum, irrespectively of the physics of the model, provided it can be described in the short-range dimer space. In fact, quantum numbers of these doublets of degenerate states are fixed by symmetry.

We decompose an eigenstate $|\psi_0\rangle$ on the two topological subspaces defined relatively to the cut Δ:

$$|\psi_0\rangle = \left|\psi_0^+\right\rangle + \left|\psi_0^-\right\rangle \quad (26)$$

where $\left|\psi_0^\pm\right\rangle$ belong respectively to the sets $\mathcal{E}_\Delta^\pm$. $|\psi_0\rangle$, as an eigenstate of the Hamiltonian with periodic boundary conditions, belongs to an irreducible representation of the translation group. In the following we will also assume an $\mathcal{R}_\pi$ and Σ_y invariance of the Hamiltonian and that, for simplicity, $|\psi_0\rangle$ transforms under a one-dimensional representation under $\mathcal{R}_\pi$ and Σ_y.

$$\begin{aligned}\mathcal{T}_x|\psi_0\rangle &= \mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{u}}|\psi_0\rangle\\ \mathcal{R}_\pi|\psi_0\rangle &= \rho_0^\pi|\psi_0\rangle\\ \Sigma^y|\psi_0\rangle &= \sigma_0^y|\psi_0\rangle\,.\end{aligned} \quad (27)$$

In the thermodynamic limit $\left|\psi_0^+\right\rangle$ and $\left|\psi_0^-\right\rangle$ are orthogonal and $\mathcal{T}_x$, $\mathcal{R}_\pi$ and Σ^y map $\mathcal{E}^+$ on $\mathcal{E}^-$ and reversely:

$$\begin{aligned}\mathcal{T}_x\left|\psi_0^\pm\right\rangle &= \mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{u}}\left|\psi_0^\mp\right\rangle\\ \mathcal{R}_\pi\left|\psi_0^\pm\right\rangle &= \rho_0^\pi\left|\psi_0^\mp\right\rangle\\ \Sigma^y\left|\psi_0^\pm\right\rangle &= \sigma_0^y\left|\psi_0^\mp\right\rangle\,.\end{aligned} \quad (28)$$

Let us now build the variational state:

$$|\psi_{1,\Delta}\rangle = \left|\psi_0^+\right\rangle - \left|\psi_0^-\right\rangle\,. \quad (29)$$

Equations (28) imply:

$$\begin{aligned}\mathcal{T}_x|\psi_{1,\Delta}\rangle &= -\mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{u}}|\psi_{1,\Delta}\rangle\\ \mathcal{R}_\pi|\psi_{1,\Delta}\rangle &= -\rho_0^\pi|\psi_{1,\Delta}\rangle\\ \Sigma^y|\psi_{1,\Delta}\rangle &= -\sigma_0^y|\psi_{1,\Delta}\rangle\end{aligned} \quad (30)$$

$|\psi_{1,\Delta}\rangle$ has thus a wave-vector $\mathbf{k}_1$, a rotation quantum number ρ_1^π and a reflection quantum number σ_1^y related to the quantum numbers of $|\psi_0\rangle$ by the relations:

$$\begin{aligned}\mathbf{k}_1 &= \mathbf{k}_0 + (\pi,0)\\ \rho_1^\pi &= -\rho_0^\pi\\ \sigma_1^y &= -\sigma_0^y.\end{aligned} \quad (31)$$

It is thus a state orthogonal to the ground-state (even on a finite-size system where the topological sectors are not rigorously orthogonal).

Since any local Hamiltonian has exponentially vanishing matrix elements between different sectors we have

$$\left\langle\psi_0^+\right|\mathcal{H}_0\left|\psi_0^-\right\rangle \to 0 \quad (32)$$

and $|\psi_{1,\Delta}\rangle$ is thus degenerate with the absolute ground-state, their symmetries being related by relations (31).

We have computed numerically the spectrum of the QHCD model for three different triangular samples: $N = 20 = 4\times5$, $N = 28 = 4\times7$ and $N = 30 = 6\times5$ sites. These samples are translation and $\mathcal{R}_\pi$ invariant (but without symmetry axis). As expected, the spectrum is exactly two-fold degenerate. Pairs of states satisfying equation (31) are exactly degenerate. The two ground-states, in particular, have $\mathbf{k} = (0,0)$ $\mathcal{R}_\pi = 1$ and $\mathbf{k} = (\pi,0)$ $\mathcal{R}_\pi = -1$.

4.1.3 Degeneracies in even×even samples

As remarked by Bonesteel [7] on a square lattice a $\pi/2$ rotation exchanges sector $(+,-)$ and sector $(-,+)$ but sectors $(-,-)$ and $(+,+)$ remain inequivalent. A similar phenomenon occurs on the triangular lattice where $2\pi/3$ rotations permute cyclically 3 of the 4 sectors. Consider a finite-size triangular lattice with periodic boundary conditions which has the $2\pi/3$-rotation symmetry ($N = L \times L$ for instance). One draws 3 cuts $\Delta_{1,2,3}$ which can be deduced from each other by $\pm 2\pi/3$ rotations. Of course only two of these three cuts are topologically distinct but one can associate 3 parities to any short-rang dimer configurations anyway. These three cuts separate the sample in two sets of $N/2$ sites. If, for instance, $N/2$ is even the only allowed parities are $(+,+,+)$, $(+,-,-)$ $(-,+,-)$ and $(-,-,+)$ and we recover 4 sectors as it should be. In this form it is now obvious that the three last sectors are symmetric by $2\pi/3$ rotations and will have the same ground-state energy. One can also easily determine the relations between the quantum numbers of these triplets. As result, if $N/2$ is even, sectors $(+,-)$, $(-,+)$ and $(-,-)$ will be exactly degenerate and if $N/2$ is odd $(+,-)$, $(-,+)$ and $(+,+)$ will be exactly degenerate

If we consider how $2\pi/3$ rotations and translations act on this three-dimensional manifold, one can easily show that this 3-fold degeneracy can only take two forms. a) The degenerate states can correspond to a three-dimensional representation. The only possibility is that the corresponding momentum are $\mathbf{K}_{i=1,2,3}$. b) The other possibility is that a two-dimensional representation is degenerate with a one-dimensional representation. The only possible realization is a $\mathbf{k} = 0$ $R_{2\pi/3} = 1$ state degenerate with two $\mathbf{k} = 0$ $R_{2\pi/3} = j$ and j^2 complex conjugate states. For these samples, the $2\pi/3$ rotation plays exactly the same role as the one-step translation in the case of even×odd samples.

All these properties were checked numerically in the QHCD model on the triangular lattice. We found that the second case of degeneracy is realized for this model in the $N = 28$ sample (*i.e.* the four eigenstates of the quasi-degenerate ground-state multiplet have wave vector $\mathbf{k} = (0,0)$ but belongs to different IRs of C_3). In fact, it must be so for any even×even sample at the RK point since the wave functions clearly have $\mathbf{k} = 0$ in each sector (equal amplitude superposition of all dimer coverings). This remains true as long as the gap does not close and should be verified in the whole spin-liquid phase of the model. In that case both Proposals B and B' of Section 2.1 are incorrect.

We can ask what is the result of Oshikawa's insertion of a flux in such an even×even spin-liquid. When boundary conditions are twisted, the Hamiltonian is no longer $R_{2\pi/3}$-symmetric and the 2 irreducible representations in which the topological ground-states lie ($R_{2\pi/3} = 1$ and $R_{2\pi/3} = j, j^2$) merges into a single one with momentum $\mathbf{k} = (0,0)$. For this reason, during the adiabatic process, these four energy levels cannot cross and must necessarily map onto themselves after the complete flux quantum insertion. In that case Oshikawa's procedure do not relate the different ground-states with each other.

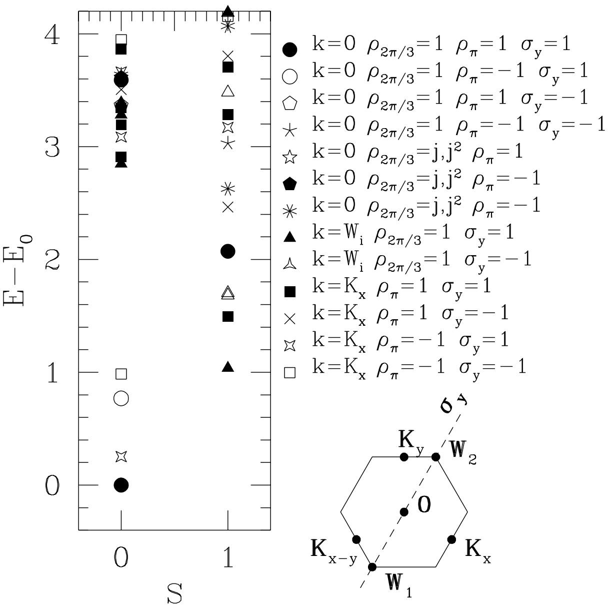

Fig. 10. First eigenstates of the multiple-spin exchange model on a 6×6 sites sample (parameters $J_2 = -2$ and $J_4 = 1$; the system is in a spin liquid phase). The first four levels with total spin $S = 0$ and some $S = 1$ are displayed. The quantum numbers of the eigenstates are displayed at the right of the figure. The ground-state belongs to the trivial representation of the group of translations, rotations and reflections (in particular $\mathbf{k} = (0,0)$, $\rho_0^\pi = 1$ and $\sigma_0^y = 1$). The first $S = 0$ excited state is three-fold degenerate (wave-vectors $\mathbf{K}_i$, $\rho_1^\pi = -1$ and $\sigma_1^y = 1$). The finite-size scaling indicates that this state collapses to the absolute ground-state in the thermodynamic limit (Tab. 2 and Fig. 9). The third and fourth energies in the $S = 0$ spin sector are also probably degenerate in the thermodynamic limit (global 8-fold degeneracy).

4.2 4-fold degeneracy in RVBSL phases

The numerical data on MSE model suggest that the ground-state degeneracy is 4 in a spin liquid phase [8], whatever may be the shape of the sample (Fig. 9). In the case of even×odd sample ($N = 30 = 6 \times 5$ in Tab. 3), the quasi-degenerate levels exactly satisfy the constraint of equation (31). In even×even samples which have the $2\pi/3$-rotation symmetry ($N = 28$ and 36 in Tab. 3 and Fig. 10), the quasi-degeneracy is observed between a $\mathbf{k} = (0,0)$

[8] A ground-state degeneracy has been evoked by Wen and co-workers [33] in *chiral* spin liquids. To our understanding, this is a bit more complex than the present case. A chiral spin liquid is a gapped phase with long-range order in chirality (local observable defined as the triple product of 3 neighboring spins). It not only breaks reflection symmetry but also time reversal symmetry and it is supposed to be $2(k)^d$ degenerate, where k is an even index related to the fractional statistics of the elementary excitations.

Table 3. Energies and quantum numbers of the first two multiplets in the spin liquid spectra of the MSE model. The wave vector is indicated in the first column, the quantum numbers ρ^π and σ^y are indicated in parenthesis after the energy. Samples $N = 28$ and 30 have no axial symmetry. Relations (31) are satisfied in the even-odd . (*) stands for data where the 2$^{\rm nd}$ multiplet does not appear clearly, due to the absence of C_3 symmetry of the sample.

N	28	30	32	36
$\mathbf{k_0} = (0, 0)$	−110.965 (1)	−117.896 (−1)	−126.608(1,1)	−142.867 (1,1)
$\mathbf{k_1} = (\pi, 0)$	−109.396 (−1)	−117.193 (1)	−125.684(−1,−1)	−142.615 (−1,1)
$\mathbf{k_0} = (0, 0)$	−108.168 (1)	−116.590 (1)	*	−142.098 (−1,1)
$\mathbf{k_1} = (\pi, 0)$	−107.633 (−1)	−116.544 (−1)	*	−141.883 (−1,−1)

state and three $\mathbf{k} = \mathbf{K}_{1,2,3}$, in perfect agreement with the arguments developed in Section 4.1.3. These are additional arguments in favor of the topological interpretation of the quasi-degeneracy observed in this model [23].

The same situation appears in the liquid phase of the QHCD model on the triangular lattice (we consider the $J = 1$ and $V = 0.95$ point to avoid difficulties with the first order transition to the staggered phase at $V = 1$). This degeneracy property was already inferred by various authors at the end of the eighties [3–5]. From the results of Section 4.1 we know that the ground-states inside each of the 4 topological sectors give rise to two energy levels (two doublets or one singlet plus a triplet depending on the sample symmetry). Indeed a quasi four-fold degeneracy of the ground-state is observed. More precisely, the two levels approach each other as the linear size is increased. For instance, this small level splitting drops from $\delta = 0.02285$ on a $16 = 4 \times 4$ sample to 0.008147 for $30 = 6 \times 5$ sites. In samples without $2\pi/3$-rotation symmetry a set of four nearly degenerate levels is observed ($N = 24 = 6 \times 4$ for instance).

We make the following assumption: a) the ground-state can be described in a short-ranged dimer basis. b) All n-dimer correlations ($n = 2,\ 3, \cdots$) are short-range and the corresponding correlation lengths are bounded. c) The Hamiltonian is local. From hypotheses a) and c) it is clear that for a large enough system the four topological sectors are not mixed in the ground-state and the spectrum can be computed separately in each sector. We do not have any symmetry operation which connects all *four* sectors and we need a physical argument to explain the fact that energies should be the same in each sector (in the thermodynamic limit). Because of their topological nature, it is not possible to determine to which sector a dimer configuration belongs by looking only at a *finite area* of the system. In other words, any dimer configuration defined over a large but finite part of the system can be equally realized in all sectors. The Hilbert space available to the system is the same over any finite region of the system. In the absence of any form of long-range order the system can therefore optimize all its correlations with an arbitrary high accuracy equally well in each sector. At this point we can only conclude that the four sector will have the energy *density* and we cannot exclude the existence of a gap between the different topological sectors. However the numerical results obtained in the QHCD and MSE models indicate that it is not the case and that the four ground-state have asymptotically the same *total* energy. We think that this should be true for a general short-range RVB spin liquid. Based on the relationship between the effect of a twist and the topological degeneracy in such systems, a vanishing gap between sectors in the thermodynamic limit is likely to be related to the complete absence of sensitivity to such a twist (Sect. 3.3).

4.3 Miscellaneous remarks on the RVB ground-state degeneracy

— *Dimers and twist operator*. The variational state $|\psi_{1,\Delta}\rangle$ can be deduced simply from $|\psi_0\rangle$ by changing the sign of the dimers crossing the cut Δ. Such an operation can also been seen as a 2π twist of the spins of column 0. As was noticed by Bonesteel [7] for the one-dimensional problem such an operation is mathematically related to action of the LSMA twist operator. From the physical point of view the reason why $|\psi_{1,\Delta}\rangle$ has asymptotically the same energy as $|\psi_0\rangle$ becomes clear: in the absence of stiffness and of long-range spin-spin correlations, the perturbation induced by the boundary term of equation (14) cannot propagate and does not change the energy of the initial state: its only effect is to change the relative phases of the different topological components of the wave function, and consequently the momentum and space symmetry quantum numbers of the initial state of the even-odd samples.

— *Fractionalization and topological degeneracy.* To our knowledge all present theoretical descriptions of fractionalized excitations in 2D magnets or related problems [34,38–41]) (we should also mention topological properties of Laughlin's wave function for fractional quantum Hall effect [36,35]) imply topological ground-state degeneracies. In such pictures, the physical operation which transforms a ground-state into another is the virtual creation of a pair of spinons (by dimer breaking) followed by its annihilation after the circulation of one of them around the torus. In such a process a π phase-shift is introduced between the topological sectors (as in the above recipe). For samples with an odd number of rows this operation connects eigenstates with different $\mathbf{k}$ vectors (and space quantum numbers) as described in equations (31). Oshikawa's adiabatic construction is of the same nature.

— *Numerical studies with a different topology.* An interesting check of the pure topological nature of this degeneracy could be obtained by studying the problem no more on a torus but on a surface with a different genus. On a sphere we expect an absence of degeneracy. Unfortunately if a lattice can be represented on an infinite plane, both the number of links L and plaquettes P depend linearly on the number of sites N and Euler's relation $P - L + N = 2 - G$ constrains the genius G to be 2! The torus is the only possible topology if we require a full translation invariance in both directions. In a recent work, Ioffe *et al.* [37] have studied the absence of sensitivity to disorder as an evidence for topological phenomenon in the liquid phase of the QHCD model on the triangular lattice. They also used *open* boundary conditions to modify the topology of the system and argue in that case that the low-energy spectrum is free of edge states which could hide the actual ground-state degeneracy.

— *Example of RVB phase with 2 spins in the unit cell.* A spin liquid state, seeming very similar to the state observed in the MSE model on the triangular lattice, has been observed in the $J_1 - J_2$ model on the hexagonal lattice [21] for $J_1 = -1,\ J_2 = 0.3$. No quasi-degeneracy of the ground-state has been noticed. It should be remarked that in this system there are 2 spins $\frac{1}{2}$ per crystallographic unit cell and no degeneracy is expected on the basis of the topological arguments.

4.4 Symmetry breaking in gapped phases

From the mathematical point of view, ground-state wave functions that break one-step translations or space group symmetries can be built from linear combinations of the degenerate ground-states of the *even-odd* samples. In a completely equivalent way, ground-state wave functions that break rotation symmetry can be built in even×even samples. One could thus superficially conclude that spontaneous symmetry breaking is possible in RVBSL, we will show below that this assumption is false.

There are many features which show that this degeneracy property is a subtle one, both from the mathematical and physical viewpoints.

The possible alternation of the spatial properties of the low-lying excitations with the parity of the number of rows of the sample (as observed in the QHCD model on the triangular lattice) is a first difficulty. The degeneracy of the RVBSL is in fact quite different from that appearing in a VBC. We do not expect the VBC ground-state degeneracy to depend on the genus of the sample, as the RVBSL does.

From the physical point of view also the two situations are quite different. An infinitesimal symmetry breaking perturbation is able to select one symmetry breaking ground-state of the VBC, but as we will show below this is impossible in the RVBSL.

Let us call $\mathcal{A}$ the extensive non-diagonal observable appearing in the VBC in the thermodynamic limit. On a columnar VBC modulated in the $\mathbf{u}$ direction, this observable is:

$$\mathcal{A} = \sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\mathbf{K}_1 \cdot \mathbf{r}_j} P_{S=0}(\mathbf{r}_j, \mathbf{r}_j + \mathbf{u}) \tag{33}$$

where $P_{S=0}(\mathbf{r}_j, \mathbf{r}_j + \mathbf{u})$ is the projector on the singlet state of two neighboring spins. $\mathcal{A}$ connects eigenstates with wave-vector $\mathbf{k}_0$ to states with wave-vector $\mathbf{k}_0 + \mathbf{K}_1$.

On a finite size sample, with periodic boundary conditions, the expectation value of $\mathcal{A}$ is zero in any eigenstate, but $\langle \mathcal{A}^2 \rangle$ could be non zero. If the order parameter $\mathcal{P}$ defined by:

$$\mathcal{P}^2 = \langle \psi_{g.s.} | \mathcal{A}^\dagger \mathcal{A} | \psi_{g.s.} \rangle / N^2 \tag{34}$$

does not vanish in the thermodynamic limit, the system has columnar dimer long-range order with wave vector $\mathbf{K}_1$.

Let us now consider a perturbation of the Hamiltonian:

$$H_\delta = H_0 - (\delta \mathcal{A} + \mathrm{h.c.})\,. \tag{35}$$

At $T = 0$, the intensive linear response on the observable $\mathcal{A}$ is measured by the susceptibility:

$$\chi = \frac{2}{N} \left\langle \psi_{g.s.} | \mathcal{A}^\dagger \frac{1}{H_0 - E_{g.s}} \mathcal{A} | \psi_{g.s.} \right\rangle. \tag{36}$$

This susceptibility is bounded from below [42]:

$$\frac{4\mathcal{P}^4 N^2}{f} < \chi \tag{37}$$

where f is the oscillator strength:

$$f = \frac{1}{N} \langle \psi_{g.s.} | \left[\mathcal{A}, [H_0, \mathcal{A}] \right] | \psi_{g.s.} \rangle \cdot \tag{38}$$

The demonstration uses the properties of the spectral decomposition associated to the operator $\mathcal{A}$:

$$S(\omega) = \frac{1}{N} \sum_{n \neq 0} \left| \langle \psi_{g.s.} | \mathcal{A} | n \rangle \right|^2 \delta(\omega - \omega_n) \tag{39}$$

where $\omega_n = E_n - E_{g.s}$

$$\mathcal{P}^2 = \frac{1}{N} \int S(\omega) \mathrm{d}\omega. \tag{40}$$

Using the Cauchy Schwartz inequality one obtains:

$$\mathcal{P}^4 \leq \frac{1}{N^2} \int \omega S(\omega) \mathrm{d}\omega \int \omega^{-1} S(\omega) \mathrm{d}\omega \tag{41}$$

where

$$\int \omega S(\omega) \mathrm{d}\omega = f/2 \tag{42}$$

$$\int \omega^{-1} S(\omega) \mathrm{d}\omega = \chi/2 \tag{43}$$

which proves inequality (37). For a short-range Hamiltonian the oscillator strength f is $\mathcal{O}(1)$ and inequality (37) implies that the $T = 0$ susceptibility associated to a finite order parameter diverges at least as the square of the sample size: any infinitesimal symmetry breaking perturbation will select a symmetry breaking state.

We will now show that for a RVBSL, where all the correlations functions are short-ranged with correlation lengths bounded by ξ, the susceptibilities of the medium remain finite in the thermodynamic limit. To do so we distinguish in equation (36) the contributions from the quasi-degenerate states of the topological multiplet (called $|\alpha_i\rangle$) from the contribution of the other states of the spectrum, above the physical gap Δ. We thus obtain the following upper bound for the susceptibility:

$$\chi = \chi_{\rm top} + \chi_{\Delta} \tag{44}$$

$$\chi_{\rm top} = \frac{2}{N}\left\langle \psi_{g.s.}|\mathcal{A}^\dagger \frac{|\alpha_1><\alpha_1|}{E_{\alpha_1}-E_{g.s.}}\mathcal{A}|\psi_{g.s.}\right\rangle \tag{45}$$

$$\begin{aligned}\chi_{\Delta} \leq \frac{2}{N\Delta}&\left[\langle\psi_{g.s.}|\mathcal{A}^\dagger\mathcal{A}\,|\psi_{g.s.}\rangle\right.\\ &\left.-\langle\psi_{g.s.}|\mathcal{A}^\dagger|\alpha_1\rangle\langle\alpha_1|\mathcal{A}|\psi_{g.s.}\rangle\right]\end{aligned} \tag{46}$$

where $|\alpha_1\rangle$ stands for the state(s) of the topological multiplet connected to the absolute ground-state by $\mathcal{A}$. Using the local properties of $\mathcal{A}$, $\mathcal{A}^\dagger|\alpha_1>$ is in the same topological sector as $|\alpha_1\rangle$ and $\langle\psi_{g.s.}|\mathcal{A}^\dagger|\alpha_1\rangle$ is at most of $\mathcal{O}(N\times 2^{-L/2})$ (see paragraph 4.1.1). As $E_{\alpha_1}-E_{g.s.}$ is supposed to decrease as $\exp(-L/\xi)$ (see Fig. 9) $\chi_{\rm top}$ goes to a constant when the size of the sample goes to infinity, provided ξ is small enough [9]. In a system with exponentially decreasing correlations, $\mathcal{P}^2$ decreases as $1/N$ and χ_{Δ} is trivially constant at the thermodynamic limit.

In such a phase an infinitesimal field cannot induce a symmetry breaking and there could not be any *spontaneous* symmetry breaking.

5 Conclusion

The main statements of this paper could be summarized in the following way. In any situations the ground-state of an $SU(2)$ Hamiltonian with one spin-$\frac{1}{2}$ per unit cell is degenerate in the thermodynamic limit. In fact the problem encompasses two very different cases:

– Either the system has a $T = 0$ macroscopic long-range order: in these situations the degeneracy of the ground-state encompasses and reveals the symmetry breakings of the macroscopic ground-states. It is algebraic in the system size in case of a continuous $SU(2)$ symmetry breaking and involves only a finite number of levels in case of a discrete symmetry breaking. This situation is not restricted to spin-$\frac{1}{2}$ Hamiltonians.
– Or the system has no long-range order at $T = 0$, it is a spin liquid. In that case the degeneracy property depends essentially on the existence of an odd-half-integer total spin in the unit cell. In this paper we have only studied the case of a unique spin-$\frac{1}{2}$ per unit cell, but in view of its topological origin we suspect the result to be true for the general case above. On the other hand, we have given one example with two spins $\frac{1}{2}$ in the unit cell and very plausibly a unique ground-state in the thermodynamic limit [21].

All these ideas had already been known by a number of authors. The really new contributions of this work are the following:

– We have analyzed the Lieb Schultz Mattis Affleck conjecture for 2D spin systems and shown by the numerical study of a counter-example that a part of this conjecture is probably incorrect (Sect. 2).
– For spin liquid systems, we have analyzed Oshikawa adiabatic construction and shown that it is equivalent to the hypothesis of an absence of stiffness of the gapped systems. From our numerical analysis, we conjecture that the spin liquids display in the thermodynamic limit a total absence of sensitivity to the boundary conditions (Sect. 3).
– We have shown that on a 2D torus with an odd number of rows, the ground-state is degenerate (which was already known), we have also demonstrated some exact relationship between the quantum numbers of these multiplets and extended the demonstration of degeneracy to some even×even systems with rotation symmetry (which is new). We have developed the idea that these spatial properties of the spin liquid spectra depend crucially on the parity of the number of rows in the sample and have developed both exact and numerical examples to support this statement. The study of the QHCD model in even×even samples is an explicit example where proposals B and B' of LSMA and Oshikawa are not verified.
– Finally we have shown why, in spite of the degeneracy of the ground-state, usual spin liquids cannot exhibit spontaneous symmetry breaking of any kind. This is different from VBCs (or chiral spin liquid).

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[9] Strictly speaking ξ should be $\leq \log 2$ but going from equation (21) to equation (22) dramatically overestimates the scalar product in the case of an RVBSL. The reason is that if two dimer coverings c^+ and c^- maximize $\langle c^+|c^-\rangle$ they only differ along a single large ($\sim L$) loop. They have different local correlations along the loop and their energy difference is of order L and it is very unlikely that their weights in the ground states $\psi^+(c^+)$ and $\psi^-(c^-)$ are *both* of order one.

Appendix A: Symmetries of spectra with twisted boundary conditions

A.1 Arbitrary twist angle ϕ

With periodic boundary conditions, spectra at momentum $\mathbf{k}$ and $-\mathbf{k}$ are identical because the Hamiltonian commutes with $\mathcal{R}_\pi$ the lattice rotation of angle π about the origin. In the presence of a non zero twist, this rotation is no longer a symmetry :

$$\mathcal{R}_\pi \tilde{H}_\phi \mathcal{R}_\pi^{-1} = \tilde{H}_{-\phi}. \quad (47)$$

If one defines the *spin flip* F by $FS^z_{\mathbf{x}} = -S^z_{\mathbf{x}}F$, then $\mathcal{R}_\pi F$ commutes with $\tilde{H}_\phi$ and $E_\phi(\mathbf{k}, S^z) = E_\phi(-\mathbf{k}, -S^z)$.

For an arbitrary twist the only symmetries of $\tilde{H}_\phi$ are:

- $\mathcal{R}_\pi F$
- spin rotations about the z axis: $\mathrm{e}^{\mathrm{i}\theta S^z_{\mathrm{tot}}}$
- translations
- other spatial symmetries which do not change the x coordinates. The lattice may have, for instance, an axis of symmetry parallel to the x direction.

All these symmetry operators commute and the Hilbert space of $\tilde{H}_\phi$ can thus be analyzed in terms of tensorial products of one-dimensional IRs.

A.2 π-twist

For π twist the spectrum has one additional symmetry. Since $U(2n\pi)\tilde{H}_\phi U(2n\pi)^{-1} = \tilde{H}_{\phi+2n\pi}$ for integer n, one finds:

$$[U(2\pi)\mathcal{R}_\pi]\, \tilde{H}_\phi \, [U(2\pi)\mathcal{R}_\pi]^{-1} = \tilde{H}_{2\pi-\phi} \quad (48)$$

and therefore

$$[U(2\pi)\mathcal{R}_\pi]\, \tilde{H}_\pi \, [U(2\pi)\mathcal{R}_\pi]^{-1} = \tilde{H}_\pi. \quad (49)$$

This new symmetry induces extra degeneracies in the spectra. Using equations (49) and (19), one obtains:

$$E_{\phi=\pi}(S^z_{\mathrm{tot}}, \mathbf{k}_1) = E_{\phi=\pi}(S^z_{\mathrm{tot}}, \mathbf{k}_2) \quad (50)$$

when

$$\mathbf{k}_1 + \mathbf{k}_2 = \left(\pi\,[L_y \bmod 2] + 2\pi\frac{S^z_{\mathrm{tot}}}{L_x}, 0\right). \quad (51)$$

If $S^z_{\mathrm{tot}} = 0$ this condition reduces to:

$$\mathbf{k}_1 + \mathbf{k}_2 = \begin{cases} (\pi, 0) & L_y \text{ odd} \\ (0,0) & L_y \text{ even.} \end{cases} \quad (52)$$

Of course, these degeneracies are explicitly verified in all our numerical spectra.

Finally, we address the issue of the property in the $F\mathcal{R}_\pi$ transform of the eigenstates which are degenerate at $\phi = \pi$ (the quantum number associated to this property is $\eta = \pm 1$). For this purpose we compute the commutation of $F\mathcal{R}_\pi$ with $U(2\pi)$:

$$F\mathcal{R}_\pi U(2\pi) = U(2\pi) F\mathcal{R}_\pi \exp\left(2\mathrm{i}\pi S^z_{\mathrm{tot}} + 2\mathrm{i}\pi S^z_0\right) \quad (53)$$

$$F\mathcal{R}_\pi U(2\pi) = U(2\pi) F\mathcal{R}_\pi (-1)^{L_x L_y + L_y} \quad (54)$$

where $S^z_0 = \sum_{p=0}^{L_y-1} S^z_{n=0,p}$ involves a sum over all the spins of the column 0. Since $\mathcal{R}_\pi$ trivially commutes with η we find that each doublet of degenerate eigenstates obey $\eta_1 = (-1)^{L_x L_y + L_y}\eta_2$.

Appendix B: The multiple-spin exchange model and the type I spin liquid phase

The multiple spin exchange (MSE) Hamiltonian is an effective Hamiltonian which describes the exchange of localized fermions on a lattice. In terms of multiple-spin permutation operators, the 2−, 3− and 4−spin exchanges can be reduced to :

$$H = J_2^{\mathrm{eff}} \sum P_2 + J_4 \sum \left(P_4 + P_4^{-1}\right) \quad (55)$$

where J_2^{eff} (resp. J_4) measures a combination of the tunneling amplitude of 2− and 3−spin exchanges (resp. 4−spin exchange). The permutation operators can be rewritten in terms of usual spin operators: $P_{ij} = 2\mathbf{S}_i \cdot \mathbf{S}_j + 1/2$. $P_{1234} + \text{h.c.}$ is a polynomial of degree two in $\mathbf{S_i} \cdot \mathbf{S_j}$. A complete expression in terms of spin operators can be found in reference [23]. This Hamiltonian has a spin-gap phase in a wide range of ferromagnetic J_2^{eff}, and antiferromagnetic J_4 couplings [25]. The spin liquid spectra studied in the present work are at the specific point $J_2^{\mathrm{eff}} = -2$ and $J_4 = 1$, where the spin gap is of order 0.8.

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