Nodal portraits of random waves

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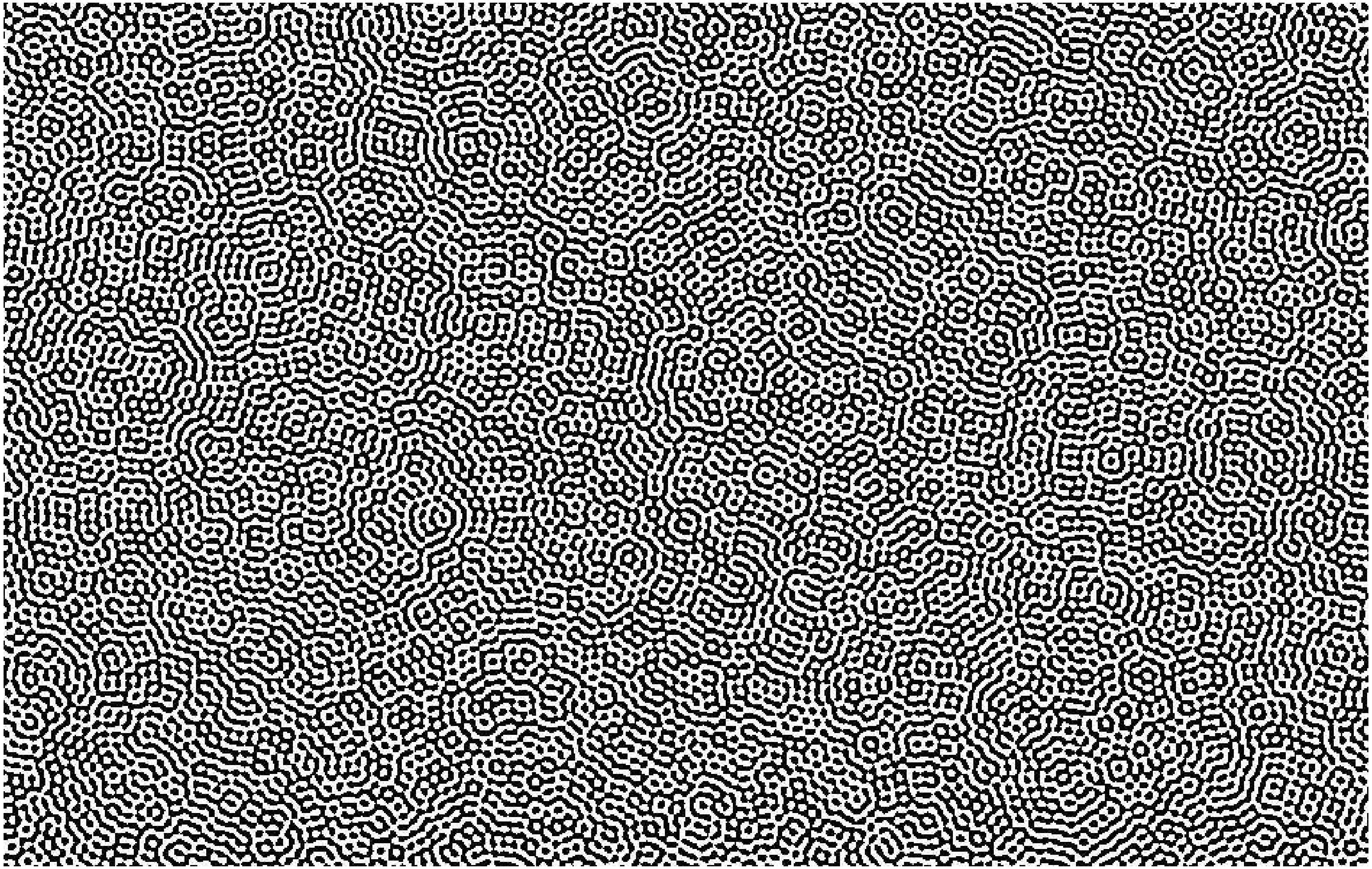


Nodal portrait of the Gaussian plane monochromatic wave provided by Alex Barnett

I: Smooth Gaussian functions (= processes = fields = waves)

T Riemannian manifold without boundary ($T = \mathbb{R}^m$, or T is a compact manifold)

$\big(\Omega, \mathcal{A}, \mathcal{P}\big)$ probability space

Definition: C^p-random Gaussian function $f\colon T \times \Omega \to \mathbb{R}^1$

(a) $\forall t \in T \quad \forall B \in \mathtt{Borel}(\mathbb{R}^1) \quad f(t)^{-1}(B) \in \mathcal{A}$ (measurability)

(b) $\forall n \in \mathbb{N} \quad \forall t_1, ..., t_n \in T \quad \forall c_1, ..., c_n \in \mathbb{R}^1$

$\sum c_i f(t_i)$ Gaussian random variable (maybe, degenerate)

(c) $\forall^{\text{a.e.}} \omega \in \Omega \quad f \in C^p(T) \quad$ (wlog, $\forall \omega \in \Omega$)

Claim: $f\colon \big(\Omega, \mathcal{A}\big) \to \big(C^p(T), \mathtt{Borel}(C^p(T))\big)$ *measurable map* $\square$

Definition: $\gamma_f = f_* \mathcal{P}$ Gaussian measure on $C^p(T)$

Remark: for $p_1 > p_2$, $C^{p_1}(T) \in \mathtt{Borel}(C^{p_2}(T))$

Definition: f and g are equivalent if $\gamma_f = \gamma_g$. We do not distinguish between equivalent measures

Definition: $K_f(t,s) = \mathcal{E}\{f(t)f(s)\}$ covariance of f

Hermitean-positivity on $T \times T$: $\sum_{i,j} c_i c_j K(t_i, t_j) = \mathcal{E}\left|\sum_i c_i f(t_i)\right|^2 \geq 0$

Knowing the Hermitean-postive kernel K, we can recover unique Gaussian function f with $K_f = K$. Indeed, K defines finite dimensional Gaussian distributions of $\big(f(t_1), ..., f(t_n)\big)$, which, by Kolmogorov's theorem, define f.

Another approach uses reproducing kernel Hilbert spaces ("RKHS", for short)
Note: all Hilbert spaces we deal with are real and separable

Suppose: $\forall s \in T \quad g \mapsto g(s)$ is a bounded functional on $\mathcal{H}$
$\Longrightarrow g(s) = \langle g, K_{\mathcal{H}}(\,.\,, s)\rangle_{\mathcal{H}}$, $s \in T$, $g \in \mathcal{H}$.
Then $K_{\mathcal{H}}$ is called the reproducing kernel on $\mathcal{H}$ ("repro-kernel", for short)

Lemma: *Given continuous Hermitean-positive kernel K, there exists a unique RKHS $\mathcal{H}$ of functions on T with $K_{\mathcal{H}} = K$* $\square$

Remark: for any o.n.b. $\{e_i\}$ in $\mathcal{H}$, $K_{\mathcal{H}}(t,s) = \sum_i e_i(t)e_i(s)$ (convergence in $\mathcal{H}$). In particular, $K(t,t) = \sum_i e_i^2(t)$

We take an o.n.b. $\{e_i\}$ in $\mathcal{H}$, take independent standard Gaussian r.v.'s ξ_i, and put

$$f(t) \stackrel{\text{def}}{=} \sum_i \xi_i e_i(t) \qquad (*)$$

Lemma: *Given $t \in T$, the series converges in $L^2(\Omega)$ and a.s.* $\square$

That is, the series $(*)$ gives us a Gaussian function f with $K_f = K_{\mathcal{H}}$. What about its smoothness?

Lemma: *Suppose $K \in C^{2p+\epsilon}(T \times T)$ with $\epsilon > 0$ Then, a.s., f is C^p-smooth, and the series $(*)$ converges locally uniformly with p derivatives* $\square$

In what follows, we need only C^2-smoothness of f. In all interesting examples, which I am aware of, everything is C^∞-smooth (and even real-analytic).

II: Zero sets of translation-invariant Gaussian functions

(Euclidean case: $T = \mathbb{R}^m$)

<u>Definition</u> Translation-invariance (= stationarity = homogeneity):
$\forall n \in \mathbb{N}\ \forall u_1, ..., u_n \in \mathbb{R}^m\ \forall v \in \mathbb{R}^m$ *the random vectors* $\big(F(u_1), ..., F(u_n)\big)$ *and* $\big(F(u_1+v), ..., F(u_n+v)\big)$ *have the same (Gaussian) distribution*

Then $K(u,v) = \mathcal{E}\left\{F(u)F(v)\right\} = \mathcal{E}\left\{F(u-v)F(0)\right\} = k(u-v)$.
By Bochner's theorem,

$$k(u) = \int_{\mathbb{R}^m} e^{2\pi \mathrm{i} u\cdot\lambda}\, \mathrm{d}\rho(\lambda) = \int_{\mathbb{R}^m} \cos\left(2\pi u \cdot \lambda\right)\, \mathrm{d}\rho(\lambda)$$

$\rho \in M^+_{\mathtt{sym}}(\mathbb{R}^m)$ is the spectral measure of F.

In this case, the RKHS $\mathcal{H} = \mathcal{F}L^2_{\mathtt{sym}}(\rho)$, $\mathcal{F}$ stands for the Fourier transform

- The spectral measure contains all information about the random function F

C^2-smoothness of F: for some $p > 2$, $\displaystyle\int_{\mathbb{R}^m} |\lambda|^{2p}\, \mathrm{d}\rho(\lambda) < \infty$

Examples:

- $\rho =$ Lebesgue measure on $\mathbb{S}^{m-1}$, "Helmholtz wave".

The random Gaussian function F satisfies the Helmholtz equation $\Delta F + \kappa^2 F = 0$

- $\rho =$ Lebesgue measure on $[-1,1]^m$, "Paley-Wiener wave"

$k(u) = \prod_{j=1}^{m} \frac{\sin(2\pi u_j)}{2\pi u_j}$. The RKHS $\mathcal{H}$ is the m-dim Paley-Wiener space

- $\rho =$ standard Gaussian measure on $\mathbb{R}^m$, "Fock-Bargmann wave"

The RKHS $\mathcal{H}$ is the m-dim Fock-Bargmann space

In these examples, F has an analytic continuation to the whole $\mathbb{C}^m$. Unfortunately, it seems that up to now, this was not of much help in understanding topology of the zero set of $F\big|_{\mathbb{R}^m}$

Notation: $Z(F) = F^{-1}\{0\}$ the (random) zero set of F

$N(R; F)$ the number of connected components of $Z(F)$ that are contained in the open ball $B(R) = \{x \colon |u| < R\}$;

Theorem I (F.Nazarov - M.S.): *Suppose that the spectral measure ρ has no atoms and is not supported by a hyperplane.*

(i) *Then there exists a constant $\nu(\rho) \geq 0$ s.t.*

$$\lim_{R\to\infty} \frac{N(R; F)}{\mathrm{vol}B(R)} = \nu(\rho) \quad \text{a.s. and in mean.}$$

(ii) *The limiting constant $\nu(\rho)$ is <u>positive</u> provided that*

$(*)$ $\exists$ *a compactly supported Hermitean-symmetric measure μ with* $\mathrm{spt}(\mu) \subset \mathrm{spt}(\rho)$ *and a bounded domain $G \subset \mathbb{R}^m$ s.t. $\widehat{\mu}\big|_{\partial G} < 0$, while for some $u_0 \in G$, $\widehat{\mu}(u_0) > 0$.*

Here, $\widehat{\mu}$ stands for the Fourier transform of μ. Hermitean positivity means: $\mu(-E) = \overline{\mu(E)}$, that is, $\widehat{\mu}$ is real-valued

How to check condition $(*)$?

$(*)$ $\exists$ a compactly supported Hermitean-symmetric measure μ with $\mathrm{spt}(\mu) \subset \mathrm{spt}(\rho)$ and a bounded domain G s.t. $\widehat{\mu}\big|_{\partial G} < 0$ while for some $u_0 \in G$, $\widehat{\mu}(u_0) > 0$.

A simple and crude sufficient condition:

- $\mathrm{spt}(\rho)$ *contains a sphere centered at the origin.*

Proof: take $\mu =$ the Lebesgue measure on that sphere $\Longrightarrow$ $\widehat{\mu}$ is radially symmetric, vanishes on concentric spheres with radii tending to ∞ $\square$

Using a little bit of harmonic analysis, one can go further:

- $\mathrm{spt}(\rho)$ *contains an open subset of a sphere centered at the origin*

In the planar case $(m = 2)$, there is another simple sufficient condition:

- $\mathrm{spt}(\rho)$ *contains a compact set that cannot be covered by finitely many segments*

Proof: Take $\lambda_1, \lambda_2 \in \mathrm{spt}(\rho)$, $\lambda_2 \neq c\lambda_1$, and consider

$$\cos\big(\lambda_1 \cdot x\big) + \cos\big(\lambda_2 \cdot x\big) = 2\cos\Big(\frac{\lambda_1 + \lambda_2}{2} \cdot x\Big)\cos\Big(\frac{\lambda_1 - \lambda_2}{2} \cdot x\Big)$$

Then add an accurately chosen small trigonometric sum with frequencies at $\mathrm{spt}(\rho)$ to create a bounded component around the origin. $\square$

Basics from the ergodic theory needed for the proof of Theorem I:

Suppose $\mathbb{R}^m$ acts by measure-preserving transformations $\{\tau_v\}_{v\in\mathbb{R}^m}$ on a probability space $(\Omega, \mathfrak{S}, \mathcal{P})$

WIENER'S ERGODIC THEOREM: *Suppose that $\Phi \in L^1(\mathcal{P})$ and that $(v,\omega) \mapsto \Phi(\tau_v\omega)$ is $\mathtt{Borel}(\mathbb{R}^m) \times \mathfrak{S}$ measurable. Then the limit*

$$\lim_{R\to\infty} \frac{1}{\mathrm{volB(R)}} \int_{B(R)} \Phi(\tau_v\omega)\, \mathrm{dvol}(v) = \bar{\Phi}(\omega)$$

exists with probability 1 *and in $L^1(\mathcal{P})$. The limiting r.v. $\bar{\Phi}$ is τ-invariant, which means that $\bar{\Phi}\circ\tau_v = \bar{\Phi}$, $v \in \mathbb{R}^m$.*

The action of $\mathbb{R}^m$ is *ergodic* if for each τ-invariant set $A \in \mathfrak{S}$, either $\mathcal{P}(A) = 0$ or $\mathcal{P}(A) = 1$. In this case, the limiting r.v. $\bar{\Phi}$ is constant. Due to the L^1-convergence, $\bar{\Phi} = \mathcal{E}\{\Phi\}$.

Basics from the ergodic theory (continuation):

In our set-up, $\Omega = C^2(\mathbb{R}^m)$ (countably normed space)

$\mathfrak{S} = \mathtt{Borel}(C^2(\mathbb{R}^m))$

$\mathcal{P} = \gamma_F$ gaussian measure on $C^2(\mathbb{R}^m)$ generated by F

$\overline{\mathfrak{S}}_F$ the Lebesgue completion of $\mathfrak{S}$ w.r.t. γ_F

$\mathbb{R}^m$ acts by shifts: $\tau_v G(u) = G(u+v)$

Fomin-Grenander-Maruyama Theorem: *The action of the shifts is ergodic provided that the spectral measure has no atoms.*

Conclusion: *In assumptions of Theorem I, for any r.v.* $\Phi \in L^1(\gamma_F)$ *such that the function* $(v, G) \mapsto \Phi(\tau_v G)$ *is* $\mathtt{Borel}(\mathbb{R}^m) \times \mathfrak{S}_F$ *measurable,*

$$\lim_{R\to\infty} \frac{1}{\mathrm{volB(R)}} \int_{B(R)} \Phi(\tau_v G)\,\mathrm{dvol}(v) = \mathcal{E}\{\Phi\} \qquad \text{a.s. and in } L^1(\gamma_F)$$

Now, we are ready to prove Theorem I

III: Proof of Theorem 1

Step 1: Integral geometric sandwich

Notation: $N(u,r;F)$ the number of connected components of $Z(F)$ contained in the open ball $B(u,r)$, $N^*(u,r;F)$ the number of connected components of $Z(F)$ that intersect the closed ball $\bar{B}(u,r)$

"Sandwich estimate": for $0 < r < R < \infty$,

$$\int_{B(R-r)} \frac{N(u,r;F)}{\mathrm{vol}B(r)}\,\mathrm{dvol}(u) \le N(R;F) \le \int_{B(R+r)} \frac{N^*(u,r;F)}{\mathrm{vol}B(r)}\,\mathrm{dvol}(u)$$

Observe that $N^*(u,r;F) - N(u,r;F) \le \#\text{critical pts of } F\big|_{\partial\mathbb{B}(u;r)} \stackrel{\text{def}}{=} \mathfrak{N}(u,r;F)$, and that $N(u,r;F) = N(r;\tau_u F)$, $\mathfrak{N}(u,r;F) = \mathfrak{N}(r;\tau_u F)$. Thus

$$\frac{1-o(1)}{\mathrm{vol}B(R-r)} \int_{B(R-r)} \frac{N(r;\tau_u F)}{\mathrm{vol}B(r)}\,\mathrm{dvol}(u) \le \frac{N(R;F)}{\mathrm{vol}B(R)}$$

$$\le \frac{1+o(1)}{\mathrm{vol}B(R+r)} \int_{B(R+r)} \frac{N(r;\tau_u F) + \mathfrak{N}(r;\tau_u F)}{\mathrm{vol}B(r)}\,\mathrm{dvol}(u)\,, \qquad R \to \infty,\ r \text{ fixed}$$

Step 2. Applying ergodic theorem:

Fix $r > 0$ and apply ergodic theorem to the functions $G \mapsto N(r; G)$, $G \mapsto \mathfrak{N}(r; G)$. We see that for each $r > 0$,

$$\frac{\mathcal{E}N(r;F)}{\mathrm{vol}B(r)} \le \varliminf_{R\to\infty} \frac{N(R;F)}{\mathrm{vol}B(R)} \le \varlimsup_{R\to\infty} \frac{N(R;F)}{\mathrm{vol}B(R)} \le \frac{\mathcal{E}N(r;F) + \mathcal{E}\mathfrak{N}(r;F)}{\mathrm{vol}B(r)}$$

Step 3: The Kac-Rice bound for the number of critical points:

$B \subset \mathbb{R}^m$ a ball, $f\colon \bar{B} \to \mathbb{R}^1$ Gaussian $C^2(\bar{B})$-function
$\mathfrak{N}(\bar{B}; f)$ the number of critical points of f in $\bar{B}$

Lemma: *Suppose* $M = \sup_{\bar{B}} \mathcal{E}\|\nabla^2 f\|^2 < \infty$ *and* $\kappa = \inf_{\bar{B}} \det \mathtt{Cov}[\nabla f, \nabla f] > 0$. *Then*

$$\mathcal{E}\mathfrak{N}(\bar{B}; f) \lesssim_m M^{m/2} \kappa^{-1/2} \mathrm{vol}(B)$$

We apply this estimate to restriction of F to spherical caps on $\partial\mathbb{B}(r)$:

Corollary: $\mathcal{E}\mathfrak{N}(r; F) \lesssim \mathrm{vol}_{m-1}\partial\mathbb{B}(r)$

Hence, $\lim_{R\to\infty} \frac{N(R;F)}{\mathrm{vol}B(R)}$ exists (a.s. and in L^1) and equals $\lim_{r\to\infty} \frac{\mathcal{E}N(r;F)}{\mathrm{vol}B(r)} =: \nu(\rho)$

Idea of the proof of Kac-Rice bound:

$g\colon \bar{B} \to \mathbb{R}^m$ C^1-vector field ($g = \nabla f$)

Let $X(\epsilon, \delta) = \{x \in \bar{B}\colon |g(x)| < \delta(|\mathrm{d}g| + \epsilon)\}$ ($\mathrm{d}g$ the differential of g)

Suppose $g(x_0) = 0 \Longrightarrow B(x_0, \delta) \cap \bar{B} \subset X(\epsilon, \delta)$ provided that δ is small
($\sup\{|\mathrm{d}g(x_1) - \mathrm{d}g(x_2)|\colon |x_1 - x_2| < \delta\} < \epsilon$ suffices)

Hence, if g vanishes at N different points, then $N\delta^m \lesssim \mathrm{vol}X(\epsilon, \delta)$ for small δ's
$\Longrightarrow N \lesssim \limsup_{\delta \to 0} \delta^{-m}\mathrm{vol}X(\epsilon, \delta)$

We apply this argument to the Gaussian vector field $g = \nabla f$, and find an upper bound for $\delta^{-m}\mathcal{E}\{\mathrm{vol}X(\epsilon, \delta)\}$ *uniform* in δ and ϵ

For this, the uniform non-degeneration of the covariance matrix
$\mathtt{Cov}[\nabla f(x), \nabla f(x)]$ is essential. $\square$

Step 4: Positivity of $\nu(\rho)$:

Recall: by the sandwich estimate, $\forall r > 0\ \nu(\rho) \geq \dfrac{\mathcal{E}N(r;F)}{\mathrm{vol}B(r)}$

$\Longrightarrow$ it suffices to show that $\exists r_0 > 0$ s.t. $\mathcal{P}\big\{N(r_0;F)\big\} > 0$

Recall condition $(*)$: $\exists$ a compactly supported Hermitean-symmetric measure μ with $\mathrm{spt}(\mu) \subset \mathrm{spt}(\rho)$ and a bounded domain $G \subset \mathbb{R}^m$ s.t. $\widehat{\mu}\big|_{\partial G} < 0$ while for some $u_0 \in G$, $\widehat{\mu}(u_0) > 0$.

Hence, the following Gaussian lemma does the job:

Lemma: *Let μ be a Hermitean symmetric compactly supported measure with* $\mathrm{spt}(\mu) \subset \mathrm{spt}\rho$*. Then $\forall$ ball $B \subset \mathbb{R}^m$ and $\forall \epsilon > 0$,*

$$\mathcal{P}\big\{\|F - \widehat{\mu}\|_{C(\bar{B})} < \epsilon\big\} > 0\,.$$

Applying this lemma with μ from condition $(*)$ and with a ball $B(r_0)$ s.t. $\bar{G} \subset B(r_0)$, we see that with a positive probability the zero set $Z(F)$ has a bounded connected inside $B(r_0)$, whence, $\mathcal{P}\big\{N(r_0;F)\big\} > 0$ $\square$

LEMMA: *Let μ be a Hermitean symmetric compactly supported measure with* $\mathrm{spt}(\mu) \subset \mathrm{spt}\rho$. *Then $\forall$ ball $B \subset \mathbb{R}^m$ and $\forall \epsilon > 0$,*

$$\mathcal{P}\left\{ \|F - \widehat{\mu}\|_{C(\bar{B})} < \epsilon \right\} > 0\,.$$

Proof: Recall that $F(u) = \sum_j \xi_j e_j(u)$, $\{e_j\}$ is an O.N.B. in $\mathcal{H} = \mathcal{F}L^2_{\mathtt{sym}}(\rho)$, ξ_j's are independent standard Gaussian r.v.'s, and a.s. the series converges uniformly in B

$\Longrightarrow$ For every $h \in L^2_{\mathtt{sym}}(\rho)$, $\mathcal{P}\left\{ \|F - \widehat{h\,\mathrm{d}\rho}\|_{C(\bar{B})} < \epsilon \right\} > 0$

Using that

- Every Hermitean symmetric compactly supported measure with $\mathrm{spt}(\mu) \subset \mathrm{spt}\rho$ can be weakly approximated by measures of the form $h\,\mathrm{d}\rho$, h is a compactly supported function in $L^2_{\mathtt{sym}}(\rho)$

- Weak convergence of compactly supported measures yields locally uniform convergence of their Fourier transforms.

we complete the proof of Lemma and finish off the proof of Theorem I. $\square$

Part IV: Some questions

• Nature of the limiting constant $\nu(\rho)$

• Growth of the variance of $N(R;F)$ (presumably, $\sim$ const R^m)

• Exponential concentration of $N(R;F)/\mathrm{vol}B(R)$ around $\nu(\rho)$: show that for each $\epsilon > 0$,

$$\mathcal{P}\left\{\left|\frac{N(R;F)}{\mathrm{vol}B(R)} - \nu(\rho)\right| > \epsilon\right\} \le C(\epsilon)e^{-c(\epsilon)R}$$

This is open even in the 1-dimensional case (cf. Tsirelson's lecture notes `http://www.tau.ac.il/~tsirel/Courses/Gauss3/main.html`)

Obstacle: nodal domains of small volume (clumping zeroes in the 1-dim case). The exponential concentration is known for Gaussian Helmholtz waves when such nodal domains do not exist (due to equation $\Delta F + \kappa^2 F = 0$)

• Statistics of bounded components of large diameter: given $\alpha \in (0,1)$, find the asymptotics of the mean number of connected components of diameter $\simeq R^\alpha$

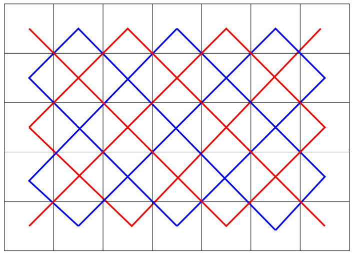
More questions: Bogomolny-Schmit bond percolation model

$2D$ Gaussian Helmholtz wave, ρ Lebesgue measure on $\mathbb{S}^1$
$\Longrightarrow \Delta F + \kappa^2 F = 0 \Longrightarrow$ local maxima are > 0, local minima are < 0

Naïve checkerboard nodal picture: the square lattice; the cells represent local maxima/minima, the sites are the saddle points; the saddle heights equal 0

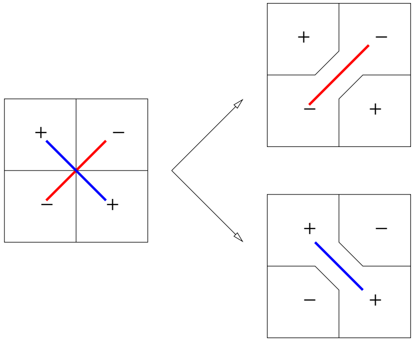
Note: asymptotic Morse relations: $N_{\mathtt{max}} + N_{\mathtt{min}} \approx N_{\mathtt{saddle}}$

Two dual square lattices:
blue one (local maxima) vertices at the cells of the grid where $F > 0$
red one (local minima) vertices at the cells of the grid where $F < 0$



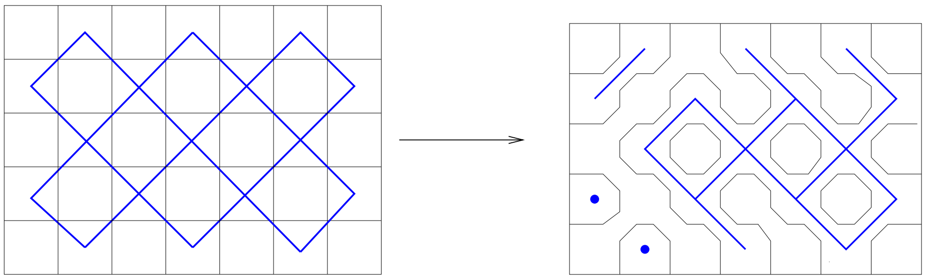
If the saddle height is positive then the bond between two neighboring maxima is open, if it is negative, then the bond is closed.

Bogomolny-Schmit assumptions: *saddle heights are uncorrelated and have equal probability being positive or negative*:



Each realization generates two graphs: the blue one whose vertices are the blue lattice points and the red one whose vertices are the red lattice points.

Each of these graphs uniquely determines the whole picture, and each of them represents the critical bond percolation on the corresponding square lattice:



Using heuristics from statistical mechanics, B-S predicted that for $R \to \infty$,

$$\mathcal{E}N(R;F) = (\nu + o(1))R^2, \qquad \text{variance of } N(R;F) = (b + o(1))R$$

with explicitly computed positive constants ν and b, and checked consistency of their predictions with numerics

Major problem: ignores correlations (which decay only as dist$^{-1/2}$)

'Minor' problem: no rigorous mathematical treatment of the critical bond percolation on the square lattice ...

Challenge: Reveal "hidden universality law" that provides the rigorous foundation for the B-S work

Question: Show that a.s. there is no infinite nodal line

Question: Show that for each $\epsilon > 0$, the probability that the set $\{x \colon F(x) > \epsilon, |x| < R\}$ has a component with diameter $\geq \epsilon R$ tends to zero.

Part V. Riemannian case

SET-UP: X smooth compact m-dim Riemannian manifold without boundary

We start with a family of finite-dimensional RKHS $\mathcal{H}_L$ of smooth functions on X, $\dim\mathcal{H}_L \to \infty$ as $L \to \infty$.

$K_L(x,y)$ is a repro-kernel of $\mathcal{H}_L$: $f(y) = \langle f(\,.\,), K_L(\,.\,,y)\rangle_{\mathcal{H}_L}$, $f \in \mathcal{H}_L$, $y \in X$

We always assume that $x \mapsto K_L(x,x)$ does not vanish on X.

Gaussian ensemble: $f_L = \sum \xi_k e_k$, $\{e_k\}$ is an ONB in $\mathcal{H}_L$, and ξ_k are independent standard Gaussian r.v.'s

The covariance of the Gaussian function f_L:

$$\mathcal{E}\{f_L(x)f_L(y)\} = \sum e_k(x)e_k(y) = K_L(x,y)$$

The distribution of f_L does not depend on the choice of the orthonormal basis $\{e_k\}$ in $\mathcal{H}_L$.

Normalization:

Wlog, we assume that the functions f_L are *normalized*, that is, $\mathcal{E} f_L^2(x) = K_L(x,x) = 1$, $x \in X$.

Otherwise, replace the functions f_L and the kernel K_L by

$$\widetilde{f}_L(x) = \frac{f_L(x)}{\sqrt{\mathcal{E} f_L^2(x)}}, \quad \widetilde{K}_L(x,y) = \frac{K_L(x,y)}{\sqrt{K_L(x,x) \cdot K_L(y,y)}} .$$

This normalization changes the Hilbert spaces $\mathcal{H}_L$ but the zero sets of the Gaussian functions f_L and $\widetilde{f}_L$ remain the same.

In basic examples, the function $x \mapsto K_L(x,x)$ is constant (that is, the norm of the point evaluation in $\mathcal{H}_L$ does not depend on the point), so the normalization boils down to the division by that constant.

Transplantation to the Euclidean space and scaling

T_xX the tangent space at x

$\exp_x : \mathbb{R}^m \to X$ the exponential map

$I_x : \mathbb{R}^m \to T_x(X)$ a linear Euclidean isometry (its choice is irrelevant for us)

Put $\Phi_x = \exp_x \circ I_x : \mathbb{R}^m \to X$, $\Phi_x(0) = x$

To scale the covariance kernel K_L at $x \in X$ in L times, put

$$K_{x,L}(u,v) \stackrel{\text{def}}{=} K_L\left(\Phi_x(L^{-1}u), \Phi_x(L^{-1}v)\right), \qquad u, v \in \mathbb{R}^m .$$

Note: $K_{x,L}(u,v)$ are covariance kernels of scaled Gaussian functions $f_{x,L}(u) \stackrel{\text{def}}{=} f_L(\Phi_x(L^{-1}u))$, $u \in \mathbb{R}^m$, that is,

$$K_{x,L}(u,v) = \mathcal{E}\{f_{x,L}(u) f_{x,L}(v)\}$$

Definition: The Gaussian ensemble (f_L) has *translation-invariant local limits* as $L \to \infty$ if for a.e. $x \in X$, there exists a Hermitean positive definite function $k_x \colon \mathbb{R}^m \to \mathbb{R}^1$, such that for each $R < \infty$,

$$\lim_{L \to \infty} \sup_{|u|,|v| \le R} |K_{x,L}(u,v) - k_x(u-v)| = 0 \,.$$

The limiting kernels $k_x(u-v)$ are covariance kernels of translation-invariant Gaussian functions $F_x \colon \mathbb{R}^m \to \mathbb{R}^1$ $(x \in X)$

$k_x = \widehat{\rho}_x, \quad \rho_x$ are probability meas ρ_x on $\mathbb{R}^m$, symmetric w.r.t. the origin

We call the function F_x *the local limiting function*, and the measure ρ_x *the local limiting spectral measure* of the family $\{f_L\}$ at the point x.

Technical assumptions: smoothness and non-degeneration

C^2-smoothness: *The Gaussian ensemble* (f_L) *is* $C^{2+\epsilon}$*-smooth if* $\exists p > 2$ *s.t. for every* $R < \infty$,

$$\limsup_{L\to\infty} \sup_{x\in X} \mathcal{E}\|f_{x,L}\|^2_{C^p(\bar{B}(R))} < \infty$$

Remark: this holds provided that there exists $p > 2$ s.t. $\forall R < \infty$ $K_{x,L} \in C^{2p}(\bar{B}(R) \times \bar{B}(R))$ uniformly in $x \in X$.

Non-degeneracy: $\liminf\limits_{L\to\infty} \inf\limits_{x\in X} \det \texttt{Cov}[\nabla f_L(x), \nabla f_L(x)] > 0\,.$

Here, $\texttt{Cov}[\nabla f_L(x), \nabla f_L(x)]$ is the covariance matrix of $\nabla f_L(x)$.

Note: if (f_L) is $C^{2+\epsilon}$-smooth, non-degenerate and has translation-invariant local limits, then limiting spectral measures ρ_x are not supported by hyperplanes and satisfy the moment condition $\displaystyle\int_{\mathbb{R}^m} |\lambda|^p \, \mathrm{d}\rho_x(\lambda) < \infty$. This holds uniformly w.r.t. $x \in X$.

$N(f_L)$ number of components of the zero set of the function f_L.
$\nu(\rho)$ limiting constant from Theorem I. Put $\bar{\nu}(x) = \nu(\rho_x)$.

THEOREM II (F.Nazarov-M.S.): *Suppose that (f_L) is a $C^{2+\epsilon}$-smooth non-degenerate Gaussian ensemble on X that has translation-invariant local limits. Suppose that local limiting spectral measures ρ_x have no atoms. Then $\bar{\nu} \in L^\infty(X)$, and*

$$\lim_{L\to\infty} \mathcal{E}\Big\{\Big| L^{-m} N(f_L) - \int_X \bar{\nu}\,\mathrm{d}\,\mathrm{vol}\Big|\Big\} = 0\,.$$

Remark: The measure $\bar{\nu}\,\mathrm{dvol_X}$ does not depend on the choice of the Riemannian metric on X. The change of the scalar products in the tangent space T_xX boils down to the counting the number of components of the zero set of the limiting translation-invariant Gaussian function F_x in concentric ellipsoids instead of balls.

Local version of Theorem II: $\forall^{\text{a.e.}} x \in X \ \forall \epsilon > 0$,

$$\lim_{R\to\infty} \lim_{L\to\infty} \mathcal{P}\Big\{ \Big| \frac{N\big(x, R/L; f_L\big)}{\mathrm{vol} B(R)} - \bar{\nu}(x) \Big| > \epsilon \Big\} = 0 \qquad (\texttt{loc})$$

Here, $N\big(x, \frac{R}{L}; f_L\big) = N\big(R; f_{x,L}\big)$ is a number of connected components of the zero set $Z(f_L)$ containing in the open ball centered at $x \in X$ of radius R/L, $\mathrm{vol}B(R)$ is the Euclidean volume of a ball of radius R.

Idea of the proof: $x \in X$, $R \gg 1$ are fixed. Let $L \to \infty$. Then $\mathcal{E}\big\{ f_{x,L}(u) f_{x,L}(v) \big\} \to \mathcal{E}\big\{ F_x(u) F_x(v) \big\}$ uniformly in $u, v \in \bar{B}(R)$.

1. *Coupling*: $\exists$ Gaussian random functions $\widetilde{f}_{x,L}$, $\widetilde{F}_x$ *defined on the same probability space*, s.t. $\widetilde{f}_{x,L}$ is equidistributed with $f_{x,L}$, $\widetilde{F}_x$ is equidistributed with F_x, and $\mathcal{E}\| \widetilde{f}_{x,L} - \widetilde{F}_x \|_{C^1(\bar{B}(R))}$ is small.

2. With high probability, $\min_{\bar{B}(R)} \max\big(|f_{x,L}|, |\nabla f_{x,L}| \big)$ is NOT small. This is due to statistical independence of $f_{x,L}$ and $\nabla f_{x,L}$

1. & 2. & some basic calculus yield $N(R-1; F_x) \le N(R; f_{x,L}) \le N(R+1; F_x)$ (with high probability). Then, applying Theorem I, we get ($\texttt{loc}$). $\square$

Theorem II is "an integrated version" of the local result:

1. Put $\Omega(\epsilon, x, R, L) = \{ \big| N(x, R/L, f_L)/\mathrm{vol}\,\mathrm{B(R)} - \bar{\nu}(x) \big| > \epsilon \}$
By (loc), $\lim\limits_{R\to\infty} \lim\limits_{L\to\infty} \mathcal{P}\big(\Omega(\epsilon, x, R, L)\big) = 0$.

Egorov's theorem $\Longrightarrow$ $\forall \eta > 0\ \exists X_\eta \subset X$ with $\mathrm{vol}\, X_\eta \ge (1-\eta)\,\mathrm{vol}\, X$ s.t.
$N(x, R/L, f_L)/\mathrm{vol}\,\mathrm{B(R)} \to \bar{\nu}(x)$ as $L \to \infty$, uniformly in $x \in X_\eta$

2. Integral-geometric sandwich ($R \gg 1$, $0 < \delta \ll 1$ are fixed, $L \to \infty$):

$$(1-\delta) \int_X \Big[\frac{N(x, R/L, f_L)}{\mathrm{vol}\, B(R/L)} - \bar{\nu}(x) \Big] \,\mathrm{d\,vol}\,(x) \le N(f_L)$$
$$\le (1+\delta) \int_X \Big[\frac{N^*(x, R/L, f_L)}{\mathrm{vol}\, B(R/L)} - \bar{\nu}(x) \Big] \,\mathrm{d\,vol}\,(x)$$

To return from N^* to N, we need to control the mean number of components of diameter $\gg R/L$ (using Kac-Rice bound).

3. We get $L^{-m} N(f_L) \approx \displaystyle\int_X \Big[\frac{N(x, R/L, f_L)}{\mathrm{vol}\, B(R)} - \bar{\nu}(x) \Big] \,\mathrm{d\,vol}\,(x)$.
To treat the integrals over $X \setminus X_\eta$, we need to control the mean number of small components (volume of the corresponding nodal domain is $\ll L^{-m}$) $\square$

Part VI. Examples: 4 ensembles satisfying conditions of Theorem II

In these examples, the function $K_L(x,x)$ is constant, the limiting translation-invariant kernel k_x does not depend on $x \in X$, is real analytic, and is not supported by a hyperplane.

The limiting spectral measure satisfies condition $(*)$ in Theorem I that yields positivity of the limiting constant $\nu(\rho)$.

The scaled kernels $K_{x,L}(u,v)$ converge to the limiting kernel $k(u-v)$ with partial derivatives of any order, and the convergence is uniform in $x \in X$. This yields smoothness and non-degeneracy of the ensembles (f_L).

1. Trigonometric ensemble $X = \mathbb{T}^m$ (m-dim torus)

$\mathcal{H}_{n,m} \subset L^2(\mathbb{T}^m)$ subspace of trigonometric polynomials in m variables of degree $\leq n$ in each of the variables.

The repro-kernel is the product of m Dirichlet kernels

$$K_{n,m}(x,y) = \prod_{j=1}^{m} \frac{\sin\left[\pi(2n+1)(x_j - y_j)\right]}{(2n+1)\sin\left[\pi(x_j - y_j)\right]},$$

scaling parameter $L = n$ (the degree)

After scaling, $K_{x,L}(u,v)$ converges with partial derivatives of any order to $k(u-v)$, $k(u) = \prod_{j=1}^{m} \frac{\sin 2\pi u_j}{2\pi u_j}$.

Limiting function F: the Paley-Wiener wave.

Limiting spectral measure $\rho =$ Lebesgue measure on the unit cube in $\mathbb{R}^m$

2. Arithmetic random waves $X = \mathbb{T}^m$

$\mathcal{H}_\lambda$ is the subspace of $L^2(\mathbb{T}^m)$ consisting of trigonometric polynomials of the form

$$\operatorname{Re} \sum_{\nu \in \mathbb{Z}^m :\ |\nu| = \lambda} c_\nu e^{2\pi \mathrm{i}(\nu \cdot x)},$$

$\dim \mathcal{H}_\lambda =$ the number of ways to represent the integer λ^2 as a sum of squares of integers. The repro-kernel is $K_\lambda(x-y)$ where

$$K_\lambda(x) = \frac{1}{\dim \mathcal{H}_\lambda} \sum_{\nu \in \mathbb{Z}^m :\ |\nu| = \lambda} \cos(2\pi \nu \cdot x).$$

The scaling parameter equals λ.

For $m \geq 5$, the limiting kernel k is the Fourier transform of the Lebesgue measure on the unit sphere in $\mathbb{R}^m$. The limiting function F is the Helmholtz wave. For $2 \leq m \leq 4$, this holds under additional arithmetic restrictions on λ.

It is curious that in the case $m = 3$, convergence to the limiting kernel follows from the Linnik equidistribution theorem for the number of lattice points on the three-dimensional sphere (cf. recent work of Bourgain-Rudnick)

3. Spherical ensemble: $X = \mathbb{S}^m$ (m-dim sphere)

$\mathcal{H}_{n,m} \subset L^2(\mathbb{S}^m)$ subspace spanned by polynomials in $m+1$ variables of degree $\le n$, restricted on $\mathbb{S}^m$.

The repro-kernel:

$$K_{n,m}(x,y) = c(n,m) P_n^{(\frac{m}{2},\frac{m}{2}-1)}(x \cdot y), \qquad x, y \in \mathbb{S}^m$$

where $P_n^{(\alpha,\beta)}$ are Jacobi polynomials of degree n and of index (α,β); i.e., polynomials orthogonal on $[-1,1]$ with the weight $(1-x)^\alpha(1+x)^\beta$.

Mehler-Heine asymptotics:

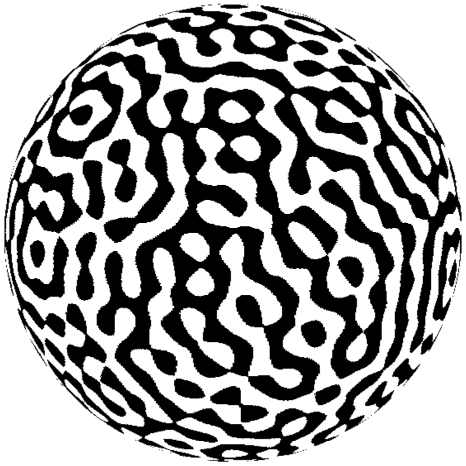
$$\lim_{n\to\infty} n^{-\frac{m}{2}} P_n^{(\frac{m}{2},\frac{m}{2}-1)}\left(\cos\frac{z}{n}\right) = \left(\frac{z}{2}\right)^{-\frac{m}{2}} J_{\frac{m}{2}}(z)$$

where $J_{\frac{m}{2}}(z)$ is Bessel's function, and the convergence is locally uniform in $\mathbb{C}$.

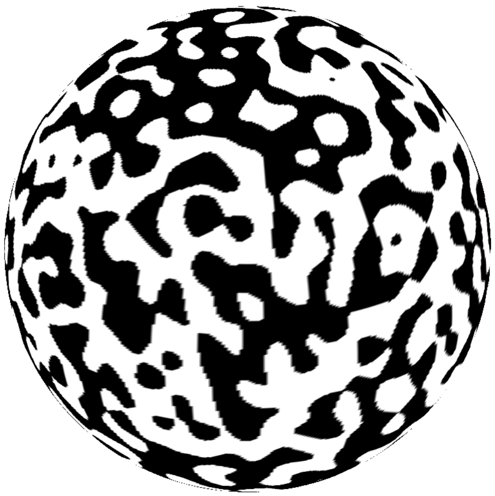
Scaling parameter $L = n$ (the degree)
Limiting spectral measure $\rho =$ Lebesgue measure on the unit ball in $\mathbb{R}^m$

Nodal portraits created by Alex Barnett ("elliptic regularity in action"):

Gaussian spherical harmonic of degree 40



Gaussian linear combination of spherical harmonic of degrees ≤ 40



4. Kostlan ensemble: Homogeneous polynomials of degree n in $m+1$ variables restricted to $X = \mathbb{S}^m$

The scalar product $\langle f, g \rangle = \sum_{|J|=n} \binom{n}{J}^{-1} f_J g_J$, where

$$f(X) = \sum_{|J|=n} f_J X^J, \quad g(X) = \sum_{|J|=n} g_J X^J, \qquad X^J = x_0^{j_0} x_1^{j_1} x_2^{j_2} \ldots x_m^{j_m},$$

$J = (j_0, j_1, j_2, \ldots j_m)$, $|J| = j_0 + j_1 + j_2 + \ldots + j_m$, $\binom{n}{J} = \frac{n!}{j_0! j_1! j_2! \ldots j_m!}$.

Complexification: after continuation of the homogeneous polynomials f and g to $\mathbb{C}^{m+1}$, the scalar product coincides with the one in the Fock-Bargmann space

$$\langle f, g \rangle = c_{n,m} \int_{\mathbb{C}^{m+1}} f(Z) \overline{g(Z)} e^{-|Z|^2} \, \mathrm{dvol}(Z)$$

It is known that the complexified Kostlan ensemble is *the only unitarily invariant* Gaussian ensemble of homogeneous polynomials.

Kostlan ensemble (continuation):

In the homogeneous coordinates, the covariance kernel equals $\left(\frac{X \cdot Y}{|X|\,|Y|}\right)^n$.
In the chart $x_0 = y_0 = 1$, we get $\left(\frac{1+(x \cdot y)}{\sqrt{1+|x|^2}\,\sqrt{1+|y|^2}}\right)^n$.

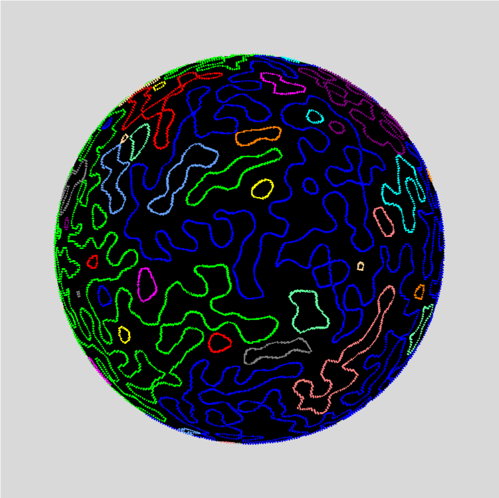
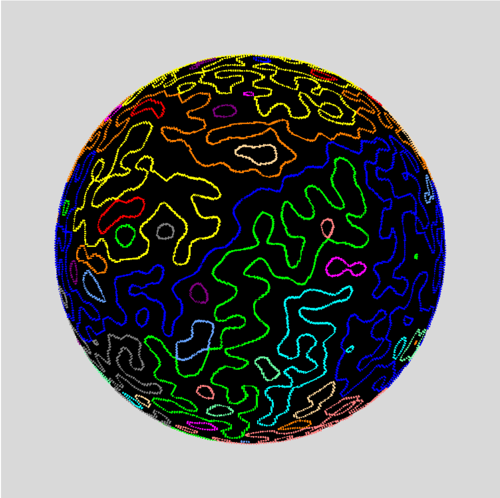
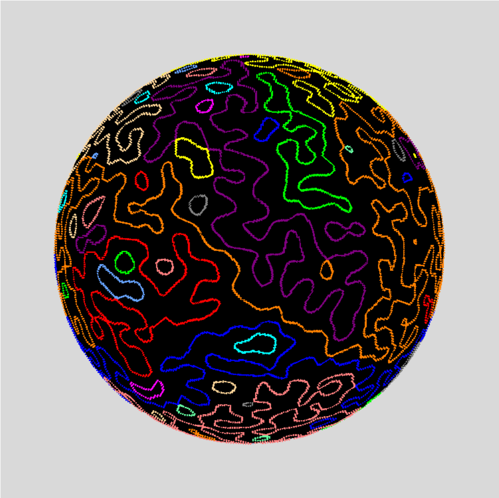
The features:

- $L = \sqrt{n}$ (square root of the degree, not the degree, as in previous examples)
- very rapid decay of the covariance away from the diagonal.

The limiting spectral measure is the Gaussian measure on $\mathbb{R}^m$ with the density $\exp\left[(x \cdot y) - \frac{1}{2}|x|^2 - \frac{1}{2}|y|^2\right]$. The limiting function F is the Fock-Bargmann wave.

Asymptotic distribution of of the number of components in Kostlan ensemble was recently studied by D.Gayet and J-Y.Welschinger, and by P.Sarnak and I.Wigman.

Nodal lines of Kostlan ensemble of degree 56 on $\mathbb{S}^2$

Nodal portraits created by Maria Nastasescu

Part VII. Another approach to the statistics of the number of components of the zero set

Ensemble of spherical harmonics of large degree:

$\mathcal{H}_n$ RKHS of 2D spherical harmonics of degree n on the sphere $\mathbb{S}^2$
$\Delta f + \lambda_n f = 0$, $\lambda_n = n(n+1)$

f_n Gaussian random spherical harmonics of degree n

Repro-kernel $P_n(\cos\Theta(x,y))$, P_n Legendre poly of deg n, $P_n(1) = 1$
$\Theta(x,y)$ angle between $x, y \in \mathbb{S}^2$

Scaling coefficient $L = n$, the limiting kernel $J_0(|u-v|)$
The limiting spectral measure: the Lebesgue measure on the circumference $\mathbb{S}^1$
The limiting translation-invariant Gaussian function: Helmholtz wave

In this case, we can prove much more:

Theorem III (F.Nazarov-M.S.): $\exists \nu > 0\ \forall \epsilon > 0\ \exists C(\epsilon), c(\epsilon) > 0$ s.t.

$$\mathcal{P}\left\{ \left| n^{-2} N(f_n) - \nu \right| > \epsilon \right\} < C(\epsilon) e^{-c(\epsilon) n}$$

Gaussian isoperimetry (Sudakov-Tsirelson, Chr. Borell)

Notation: $K_{+\rho}$ ρ-neighbourhood of the set K
γ_d standard Gaussian measure in $\mathbb{R}^d$ (normalized by $\mathcal{E}\{|x|^2\} = 1$)

THEOREM: $\Sigma \subset \mathbb{R}^d$ *Borel set,* $\Pi \subset \mathbb{R}^d$ *affine half-space with* $\gamma_d(\Sigma) = \gamma_d(\Pi)$*. Then* $\forall \rho > 0$ *we have* $\gamma_d(\Sigma_{+\rho}) \geq \gamma_d(\Pi_{+\rho})$

EXERCISE: $\gamma_d(\Pi_{+\rho}) \leq \frac{3}{4} \Longrightarrow \gamma_d(\Pi) \leq 2e^{-c\rho^2 d}$.

COROLLARY (P.Levy's concentration of Gaussian measure on $\mathcal{H}_n$): $G \subset \mathcal{H}_n$ *Borel set s.t.* $\mathcal{P}(G_{+\rho}) \leq \frac{3}{4} \Longrightarrow \mathcal{P}(G) \leq 2e^{-c\rho^2 n}$.

Uniform lower semicontinuity of $f \mapsto N(f_n)/n^2$

FUNDAMENTAL LEMMA: *$\forall \epsilon > 0$ $\exists \rho > 0$ and $\exists E_n \subset \mathcal{H}_n$ with $\mathcal{P}(E_n) \leq C(\epsilon)e^{-c(\epsilon)n}$ s.t. $\forall f \in \mathcal{H}_n \setminus E_n$ and $\forall g \in \mathcal{H}_n$ satisfying $\|g\|_{L^2(\mathbb{S}^2)} \leq \rho$, we have $N(f+g) \geq N(f) - \epsilon n^2$.*

Together with Levy's concentrarion this yields the exponential concentration of the r.v. $N(f_n)/n^2$ around its median m_n:

Let $G = \big\{ f \in \mathcal{H}_n : N(f)/n^2 < m_n - \epsilon \big\}$. Then, by Lemma, $G_{+\rho} \subset \big\{ f \in \mathcal{H}_n : N(f)/n^2 < m_n \big\} \bigcup E_n \Longrightarrow \mathcal{P}(G_{+\rho}) < \frac{1}{2} + \mathcal{P}(E_n) < \frac{3}{4}$.

Similarly, let $G = \big\{ f \in \mathcal{H}_n : N(f)/n^2 > m_n + \epsilon \big\}$. Then, by Lemma, $\big(G \setminus E_n\big)_{+\rho} \subset \big\{ f \in \mathcal{H}_n : N(f)/n^2 > m_n \big\} \Longrightarrow \mathcal{P}(\big(G \setminus E_n\big)_{+\rho}) < \frac{1}{2}$.

Now, Levy's concentration yields $\mathcal{P}\big\{ \big| n^{-2} N(f_n) - m_n \big| > \epsilon \big\} < C(\epsilon) e^{-c(\epsilon)n}$. $\square$

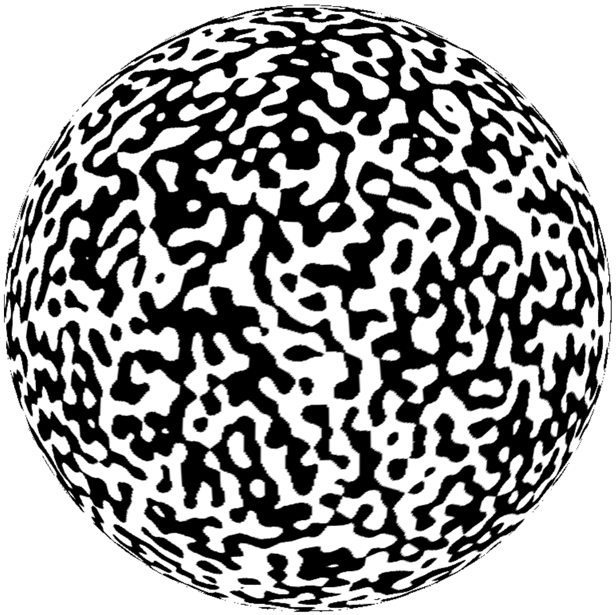
This yields Theorem III, since we already know (from Theorems II and I) that m_n tends to a positive constant ν. $\square$

More questions:

• Nothing is known about *the number of connected components of the nodal set for 'randomly chosen' high-energy Laplace eigenfunction* f_λ *on an arbitrary compact surface* M *without boundary endowed with a smooth Riemannian metric* g.

It's tempting to expect that Theorem III models what is happening when M is the two-dimensional sphere $\mathbb{S}^2$ endowed with a generic Riemannian metric g that is sufficiently close to the constant one.

Instead of perturbing the 'round metric' on $\mathbb{S}^2$, one can add a small (random) potential to the spherical Laplacian. The question remains just as hard.



The End