Summer school on Quantum Chaos

Sponsored by IAMP, EMS & ESI

Introduction: Zeev Rudnick

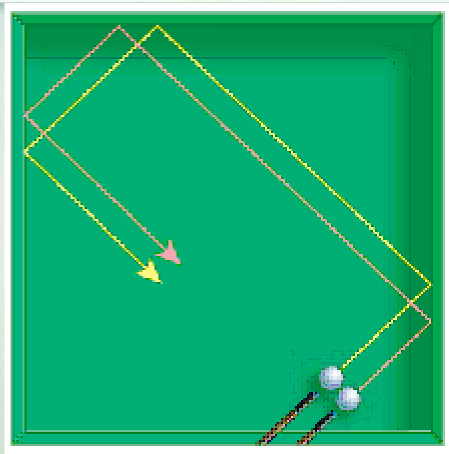
What is Quantum Chaos ?

- “Quantum chaos” = study of energy levels and stationary states of quantum systems in the semiclassical limit $\hbar \to 0$.
- This week we will explore some aspects of subject.

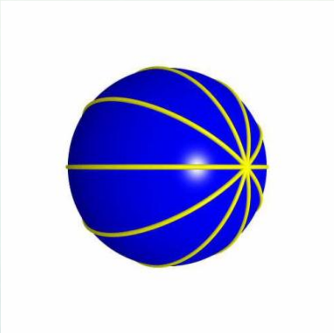
Models of classical mechanics

planar billiards:

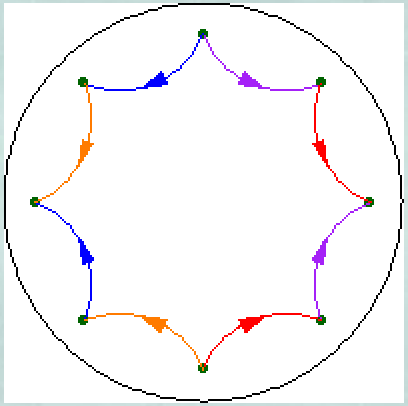
angle of reflection = angle of incidence

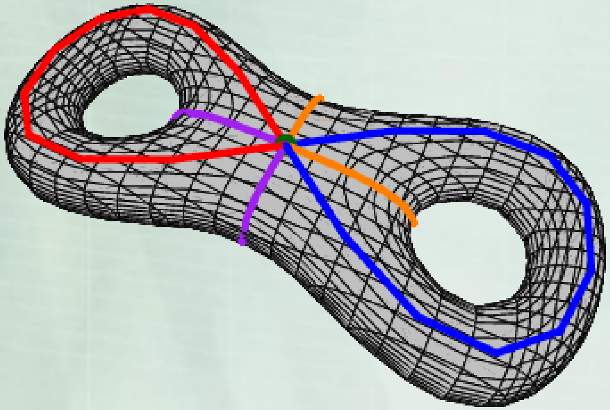


Geodesic flow on a surface



Geodesic flow on **negatively curved** surface





Rowland, Todd. "Universal Cover." From MathWorld

Regularity vs. chaos in classical dynamics

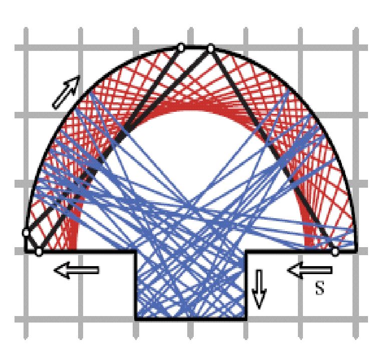
see courses by Nonnenmacher, Riviere

Classification of (conservative, Hamiltonian) dynamical systems, e;g; planar billiards, geodesic flows on surfaces

Regular (integrable):

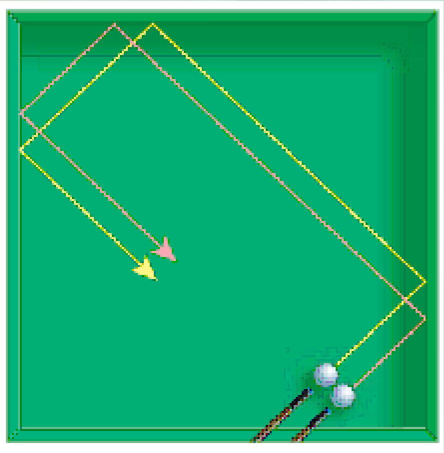
- A full set of constants of motion
- dynamics confined to **invariant tori** in phase space.
- Linear separation of trajectories

…mixed ….



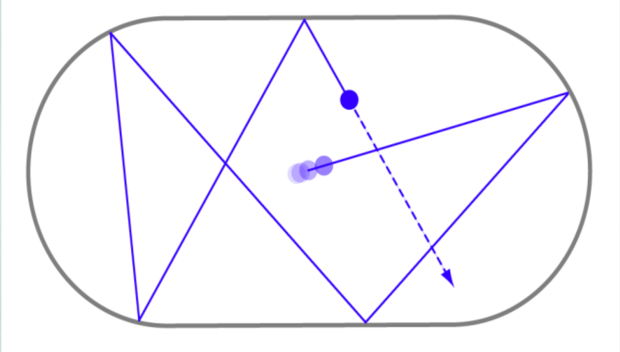
Chaotic:

- Typical orbits densely cover all of available phase space (ergodicity)
- Exponential divergence of nearby trajectories (hyperbolicity).

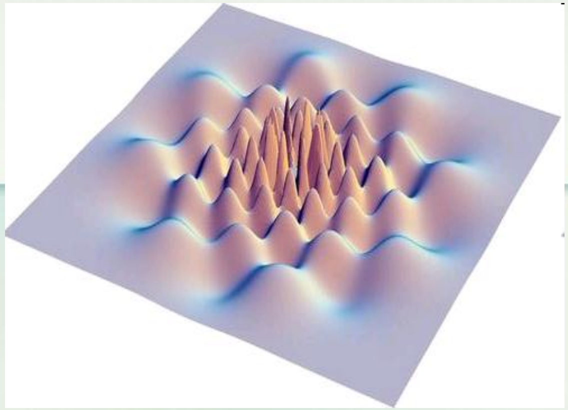


… pseudo-integrable

Rudnick



Quantum mechanics



A particle at time t is described by its **wave function** $\Psi(q,t)$

$|\Psi(q,t)|^2$ = probability density of particle in state Ψ

Time evolution is described by **Schrödinger's equation** :

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2}\Delta\Psi$$

$\hbar$ =1.054 × 10^{-34} J-s

Stationary states: $\Psi(q,t) = \psi(q)e^{-itE/h}$ with $\psi(q)$ an **eigenfunction** of Δ

$$-\frac{\hbar^2}{2}\Delta\psi = \mathbf{E}\psi$$

E=energy level

The semiclassical limit h→0 & the correspondence principle:

"*classical mechanics is a special case of quantum mechanics*".

If so, then:

How is the dichotomy "regular vs. chaotic" manifested in Quantum Mechanics ?

Statistics of eigenfunctions

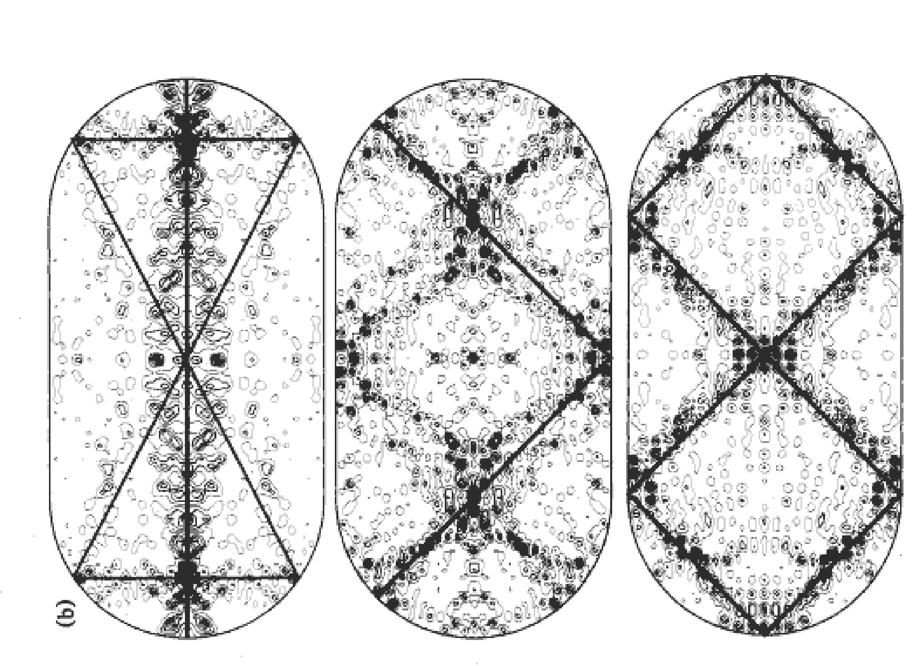
<u>The semi-classical eigenfunction hypthesis</u> of M.V. Berry and A..Voros (~ 1977):

"*Each semi-classical eigenstate has a Wigner function concentrated on the region explored by a <u>typical</u> orbit over infinite times*".

In particular, for chaotic systems, "The wave functions cover phase space uniformly"

However….."scars" were found by Heller and by McDonald & Kauffman (1984-88)

Scars: Concentration of eigenfunctions on unstable periodic orbits (controversial)



E. Heller: Scarred stadium mode

A mathematical formulation

see Hassel's course

For a particle with wave function ψ, the expectation values of its position coordinates q_1 , q_2 are given by

$$\langle q_1 \rangle_\psi := \int q_1 |\psi(q_1, q_2)|^2 \, dq_1 dq_2$$

"Likewise", for any classical observable a(q,p) of position $q=(q_1, q_2)$ and momentum $p=(p_1, p_2)$, one can define a (pseudo-differential) operator Op(a) so that the expected value of the observable a "at the state ψ" is the diagonal matrix element <Op(a)ψ,ψ>

A possible interpretation of the statement that "wave functions cover phase space uniformly" is that the matrix elements converge to the classical average of a:

$$\langle Op(a)\psi, \psi \rangle \xrightarrow[E_\psi \to \infty]{} \iint a(q, p) \, dp dq$$

Quantum Ergodicity (Schnirelman)

Schnirelman (1974): For a Riemannian manifold M with **ergodic** geodesic flow, "**most**" eigenfunctions cover phase space uniformly: if u_n is an ONB consisting of eigenfunctions of the Laplacian, then there is a subsequence of density one s.t. for all observables a(p,q)

$$\langle Op(a)u_n, u_n \rangle \xrightarrow[n\to\infty]{} \int_{S^*M} a$$

Zelditch (1987), Colin de Verdiere (1985)

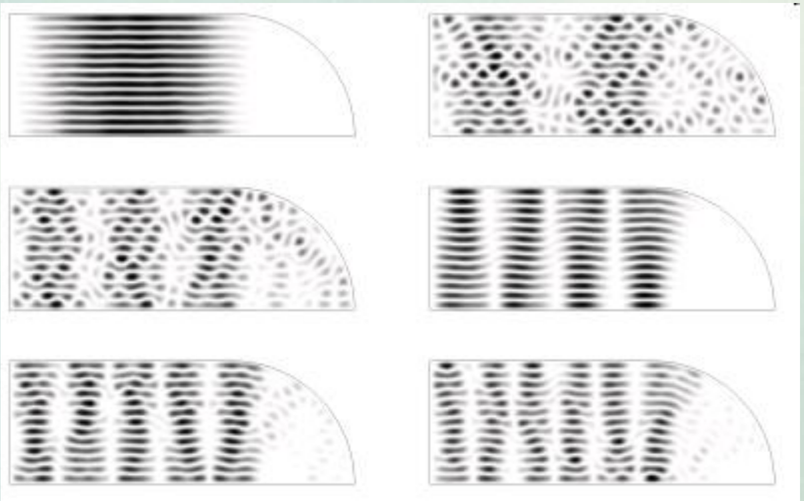
see Hassel's course

One interpretation of "Scars" is as possible **exceptional** subsequences

ZR & Sarnak (1994): Conjecture that for **negatively curved** manifolds, **no** exceptional subsequence - **Quantum Unique Ergodicity** (QUE).

see Einsiedler's course

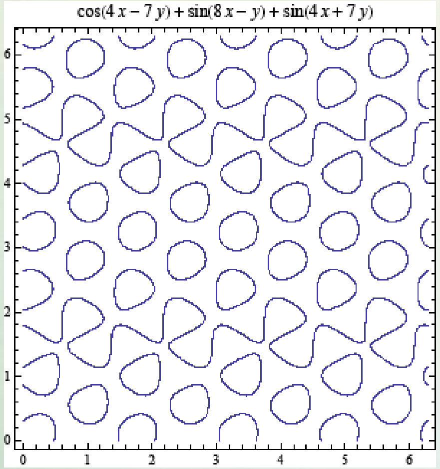
Bouncing ball modes in the stadium billiard



Hassel (2010): BBM exist for "most" stadia

Nodal lines of eigenfunctions

The **nodal line** of φ is the set $\varphi^{-1}(0)=\{x: \varphi(x)=0\}$.



nodal lines of eigenfunctions on the flat torus

Question: How do nodal lines of eigenfunctions vary as we increase the eigenvalue ?

see Zelditch's talk

Random wave models –used to predict features such as the statistics of the number of nodal domains of eigenfunctions

see Sodin's course

Statistics of the eigenvalues

A major insight of Quantum Chaos the **statistics** of the energy spectrum falls into a few **universality classes**, described by **simple statistical models**, depending on the coarse classification of the dynamics of the classical limit of the system.

One popular statistical measure is the level spacing distribution P(s) :=limiting distribution of the normalized gaps δ_j between adjacent levels

$$\delta_n := \frac{E_{n+1} - E_n}{\textbf{mean spacing}}$$

$$\frac{1}{N}\#\{n \le N : \delta_n < x\} \xrightarrow[N\to\infty]{} \int_0^x P(s)ds$$

Statistical models: Poisson vs RMT

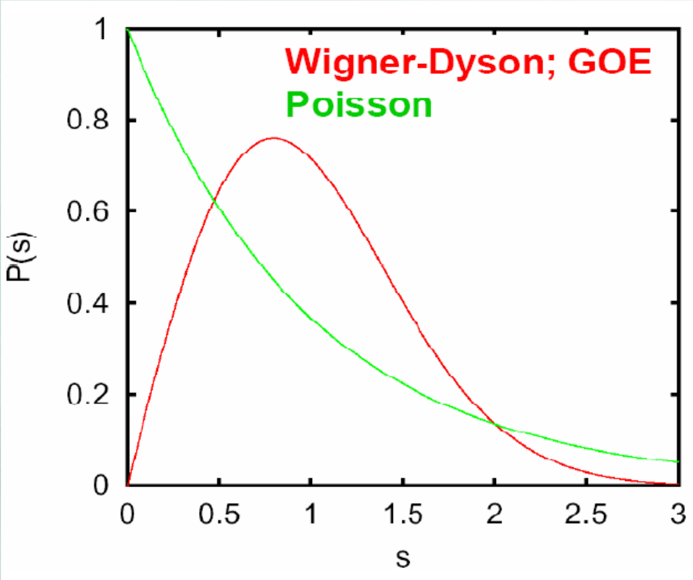
Statistics of the energy levels in the local regime can be can be compared to some simple probabilistic models:

a) Uncorrelated levels: Take the levels E_n to be independent, uniform in [0,1] (homogeneous Poisson process on the line with intensity 1).

Here the level spacing distribution P(s)=exp(-s)

b) E_n are the eigenvalues of a **random** symmetric matrix (GOE)
NxN symmetric matrices $H=H^T$,
matrix elements=independent real Gaussians
P(s) was computed by Gaudin and Mehta (1960's)

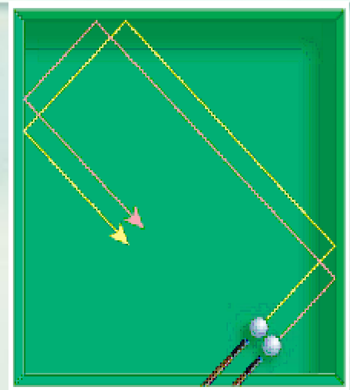
see Keating's course



Integrable vs. chaotic: universality conjectures for the level spacing distribution P(s)

Berry & Tabor 1977: For **integrable** dynamics expect* **Poisson** statistics

see Marklof's talk



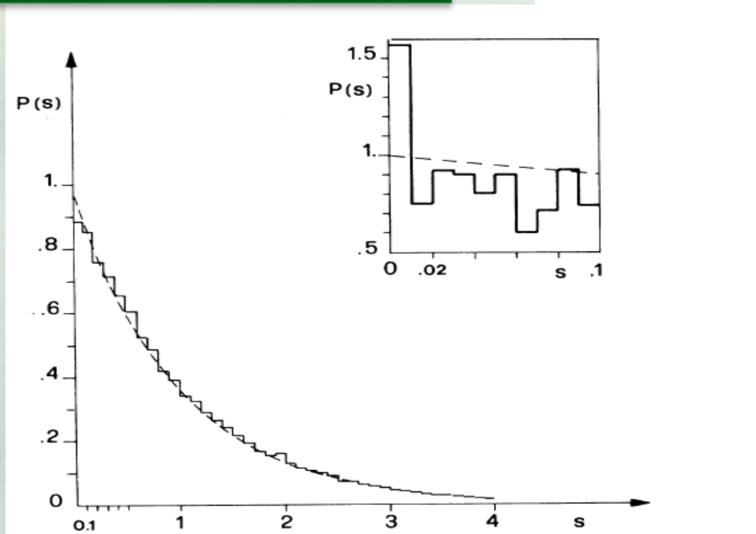
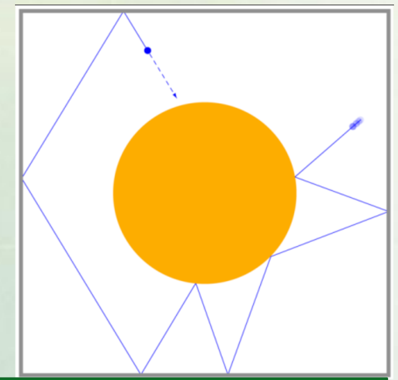


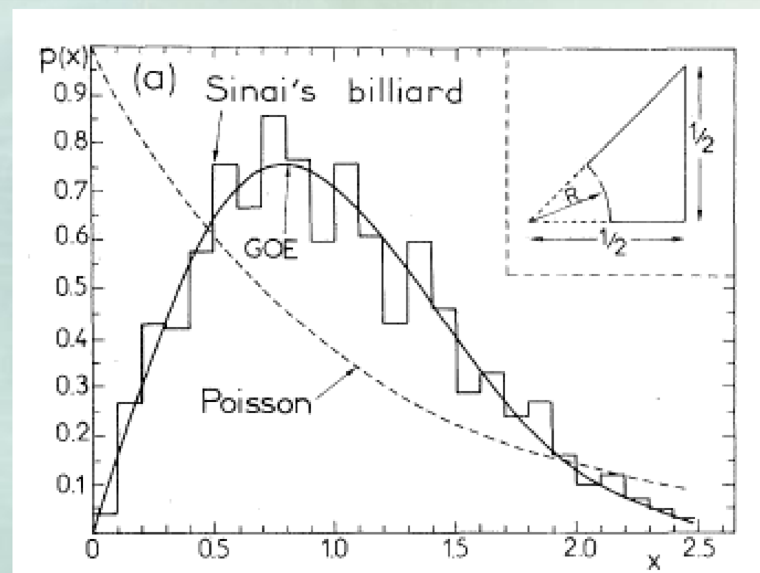
FIG. 1. Level-spacing distribution obtained from the first 100 000 levels (1) with $\alpha = \pi/3$. The dotted line is the Poisson distribution $P(s) = e^{-s}$.

rectangular billiard, aspect ratio =$\sqrt{\pi/3}$

Bohigas, Giannoni & Schmit (1984): for **chaotic** dynamics expect* **GOE** statistics



see Keating's course, Muller's talk



*="generically"

Have a good week!