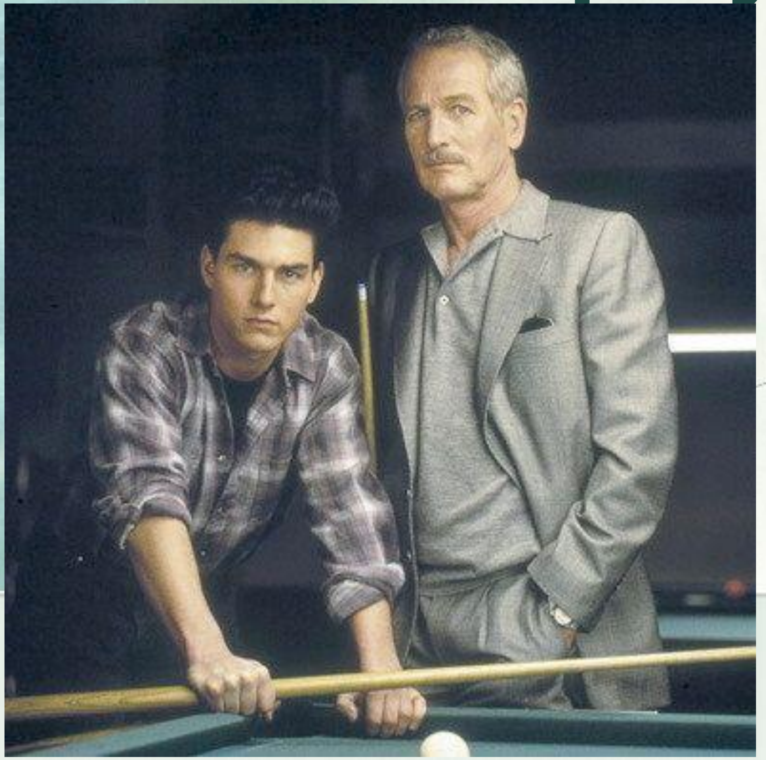
Pseudo-Integrable systems: Billiards in rational polygons



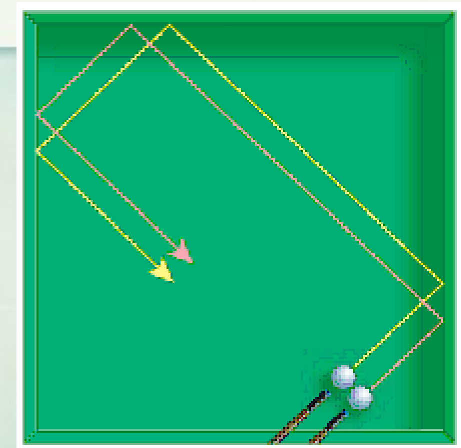
Zeev Rudnick, Tel Aviv University
(joint with Jens Marklof)

Goal: eigenfunction statistics for billiards in rational polygons

We have seen that for ergodic systems, almost all stationary states (I.e. eigenfunctions of the Laplacian) are uniformly distributed in phase space (quantum ergodicity), and for negatively curved case expect no exceptional subsequences (QUE).

For integrable systems this is not necessarily true. My goal is to expolre a “pseudo-integrable” case – billiards in rational polygons.

Billiards: classical mechanics

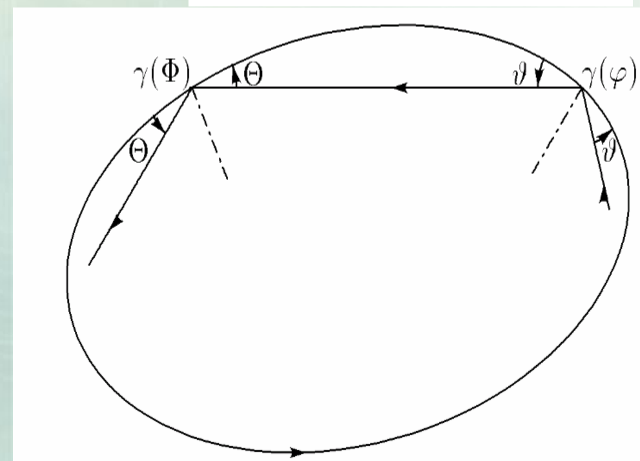


planar billiards: motion in planar domain B

angle of reflection = angle of incidence

Phase space (description of a particle):

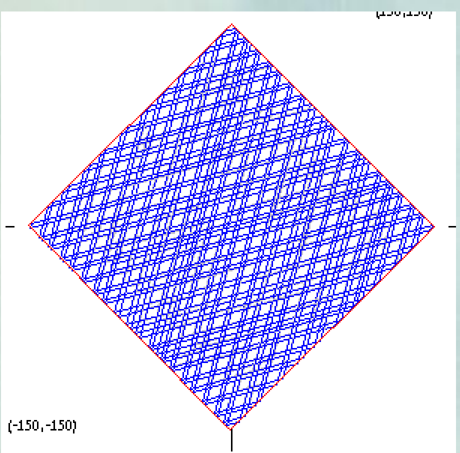
$B \times S^1$ = { **position** x of point & **direction** vector v of motion}

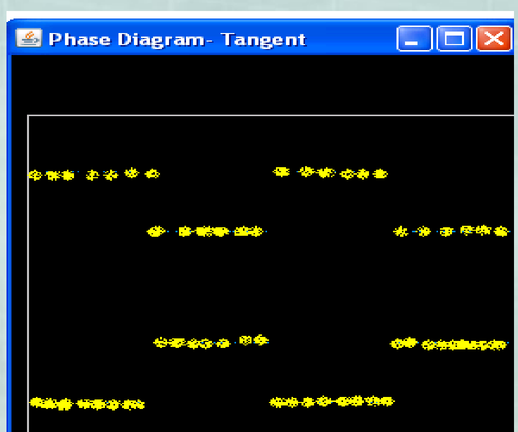


Reduced phase space (Birkhoff):

only **impacts** matter! parametrize using boundary coordinate φ and angle θ of trajectory with tangent

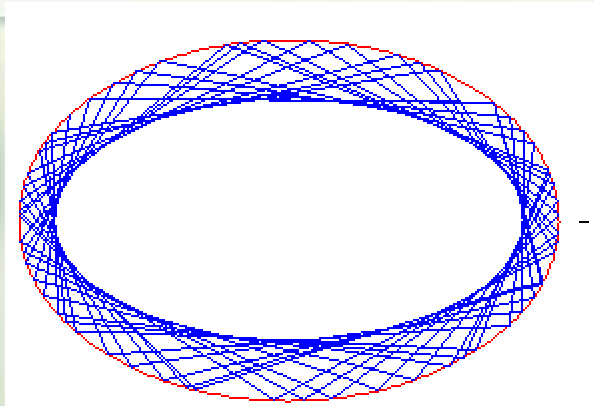
Billiard map (φ,θ)→(Φ,Θ)

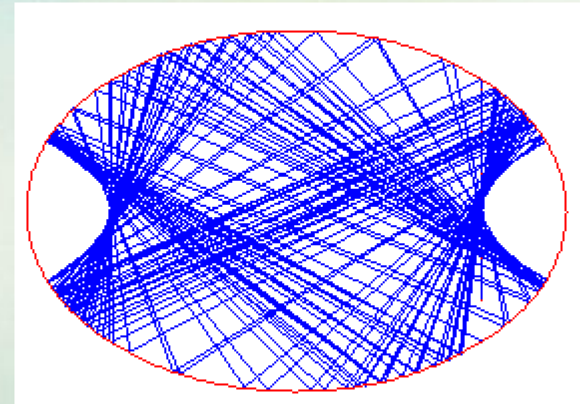


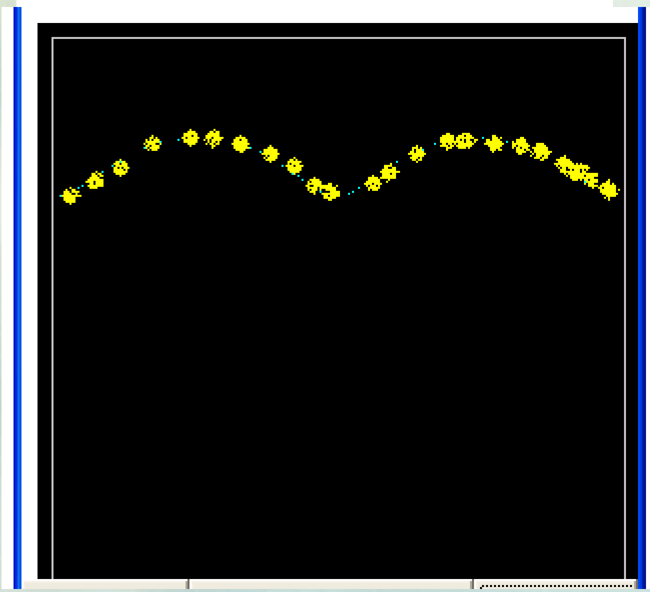


Invariant measure: $\sin\theta \, d\theta \, d\varphi$

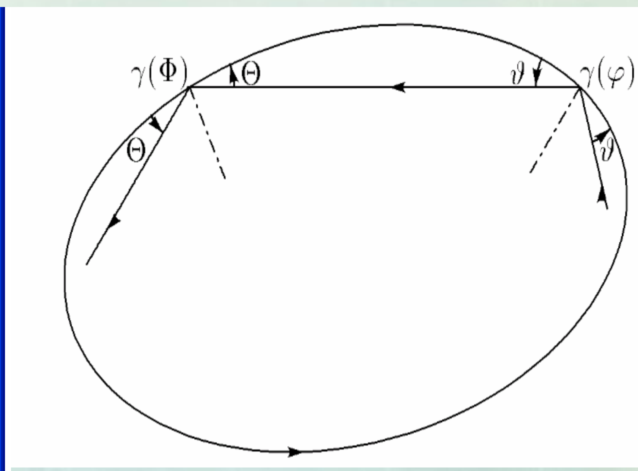
Billiards in an ellipse



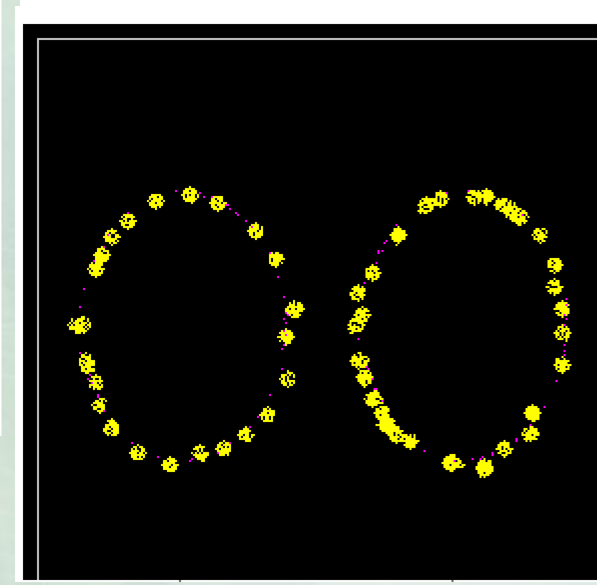




The Billiards Simulation
Bryn Mawr

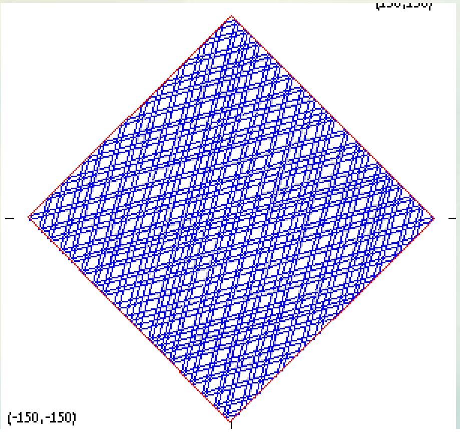


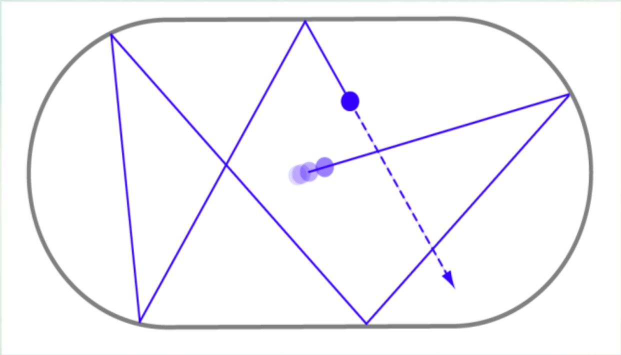
Billiard map $(\varphi,\theta)\rightarrow(\Phi,\Theta)$



Regular vs. chaotic motion

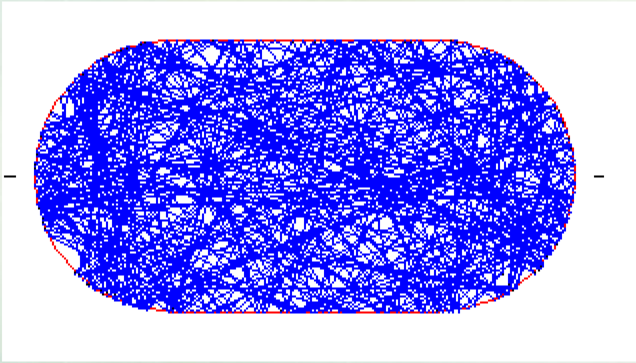
Square billiard

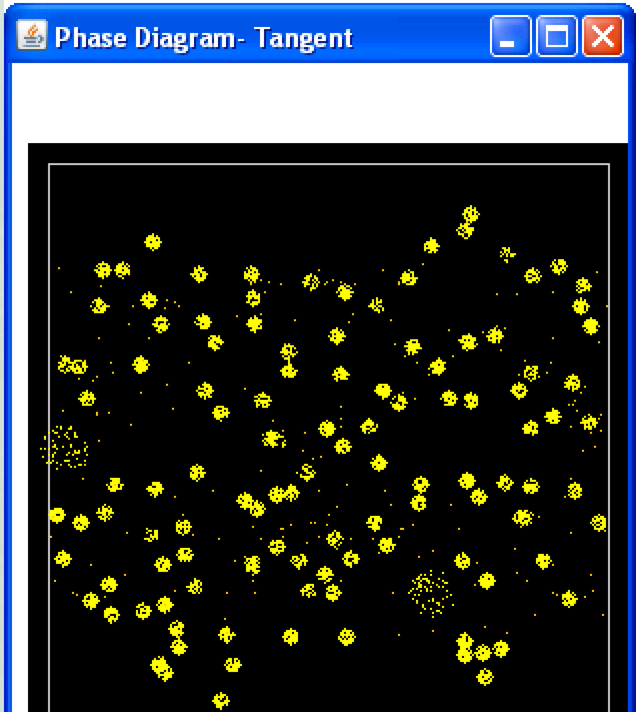




Bunimovich stadium

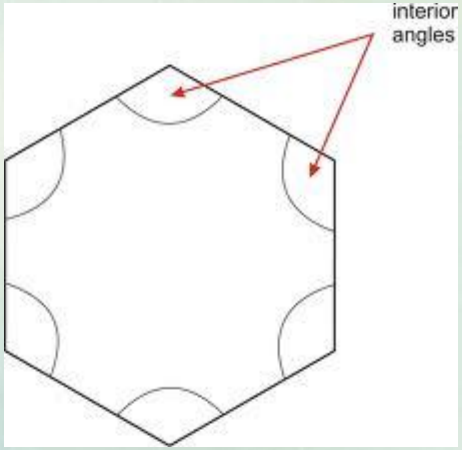
Bunimovich stadium



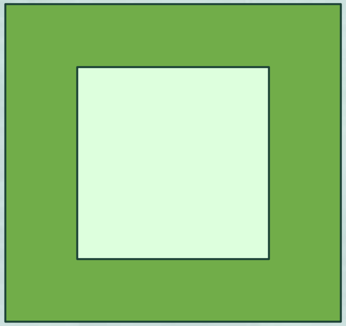


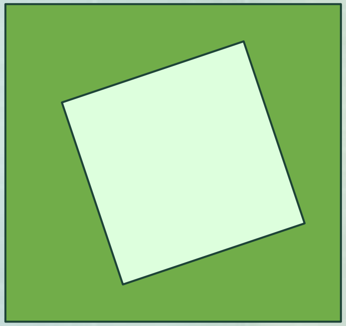
Rational polygons

A simply connected polygon is rational if all interior angles are rational multiples of π

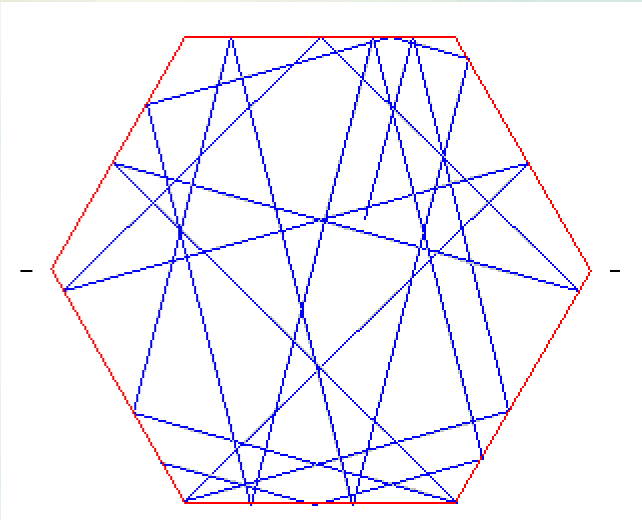


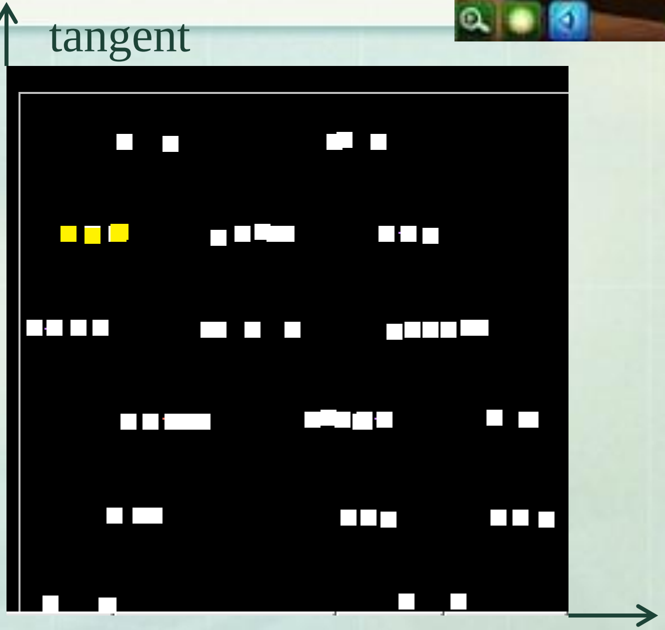
More generally: A connected polygon is **rational** if the group $\Gamma \subseteq O(2)$ generated by reflections in the sides is finite





Billiards in rational polygons





Conserved quantity: Γ-orbit of tangent angle θ

Extra constant of motion forces dynamics in phase space to be confined to invariant surfaces

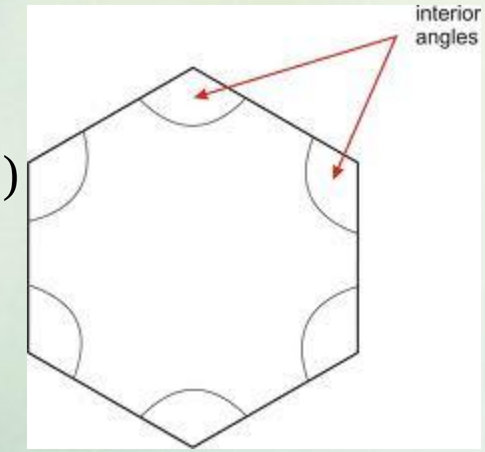
$$D_\theta = \bigcup_{\gamma \in \Gamma} D \times \gamma\theta$$

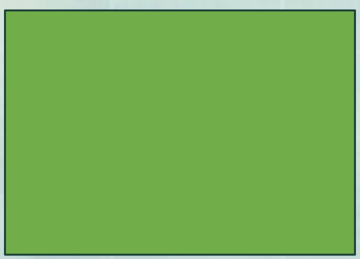
Phase space $S^*D = D x S^1$ is foliated by invariant surfaces

Genus of invariant surface

Genus of invariant surface:
interior vertex angles $=\pi\, m_i/n_i$

$$g = 1 + \frac{N}{2}\sum_i \frac{m_i - 1}{n_i}\,, \quad N = lcm(n_i)$$





g=4

g=1
integrable

g=2

Directional flows

The restriction of the flow to the invariant surface is called the “directional flow”.

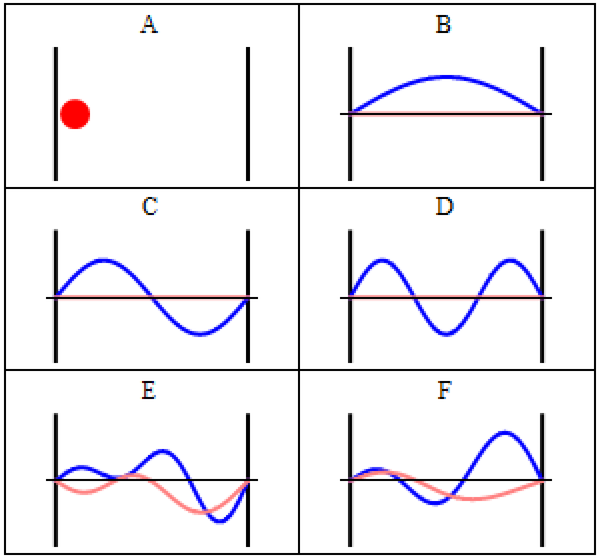
Kerckhoff, Masur & Smilie 1986: almost all directional flows are **uniquely ergodic**

- analogue of Weyl’s theorem on irrational rotations

Quantum billiards

Stationary states ↔ eigenfunctions of the Laplacian with Dirichlet boundary conditions

$$-\Delta u = Eu, \quad u|_{\partial D} = 0$$



What can we say about semiclassical measures for rational polygons?

Eigenfunctions for rational polgons

It has been observed that many eigenfunctions have clear structures related with periodic orbits – “superscars”?

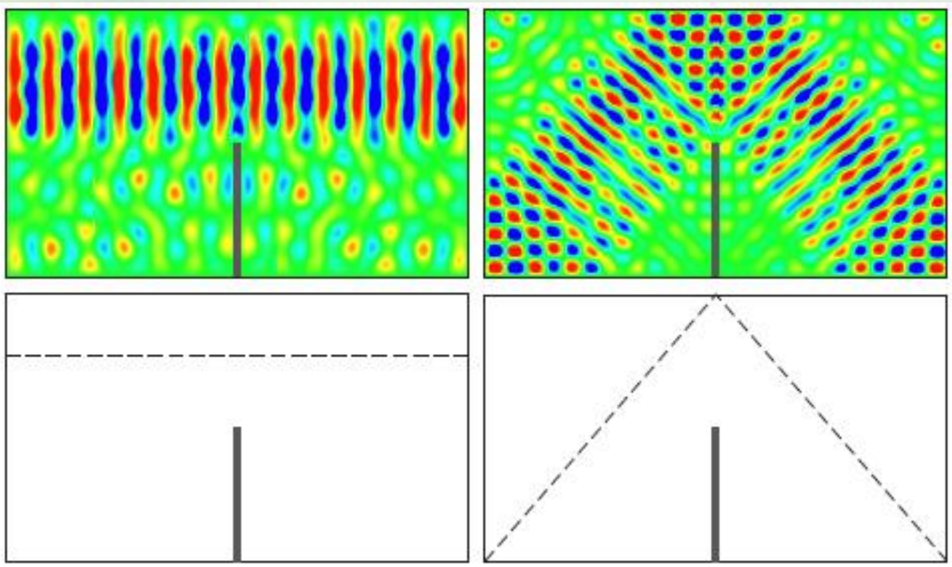


FIG. 2 (color online). The top row shows examples of experimentally obtained superscars in the barrier billiard. The color scale is the same as in Fig. 1(b). In the bottom row the

Bogomolny Deitz et al 2006

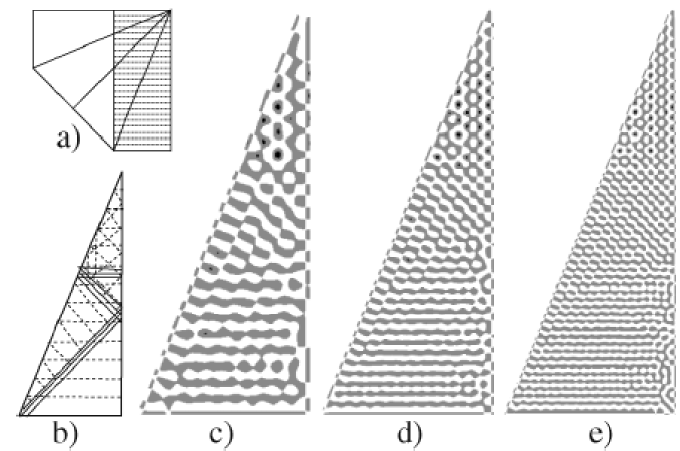


FIG. 1. (a) Unfolded scar state for the simplest POC of the right triangle with angle $\pi/8$. (b) Schematic folding of this state. Dashed lines indicate its maxima. Three solid lines show a region near SD where the unfolded scar function tends to zero. (c)–(e) Eigenfunctions with energy E close to the scar energy $E_{m,n}$. (c) $E = 407.4$; $E_{50,1} = 407.6$. (d) $E = 1015.97$; $E_{79,1} = 1016.12$. (e) $E = 1968.97$; $E_{110,1} = 1969.15$.

Bogomolny & Schmit 2004

Quantum ergodicity in configuration space

Thm (J. Marklof & ZR, 2012): For billiards in rational polygons, **almost all** eigenfunctions are uniformly distributed in **configuration space**.

i.e. given any ONB of eigenfunctions u_n, there is a density one subsequence so that for any subset A of the billiard table M,

$$\lim_{n\to\infty}\int_A |u_n(x)|^2 dx = \frac{area(A)}{area(M)}$$

Idea: Follow the (2nd) proof of Quantum Ergodicity that we saw yesterday, stopping before the last step.

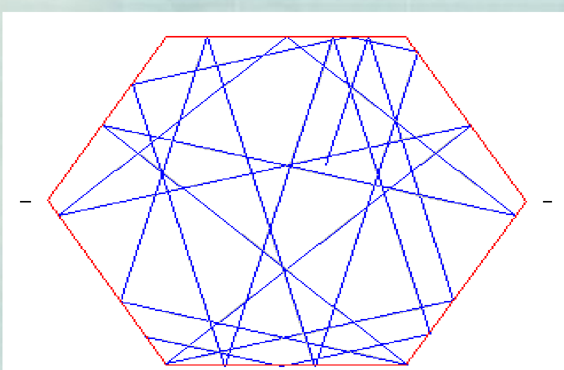
Quantum Ergodicity

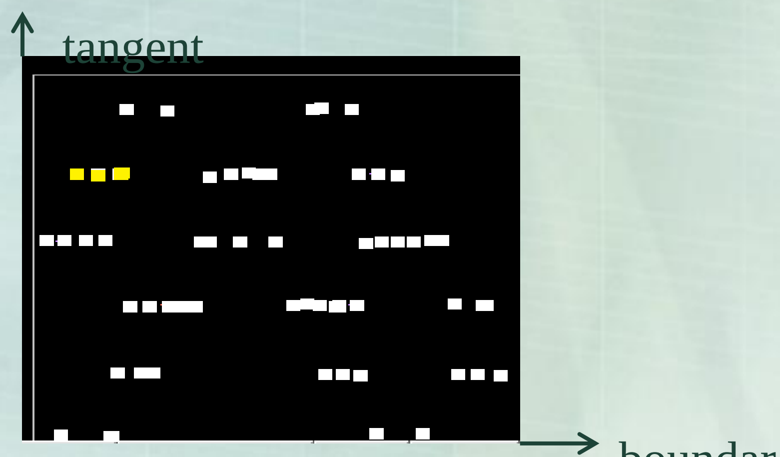
For a billiard with **ergodic** geodesic flow, "**most**" eigenfunctions cover **phase space** uniformly: If u_n is an ONB of eigenfunctions then for any observable

$$\frac{1}{N(E)} \sum_{E_n \le E} \left| \langle Op(a) u_n, u_n \rangle - \int a(p,q)\, dp\, dq \right|^2 \xrightarrow[E \to \infty]{} 0$$

Schnirelman (1974),Zelditch (1987), Colin de Verdiere (1985), Gerard & Leichtnam (1993), Zelditch-Zworski (1996).

NB: For rational billiard, dynamics is **not** ergodic!





Isotropic observables

If we are only interested in distribution of eigenfunctions in configuration space, can use observables which depend only on **position**, not on the momentum

$$a(x,\theta) = a_0(x), \quad a_0 \in C_c(D)$$

Claim: Let D be rational polygon. For isotropic observables, the average along each invariant surface equals the whole phase space average!

$$\int_{D_\theta} a \, d\mu_\theta := \frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \frac{1}{area(D)} \int_D a(x, \gamma\theta) dx = \frac{1}{area(D)} \int_D a_0(x) dx = \langle a \rangle$$

i.e. Lebesgue measure on the invariant surface in phase space projects to Lebesgue measure in configuration space

Quantum vs. classical variance

Classical variance of time-averaged observable

$$C(a,T) := \int_{S^*M} \left| \langle a \rangle_T (x,\xi) - \omega(a) \right|^2 d\mu_L(x,\xi)$$

$$\omega(a) := \int_{S^*M} a$$

space average

$$\langle a \rangle_T := \frac{1}{2T} \int_{-T}^{T} a \circ \Phi^t \, dt$$

time average

Quantum variance

$$V(a,E) := \frac{1}{N(E)} \sum_{E_n \le E} \left| \langle Op(a) u_n, u_n \rangle - \omega(a) \right|^2$$

Penultimate step in proof of Quantum Ergodicity (for **any** billiard): For all T>0,

$$\limsup_{E \to \infty} V(a,E) \le C(a,T)$$

- For **ergodic** case, the classical variance vanishes as T→∞ for **all** observables.
- We will show that for **rational polygons**, the classical variance vanishes for **isotropic** observables: $a(x,\theta)=a_0(x)$.

The classical variance for isotropic observables

Kerckhoff-Masur-Smilie: for almost all θ, and all x, the time average converges to the space average:

$$\langle a \rangle_T (x,\theta) := \frac{1}{2T} \int_{-T}^{T} a(\Phi^t(x,\theta))\, dt \xrightarrow[T\to\infty]{} \int_{D_\theta} a\, d\mu_\theta$$

For **isotropic** observables,

$$\int_{D_\theta} a\, d\mu_\theta = \frac{1}{area(D)} \int_D a_0(x) dx = \omega(a)$$

$$\lim_{T\to\infty} C(a,T) = \int_{S^*D} \left| \lim_{T\to\infty} \langle a \rangle_T (x,\theta) - \omega(a) \right|^2 d\mu_L(x,\theta) = \int_{S^*D} \left| \int_{D_\theta} a - \omega(a) \right|^2 d\mu_L =$$

$$= \int_{S^*D} \left| \omega(a) - \omega(a) \right|^2 d\mu_L = 0$$

for almost all θ

conclusion

Since for all T $$\limsup_{E\to\infty} V(a,E) \le C(a,T)$$

and $\lim_{T\to\infty} C(a,T)=0$ ⟹ $$\lim_{E\to\infty} V(a,E) = 0$$

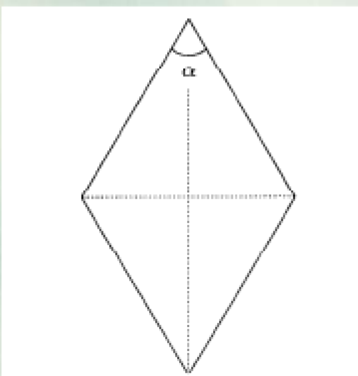
Hence there is a density one subsequence of eigenfunctions, s.t. for all $a_0 \in C(D)$

$$\lim_{n\to\infty} \int_D a_0(x)|u_n(x)|^2 dx \to \frac{1}{area(D)} \int_D a_0(x) dx$$

⟹ $$\lim_{n\to\infty} \int_A |u_n(x)|^2 dx = \frac{area(A)}{area(M)}$$

QED

Eigenvalue statistics : semi-Poisson?



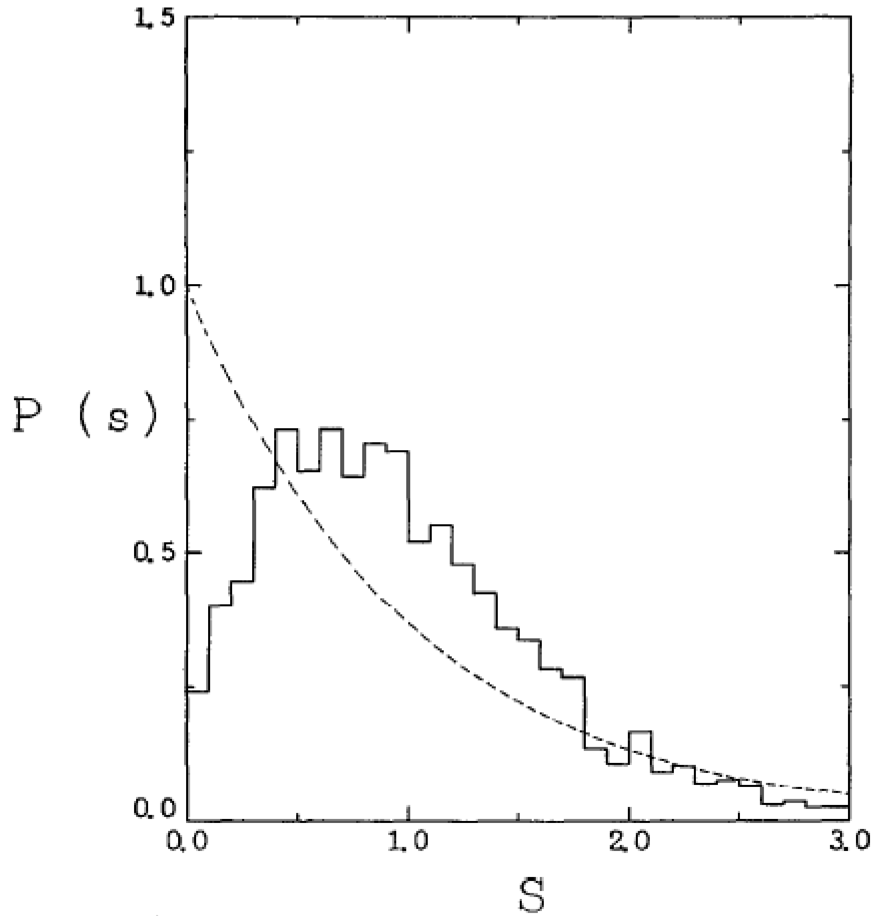
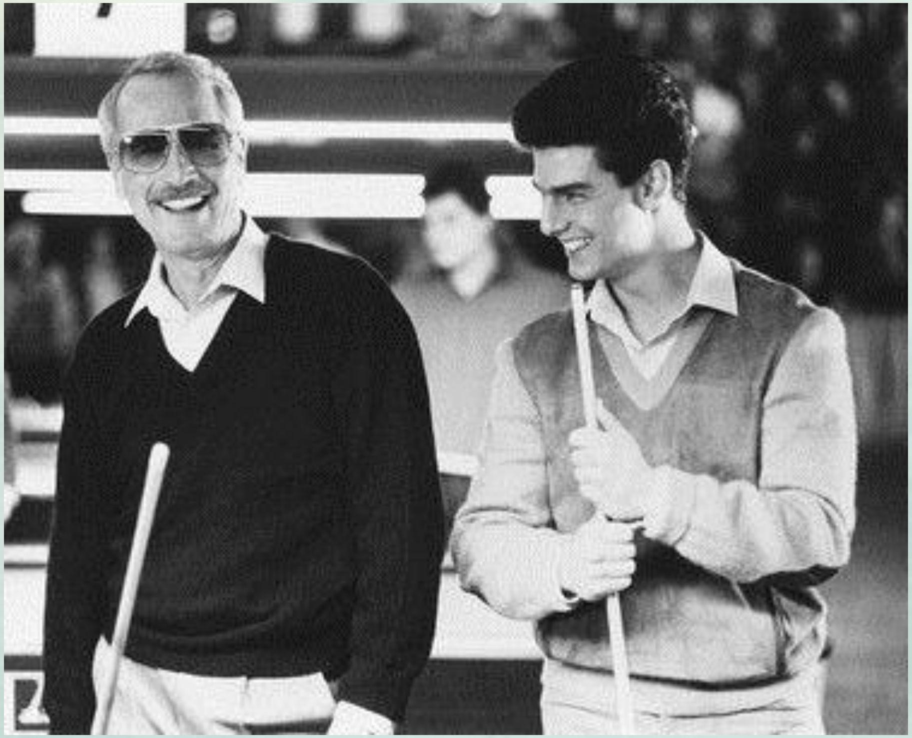


Fig. 13. The nearest-neighbour level-spacing distribution $P(s)$ for even-parity states of $\frac{1}{3}\pi$-rhombus. The broke line inserted denotes the Poisson distribution.

Summary

- Billiards in rational polygons have pseudo-integrable dynamics
- Almost all eigenfunctions are uniformly distributed in configuration space
- Other problems – spectral statistics ?

Thank you for your attention!



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