Eigenfunctions and nodal sets (real and complex)

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Nodal sets of eigenfunctions

Let (M, g) be a compact Riemannian manifold and let

$$\Delta_g = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right).$$

be its Laplace operator.
Let $\{\varphi_j\}$ be an orthonormal basis of eigenfunctions

$$\Delta \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk}$$

If $\partial M \neq \emptyset$ we impose Dirichlet or Neumann boundary conditions.
The NODAL SET of φ_j is its zero set:

$$Z_{\varphi_j} = \{x : \varphi_j(x) = 0\}.$$

A NODAL DOMAIN is a connected component of $M \backslash Z_{\varphi_j}$

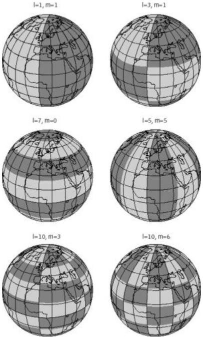
Some Intuition about nodal sets

- Algebraic geometry: Eigenfunctions of eigenvalue λ^2 are analogues on (M, g) of polynomials of degree λ. Their nodal sets are analogues of (real) algebraic varieties of this degree. The $\lambda_j \to \infty$ is the high degree limit or high complexity limit. This analogy is best if (M, g) is real analytic.
- Quantum mechanics: $|\varphi_j(x)|^2 dV_g(x)$ is the probability density of a quantum particle of energy λ_j^2 being at x. Nodal sets are the least likely places for a quantum particle in the energy state λ_j^2 to be. The $\lambda_j \to \infty$ limit is the high energy or semi-classical limit.

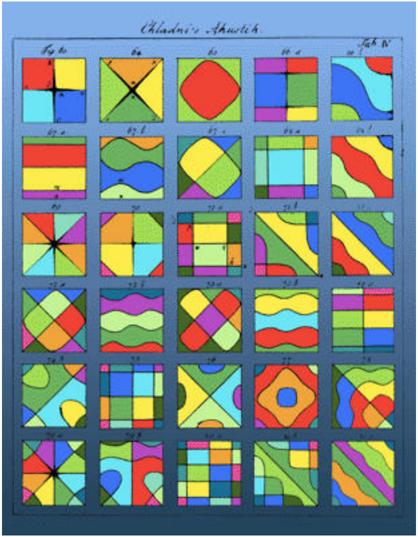
Problems

- How many nodal domains? (Courant: the nth eigenfunction has $\leq n$ nodal domains. No lower bound in general; Lewy: can be just two). How many connected components of Z_{φ_j}?
- How 'long' are nodal sets, i.e. the total length (or hypersurface volume in higher dimensions?)
- How are nodal sets distributed on the manifold?
- HOW DO ANSWERS DEPEND ON BEHAVIOR OF GEODESIC FLOW?

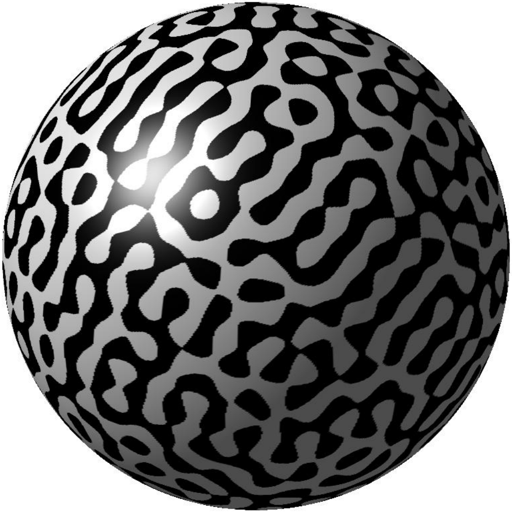
Nodal domains for $\Re Y_m^\ell$ spherical harmonics: geodesic flow integrable: Eigenfunctions coming from separation of variables



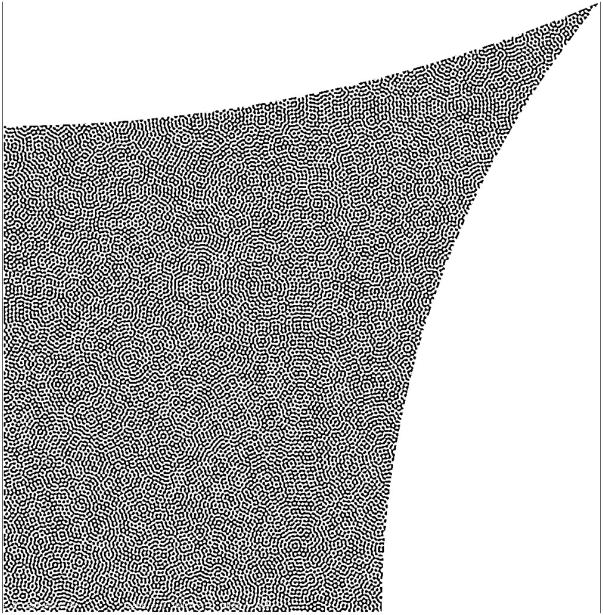
Chladni diagrams: Integrable case



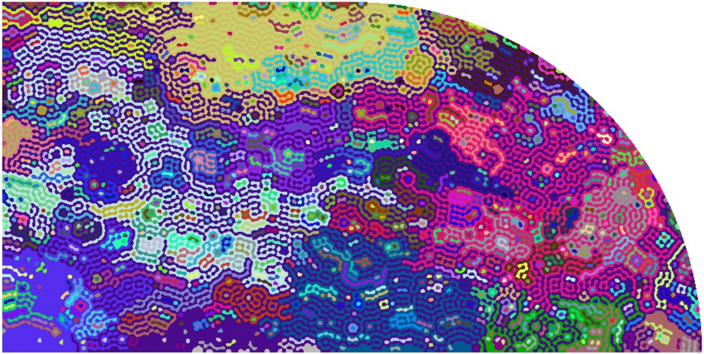
High energy nodal set: E. J. Heller, random spherical harmonic: dimension of space of spherical harmonics of degree N has dim $2N + 1$



High energy nodal set: Chaotic billiard flow



High energy nodal set: Alex Barnett// Each nodal domain is colored a random color; most are small but some are super-big (macroscopic)



Volumes of nodal hypersurfaces: real analytic case

Even the hypersurface volume is hard to study rigorously. There only exist sharp bounds in the analytic case:

THEOREM

(Donnelly-Fefferman, 1988) Suppose that (M, g) *is real analytic and* $\Delta\varphi_\lambda = \lambda^2\varphi_\lambda$. *Then*

$$c_1\lambda \leq \mathcal{H}^{n-1}(Z_{\varphi_\lambda}) \leq C_2\lambda.$$

Distribution of nodal hypersurfaces

How do nodal hypersurfaces wind around on M.?
We put the natural Riemannian hyper-surface measure $d\mathcal{H}^{n-1}$ to consider the nodal set as a *current of integration* Z_{φ_j}]: for $f \in C(M)$ we put

$$\langle [Z_{\varphi_j}], f \rangle = \int_{Z_{\varphi_j}} f(x) d\mathcal{H}^{n-1}.$$

Problems:

- How does $\langle [Z_{\varphi_j}], f \rangle$ behave as $\lambda_j \to \infty$.
- If $U \subset M$ is a nice open set, find the total hypersurface volume $\mathcal{H}^{n-1}(Z_{\varphi_j} \cap U)$ as $\lambda_j \to \infty$.
- How does it reflect dynamics of the geodesic flow?

Physics conjecture on real nodal hypersurface: ergodic case

CONJECTURE

Let (M, g) be a real analytic Riemannian manifold with ergodic geodesic flow, and let $\{\varphi_j\}$ be the density one sequence of ergodic eigenfunctions. Then,

$$\frac{1}{\lambda_j}\langle [Z_{\varphi_j}], f\rangle \sim \frac{1}{Vol(M,g)} \int_M f dVol_g.$$

Evidence: it follows from the "random wave model", i.e. the conjecture that eigenfunctions in the ergodic case resemble Gaussian random waves of fixed frequency.

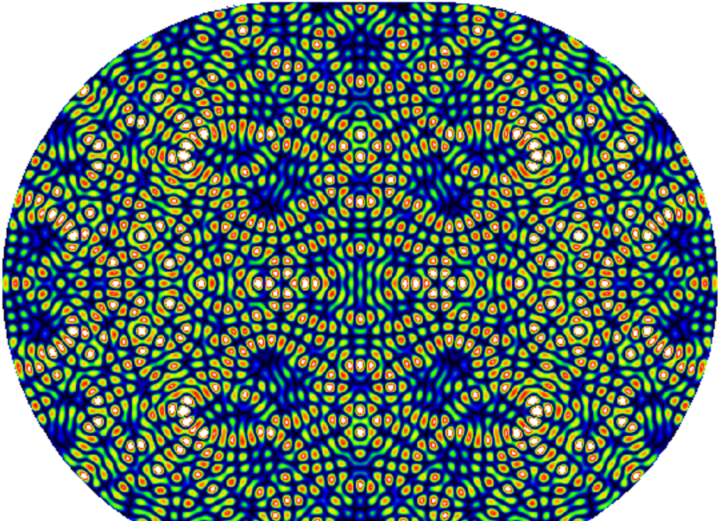
Quantum ergodicity

- Classical ergodicity: G^t preserves the unit cosphere bundle $S_g^* M$. Ergodic = almost all orbits are uniformly dense.
- On the quantum level, ergodicity of G^t implies that eigenfunctions become uniformly distributed in phase space (Shnirelman; Z, Colin de Verdière, Zworski-Z) . This is a key ingredient in structure of nodal sets. Namely,

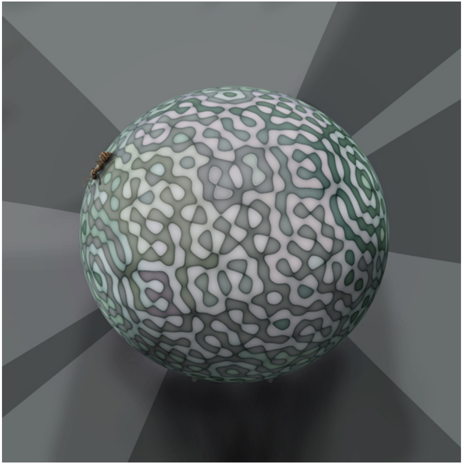
$$\int_E \varphi_j^2 dV_g \to \frac{Vol(E)}{Vol(M)}, \quad \forall E \subset M : Vol(\partial E) = 0.$$

- Equidistribution actually holds in phase space $S^* M$.
- Random wave model (Berry conjecture): when G^t is chaotic, eigenfunctions of Δ_g behave like random waves.

Intensity plot of a chaotic eigenfunction in the Bunimovich stadium



Nodal domains for a random spherical harmonics



Equidistribution in the complex domain

We want to understand equidistribution of nodal sets. Clearly not feasible for general C^∞ metrics. So we study:

- Equi-distribution theory of "complexified nodal sets" for real analytic (M, g)– i.e. complex zeros of analytic continuations of eigenfunctions into the complexification of M.
- Intersections of nodal lines and geodesics on surfaces (in the complex domain); intersection with the boundary when $\partial M \neq \emptyset$;
- The equi-distribution depends upon DYNAMICS OF GEODESIC FLOW

Real versus complex nodal hypersurfaces

The only rigorous results on distribution of nodal sets (and level sets) of eigenfunctions concern the complex zeros of analytic continuations:

$$Z_{\varphi_j^{\mathbb{C}}} = \{\zeta \in M_{\mathbb{C}} : \varphi_j^{\mathbb{C}}(\zeta) = 0\},$$

where $\varphi_j^{\mathbb{C}}$ is the analytic continuation of φ_j to the complexification $M_{\mathbb{C}}$ of M.

Equi-distribution of complex nodal sets in the ergodic case

THEOREM

(Z, 2007) Assume (M, g) is real analytic and that the geodesic flow of (M, g) is ergodic. Let $\varphi_{\lambda_j}^{\mathbb{C}}$ be the analytic continuation to phase space of the eigenfunction φ_{λ_j}, and let $Z_{\varphi_{\lambda_j}^{\mathbb{C}}}$ be its complex zero set in phase space B^*M. Then for all but a sparse subsequence of λ_j,

$$\frac{1}{\lambda_j} \int_{Z_{\varphi_{\lambda_j}^{\mathbb{C}}}} f \omega_g^{n-1} \to \frac{i}{\pi} \int_{M_\tau} f \bar{\partial}\partial \sqrt{\rho} \wedge \omega_g^{n-1}$$

As usual in quantum ergodicity, we may have to delete a sparse subsequence of exceptional eigenvalues.

Grauert tube radius $\sqrt{\rho}$

Given real analytic (M, g), complexify $M \to M_{\mathbb{C}}$.

- Complexify $r^2(x, y) \to r^2(\zeta, \bar{\zeta})$. Grauert tube function $=$

$$\sqrt{\rho} := \sqrt{-r^2(\zeta, \bar{\zeta})}.$$

Measures how deep into the complexification $\zeta \in M_{\mathbb{C}}$ is.

Examples: Torus

- Complexification of $\mathbb{R}^n/\mathbb{Z}^n$ is $\mathbb{C}^n/\mathbb{Z}^n$.
- Grauert tube function: $r(x, y) = |x - y|$ and $r_{\mathbb{C}}(z, w) = \sqrt{(z - w)^2}$. Then

$$\sqrt{\rho}(z) = \sqrt{-(z - \bar{z})^2} = 2|\Im z| = 2|\xi|.$$

- The complexified exponential map is:

$$\exp_{\mathbb{C}x}(i\xi) = x + i\xi.$$

Kähler metric on Grauert tube

- $\rho(\zeta) = -r_{\mathbb{C}}^2(\zeta, \bar{\zeta})$ is the Kähler potential of the Kähler metric $\omega_g = i\partial\bar{\partial}\rho$.
- $\sqrt{\rho}$ is singular at $\rho = 0$ (i.e. on $M_{\mathbb{R}}$):

$$(i\partial\bar{\partial}\sqrt{\rho})^n = \delta_{M_{\mathbb{R}}}, \quad i.e. \int_{M_\epsilon} f(i\partial\bar{\partial}\sqrt{\rho})^n = \int_M f dV_g.$$

Limit distribution of zeros is singular along zero section

- The Kaehler structure on $M_{\mathbb{C}}$ is $\overline{\partial}\partial\rho$. But the limit current is $\overline{\partial}\partial\sqrt{\rho}$. The latter is singular along the real domain.

- The reason for the singularity is that the zero set is invariant under the involution $\zeta \to \bar{\zeta}$, since the eigenfunction is real valued on M. The fixed point set is M and is also where zeros concentrate.

Example: the unit circle S^1

- The (real) eigenfunctions are $\cos k\theta, \sin k\theta$ on a circle.
- The complexification is the cylinder $S^1_{\mathbb{C}} = S^1 \times \mathbb{R}$.
- The complexified configuration space is similar to the phase space T^*S^1. This is always true.
- The holomorphically extended eigenfunctions are $\cos kz, \sin kz$.

Simplest case: S^1

The zeros of $\sin 2\pi kz$ in the cylinder $\mathbb{C}/\mathbb{Z}$ all lie on the real axis at the points $z = \frac{n}{2k}$. Thus, there are $2k$ real zeros. The limit zero distribution is:

$$\begin{aligned} \lim_{k\to\infty} \tfrac{i}{2\pi k} \partial\bar\partial \log|\sin 2\pi k|^2 &= \lim_{k\to\infty} \tfrac{1}{k} \textstyle\sum_{n=1}^{2k} \delta_{\frac{n}{2k}} \\ &= \tfrac{1}{\pi}\delta_0(\xi) dx \wedge d\xi. \end{aligned}$$

On the other hand,

$$\begin{aligned} \tfrac{i}{\pi}\partial\bar\partial|\xi| &= \tfrac{i}{\pi}\tfrac{d^2}{4d\xi^2}|\xi| \;\; \tfrac{2}{i} dx \wedge d\xi \\ &= \tfrac{i}{\pi}\;\tfrac{1}{2}\;\delta_0(\xi)\;\; \tfrac{2}{i} dx \wedge d\xi. \end{aligned}$$

Ergodicity of eigenfunctions in the complex domain

Ergodic eigenfunctions in the complex domain:

- Have extremal growth– $\frac{1}{\lambda} \log |\varphi_\lambda^{\mathbb{C}}|^2$ is like Siciak's maximal plurisubharmonic function on $\mathbb{C}^n$;
- Have maximal growth rate of zeros

Work in Progress: Intersections of nodal lines and geodesics

To get closer to real zeros, we "magnify" the singularity in the real domain by intersecting nodal lines and geodesics on surfaces $\dim M = 2$.
Let $\gamma \subset M^2$ be geodesic arc on a real analytic Riemannian surface. We identify it with a a real analytic arc-length parameterization $\gamma : \mathbb{R} \to M$. For small ϵ, $\exists$ analytic continuation

$$\gamma_{\mathbb{C}} : S_\tau := \{t + i\tau \in \mathbb{C} : |\tau| \leq \epsilon\} \to M_\tau.$$

Consider the restricted (pulled back) eigenfunctions

$$\gamma_{\mathbb{C}}^* \varphi_{\lambda_j}^{\mathbb{C}} \text{ on } S_\tau.$$

Intersections of nodal lines and geodesics

Let

$$\mathcal{N}^{\gamma}_{\lambda_j} := \{(t + i\tau : \gamma_H^* \varphi^{\mathbb{C}}_{\lambda_j}(t + i\tau) = 0\} \tag{1}$$

be the complex zero set of this holomorphic function of one complex variable. Its zeros are the intersection points.
Then as a current of integration,

$$[\mathcal{N}^{\gamma}_{\lambda_j}] = i\partial\bar{\partial}_{t+i\tau} \log \left| \gamma^* \varphi^{\mathbb{C}}_{\lambda_j}(t + i\tau) \right|^2 . \tag{2}$$

Equidistribution of intersections

THEOREM/CONJECTURE

Let (M,g) be real analytic with ergodic geodesic flow. Then there exists a subsequence of eigenvalues λ_{j_k} of density one such that

$$\frac{i}{\pi\lambda_{j_k}}\partial\bar{\partial}_{t+i\tau}\log\left|\gamma^*\varphi^{\mathbb{C}}_{\lambda_{j_k}}(t+i\tau)\right|^2\to\delta_{\tau=0}ds.$$

The convergence is weak convergence on $C_c(S_\epsilon)$.*

Thus, intersections of (complexified) nodal sets and geodesics concentrate in the real domain– and are distributed by arc-length measure on the real geodesic.
(Proof seems complete for periodic geodesics on surfaces when the geodesic satisfies a generic asymmetry condition; also for "random" geodesics in all dimensions)

Ideas of proofs

We now explain:

- Why it helps to work in the complex domain;
- How we relate nodal sets and geodesic flow;
- How to study intersections of nodal lines and geodesics in the ergodic case.

Why it helps to work in $M_{\mathbb{C}}$

In the complex domain we have:

1. Poincaré-Lelong formula: $Z_{\varphi_j} = \frac{i}{2\pi}\partial\bar{\partial}\log|\varphi_j^{\mathbb{C}}|^2$.
2. Compactness in L^1 of the PSH functions

$$\{\frac{i}{\lambda_j}\partial\bar{\partial}\log|\varphi_j^{\mathbb{C}}|^2\}.$$

3. L^2 norm of $|\varphi_j^{\mathbb{C}}(\zeta)|$ on Grauert tube M_τ is $e^{\lambda_j\tau}$. Easy to see from Poisson-wave kernel.
4. Control over weak* limits of $|\varphi_j^{\mathbb{C}}|^2\}$ when geodesic flow is ergodic (quantum ergodicity).

Step I: Ergodicity of complexified eigenfunctions

The first step is to prove quantum ergodicity of the complexified eigenfunctions:

Theorem

Assume the geodesic flow of (M, g) is ergodic. Then

$$\frac{|\varphi_\lambda^\epsilon(z)|^2}{||\varphi_\lambda^\epsilon||^2_{L^2(\partial M_\epsilon)}} \to 1, \quad \textit{weakly in} \;\; L^1(M_\epsilon),$$

along a density one subsquence of λ_j.

This is the analogue of what can be proved for the real eigenfunctions (Shnirelman, SZ, Colin de Verdiere).

Nodal sets (related: Shiffman-Z, Nonnenmacher)

LEMMA

We have:

$$\frac{1}{\lambda_j} \log |\varphi_\lambda^\epsilon(z)|^2 \to \sqrt{\rho}, \quad in\ L^1(M_\epsilon).$$

Combine with Poincare- Lelong:

$$[\tilde{Z}_j] = \partial\bar{\partial} \log |\tilde{\varphi}_j^{\mathbb{C}}|^2$$

to get

$$\frac{1}{\lambda_j}[\tilde{Z}_j] \to i\partial\bar{\partial}\sqrt{\rho}.$$

The exponential growth of $|\varphi_j^{\mathbb{C}}(\zeta)|$ comes directly from the eigenvalue equation

$$U(i\tau)_{\mathbb{C}}\varphi_j = e^{-\lambda_j \sqrt{\rho}(\zeta)} \varphi_j^{\mathbb{C}}.$$

Equi-distribution of intersections

So far:

$$\frac{1}{\lambda_j}\int_{Z_{\varphi^{\mathbb{C}}_{\lambda_j}}} f\omega_g^{n-1} \to \frac{i}{\pi}\int_{M_\tau} f\bar{\partial}\partial\sqrt{\rho}\wedge\omega_g^{n-1}$$

Intersections with typical geodesic:

$$\gamma_{\mathbb{C}} : S_\tau := \{t + i\tau \in \mathbb{C} : |\tau| \leq \epsilon\} \to M_\tau .$$

Then:

$$\frac{i}{\pi\lambda_{j_k}}\partial\bar{\partial}_{t+i\tau}\log\left|\gamma^*\varphi^{\mathbb{C}}_{\lambda_{j_k}}(t+i\tau)\right|^2 \to \delta_{\tau=0}ds.$$

The convergence is weak* convergence on $C_c(S_\epsilon)$.

New ingredient: quantum ergodic restriction theorem

In the real domain:

THEOREM

(J. Toth and S. Z 2010-2011; Dyatlov-Zworski, 2012) If G^t is ergodic and a geodesic H is "asymmetric" then the restrictions of $\{\varphi_j\}$ to H are quantum ergodic on H in the sense that

$$\lim_{\lambda_j \to \infty; j \in S} \langle Op_{\lambda_j}(a_0) \varphi_{\lambda_j}|_H, \varphi_{\lambda_j}|_H \rangle_{L^2(H)}$$

$$= c_n \int_{B^* H} a(s, \tau)\, \rho^H_{\partial\Omega}(s, \tau)\, ds d\tau$$

for a certain measure $\rho^H_{\partial\Omega}(s, \tau)\, ds d\tau$.

Intersections of complex zeros and geodesics

To analyze intersections of nodal lines and geodesics, we need a quantum ergodic restriction in the complex domain. It's completely different ! Analytic continuation decouples modes:

Example: Round S^2. Let Y_m^N be the usual joint eigenfunctions of Δ and rotation around the z-axis, with Y_m^N transforming by $e^{im\theta}$ under rotation. Any eigenfunction is $\varphi_N = \sum_{m=-N}^N a_{Nm} Y_m^N$. Restrict to equator: $\varphi_N|_{\varphi=0} = \sum_{m=-N}^N a_{Nm} P_m^N(1) e^{im\theta}$. Analytically continue to complex equator:

$$\varphi_N^{\mathbb{C}}|_{\gamma\mathbb{C}} = \sum_{m=-N}^{N} a_{mN} P_m^N(1) e^{im(\theta+i\eta)}.$$

Term with top m dominates! Ergodicity (or random-ness): the $a_{NN} \neq 0$, $a_{N,-N} \neq 0$. Equipartition of energy.

Complexified Poisson kernel

To connect eigenfunctions and geodesic flow, we use the Poisson kernel

$$U(i\tau, x, y) = \sum_{j=0}^{\infty} e^{-\tau\lambda_j} \varphi_j(x)\varphi_j(y).$$

It admits a holomorphic extension to $M_{\mathbb{C}}$ in $x \to \zeta$ when $\sqrt{\rho}(\zeta) < \tau$.

Theorem

(Hadamard, Mizohata; Boutet de Monvel; SZ 2011, M. Stenzel 2012) $U(i\epsilon, z, y) : L^2(M) \to H^2(\partial M_\epsilon)$ *is a complex Fourier integral operator of order* $-\frac{m-1}{4}$ *quantizing the complexified exponential map* $\exp_y(i\epsilon)\eta/|\eta|)\}$.

Euclidean case

On $\mathbb{R}^n$:

$$U(t, x, y) = \int_{\mathbb{R}^n} e^{it|\xi|} e^{i\langle \xi, x-y \rangle} d\xi.$$

Its analytic continuation to $t + i\tau, \zeta = x + ip$ is given by

$$U(t + i\tau, x + ip, y) = \int_{\mathbb{R}^n} e^{i(t+i\tau)|\xi|} e^{i\langle \xi, x+ip-y \rangle} d\xi,$$

which converges absolutely for $|p| < \tau$.
Key point:

$$U(i\tau)\varphi_{\lambda_j} = e^{-\tau\lambda_j} \varphi_{\lambda_j}^{\mathbb{C}}.$$

But $U(i\tau)\varphi_{\lambda_j}$ only changes L^2 norms by powers of λ_j. So exponential growth $= e^{\tau\lambda_j}$.