Entropic bounds for eigenfunctions. 03/08/2012

(Vienna - ESI)

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M smooth, cpct, riem manifold of dim d (without boundary)

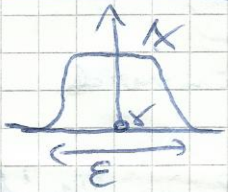
$$(\hbar \to 0) \begin{cases} -\hbar^2 \Delta \psi_\hbar = \psi_\hbar \\ \|\psi_\hbar\|_{L^2} = 1. \end{cases}$$

question: asymptotic distribution of $(\psi_\hbar)$ in T^*M as $\hbar \to 0$? especially when the involved classical dynamics is chaotic.

For this, introduce.

$\forall a \in \mathcal{C}_c^\infty(T^*M), \mu_{\psi_\hbar}(a) = \langle \psi_\hbar | Op_\hbar(a) \psi_\hbar \rangle_{L^2(M)}$

simplest chaotic object: γ is a closed hyperbolic geodesic.



χ: cut off function around γ. $\exists C > 0$ s.t.

$$\langle \psi_\hbar | Op_\hbar(\chi) \psi_\hbar \rangle \geq \frac{C}{|\log \hbar|}.$$

(Colin de Verdière-Parisse; Toth-Zelditch; Burq-Zworski; Christianson)

Concentration on these subsets cannot occur too fast.

$\to$ result sharp if no additional assumption.

Rk: can be ~~improved~~ extended to more general hyperbolic subsets (of "small dynamical" dimension)

~~Suppose now~~. Recall from S. Nonnenmacher's talk that any weak limit of $(\mu_{\psi_\hbar})_{\hbar \to 0}$ belongs to

$\mathcal{M}(S^*M, g^t) = \{$ invariant proba measures invariant under $g^t \}$

Denote $\mathcal{M}_{sc}(S^*M, g^t) = \{$weak limit of $(\mu_{\psi_\hbar})_{\hbar \to 0}\}$.

$$\subset \underbrace{\mathcal{M}(S^*M, g^t)}$$

big sets (periodic orbits) of Einsiedler / Rivière's talk

Question: What can be said on $\mathcal{M}_{sc}(S^*M, g^t)$ if the ~~geodesic~~ dynamics flow is chaotic on the whole phase space?

QUE conjecture: $\mathcal{M}_{sc}(S^*M, g^t) = \{L\}$ if M is of negative curvature

Liouville measure on S^*M

Related questions for maps of the torus.

$\rightarrow A \in S_p(2d, \mathbb{Z}) \qquad \mathcal{M}_{sc}(\mathbb{T}^{2d}, A) \subset \mathcal{M}(\mathbb{T}^{2d}, A)$

$d=1$: $\frac{1}{2}\delta_0 + \frac{1}{2}\text{Leb} \in \mathcal{M}_{sc}(\mathbb{T}^2, A)$ (F.N.dB.)

$d \geq 2$: $A = \begin{pmatrix} B & 0 \\ 0 & {}^tB^{-1} \end{pmatrix}$ "$\text{Leb}_{\mathbb{T}^d}$" $\in \mathcal{M}_{sc}(\mathbb{T}^{2d}, A)$

(Kelmer)

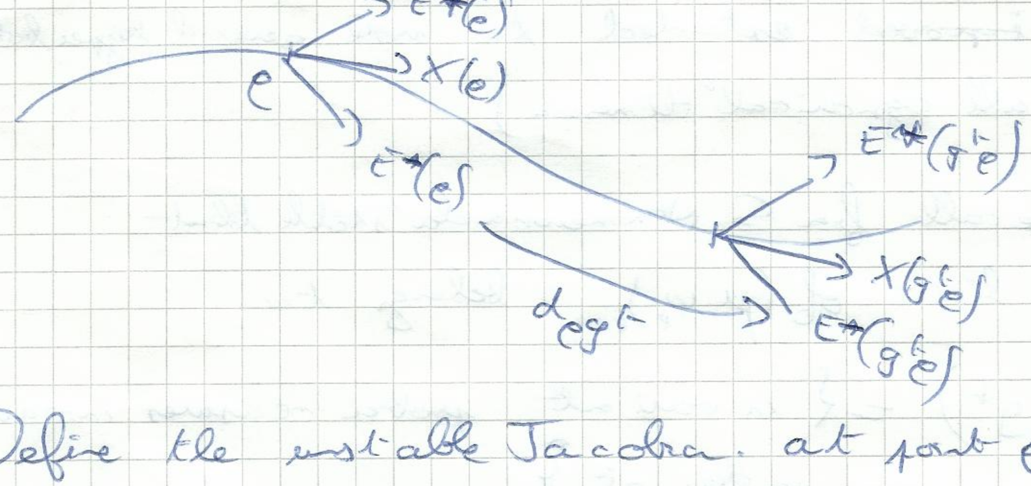
$\rightarrow$ other examples by $\begin{cases} \text{Anantharaman – Nonnenmacher} \\ \text{Gutkin} \end{cases}$

Important remark: • QUE is false for the Bunimovich stadium (ergodicity is not sufficient).

• I will not discuss the results of Einsiedler's talk

Goal: describe the ergodic properties of the elements inside $\mathcal{M}_{sc}(S^*M, g^t)$

From this point, (S^*M, g^t) has the Anosov property



Define the unstable Jacobian at point ρ.

$$\boxed{J^u(\rho) = \det\left(d_\rho g^1{}_{|E^u(\rho)}\right)}$$

Recall that for any μ in $\mathcal{M}_{sc}(S^*M, g^t)$, we defined

$h_{KS}(\mu, g) \geq 0$.

↳ evaluate the complexity of the flow from the point of view of μ.

the more μ is well distributed for the dynamics, the more $h_{KS}(\mu, g)$ will be high.

e.g.: $h_{KS}(\underbrace{\mu_x}_{\text{closed}}, g) = 0$ while $h_{KS}(L, g)$ is "high".

More precisely, Ruelle (Margulis)'s inequality

$$h_{KS}(\mu, g) \leq \int_{S^*M} \log J^u \, d\mu(\rho) \quad \forall \mu \in \mathcal{M}(S^*M, g^t)$$

with equality iff $\mu = L$. (Ledrappier and Young).

Rk: if you are able to prove that

$$h_{KS}(\mu, g) \geq \int_{S^*M} \log J^u(\rho) d\mu(\rho) \Rightarrow QUE.$$

(beware → false for quantum maps)

~~Statement of the different results~~.

Theo (Anantharaman - 2006) Write $\mu = \int_{S^*M} \mu_\rho \, d\mu(\rho)$

(ergodic decomposition of μ).

For any $\delta > 0$, $\exists \, c(\delta) > 0$ s.t.

$\forall \mu \in \mathcal{M}_{sc}(S^*M, g^t)$,

$$\mu\left(\left\{\rho : h_{KS}(\mu_\rho, g) \geq \frac{\Lambda}{2}(1-\delta)\right\}\right) \geq c(\delta) > 0.$$

where $\Lambda = \inf\left\{\int_{S^*M} \log J^u(\rho) d\mu(\rho) : \mu \in \mathcal{M}(S^*M, g^t)\right\} > 0$.

In particular,

$\forall \mu \in \mathcal{M}_{sc}(S^*M, g^t)$; $h_{KS}(\mu, g) > 0$.

Consequences:

$\rightarrow$ C invariant closed set s.t. $h_{top}(C) < \frac{\Lambda}{2}$

("small dynamical dimension")

$$\Rightarrow \forall \mu \in \mathcal{M}_{sc}(S^*M, g^t), \ \mu(C) < 1.$$

$\rightarrow$ $K \equiv -1$: $\rightsquigarrow \Lambda = d-1$

$\rightsquigarrow \forall \mu \in \mathcal{M}_{sc}(S^*M, g^t), \ \dim_H \mu \geq d.$

~~$\rightarrow \forall \mu \in$~~

Improvements of the results:

a) (Anantharaman - Nonnenmacher - Koch - 2007)

$\forall \mu \in \mathcal{M}_{sc}(S^*M, g^t)$,

$$h_{KS}(\mu, g) \geq \int_{S^*M} \log J^u(\rho)\, d\mu(\rho) - \frac{d-1}{2} \Lambda_{max}.$$

where $\Lambda_{max} = \lim_{t\to\infty} \frac{1}{t} \log \| d_\rho g^t \|$.

Rk: . lower bound can be empty if Λ_{max} too large.

. $K = -1$ $\quad h_{KS}(\mu, g) \geq \frac{d-1}{2}$.

b) (Rivière 2009/2011).

$\dim M = 2$: $\quad h_{KS}(\mu, g) \geq \frac{1}{2} \int_{S^*M} \log J^u(\rho)\, d\mu(\rho). \quad (*)$

$\dim M \geq 2$: $\quad \forall \delta > 0, \exists c(\delta) > 0$ s.t.

$$\mu\left(\left\{\rho : h_{KS}(\mu_\rho) \geq \frac{1}{2} \int_{S^*M} \log J^u \, d\mu_\rho - \delta \right\}\right) \geq c_\delta > 0.$$

Rks: 1) $(*)$ is sharp for the counterexamples and is maybe ~~optimal~~ true in any dimension. (Anantharaman - Nonnenmacher)

2) These $\neq$t results can be shown and ~~improved~~ made better in the case of quantum maps. (Brooks - Gutkin - Rivière - Anantharaman - No)

and in the higher rank case. (Anantharaman - Silberman

3) $(*)$ can be established for manifolds of $K \leq 0$ for which

L is not known to be ergodic.

4) QUE would mean: get rid of $\frac{1}{2}$ in (*).

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Idea of the proof of Anantharaman.

Fix K a compact invariant subset s.t. $h_{top}(C) \leq \frac{d\lambda_{m}}{2}$.

Fix $\sum_{i=1}^{K} P_i(x) = 1$ a smooth partition of small diameter of M. $\hat{P}_i : L^2 \to L^2$; $u \mapsto P_i \times u$.

$$U^{+\hbar n} = e^{\frac{i n \hbar \Delta}{2}}$$

n fixed:

$$\langle \underbrace{U^{-n+1}\hat{P}_{\alpha_{n-1}}U^{n-1}}_{\text{quantum evolution } \hat{P}_{n-1}(n-1)} U^{-n+2}P_{\alpha_{n-2}}U^{n-2} \dots \hat{P}_{\alpha_0}\Psi_\hbar, \Psi_\hbar \rangle$$

$$\underset{\substack{\uparrow \\ \text{Egorov} \\ + \text{composition}}}{\simeq} \langle Op_\hbar(P_{\alpha_{n-1}} \circ g^{n-1} \dots \times P_{\alpha_0}) \Psi_\hbar, \Psi_\hbar \rangle + O_n(\hbar)$$

$$\xrightarrow[\hbar \to 0]{} \mu(\underbrace{P_{\alpha_{n-1}} \circ g^{n-1} \dots \times P_{\alpha_0}}_{\text{discrete analogue of Bowen balls.}})$$

Write:

$$1 = \sum_{|\alpha|=N} \langle \hat{P}_{\alpha_{N-1}}(N-1) \dots \hat{P}_{\alpha_1}(1) \hat{P}_{\alpha_0} \Psi_\hbar, \Psi_\hbar \rangle$$

$$= \underbrace{\sum_{\substack{\text{"Bowen-balls covering} \\ C\text{"}}}}_{\swarrow} \underbrace{\langle \dots \Psi_\hbar, \Psi_\hbar \rangle}_{\downarrow} + \sum_{\substack{\text{remaining balls} \\ \text{"far from } C\text{"}}} \langle \dots \Psi_\hbar, \Psi_\hbar \rangle$$

$$| \quad | \leq C e^{-\frac{N\lambda}{2}} \hbar^{-\frac{d}{2}}.$$

number of terms in the sum. $e^{N.h_{top}(K)}$.

$\uparrow$ hyperbolic dispersion estimate

$= o(1)$ as $\hbar \to 0$ ⇐ all the mass of $\Psi_\hbar$ is far from K

but.

Problem need to take $N \simeq K \log \hbar^{-1}$ with K large.

subadditive structure. n) goes to $n := [\kappa \{\log t\}]$

with $\kappa > 0$ small.