Quantum chaos with open systems

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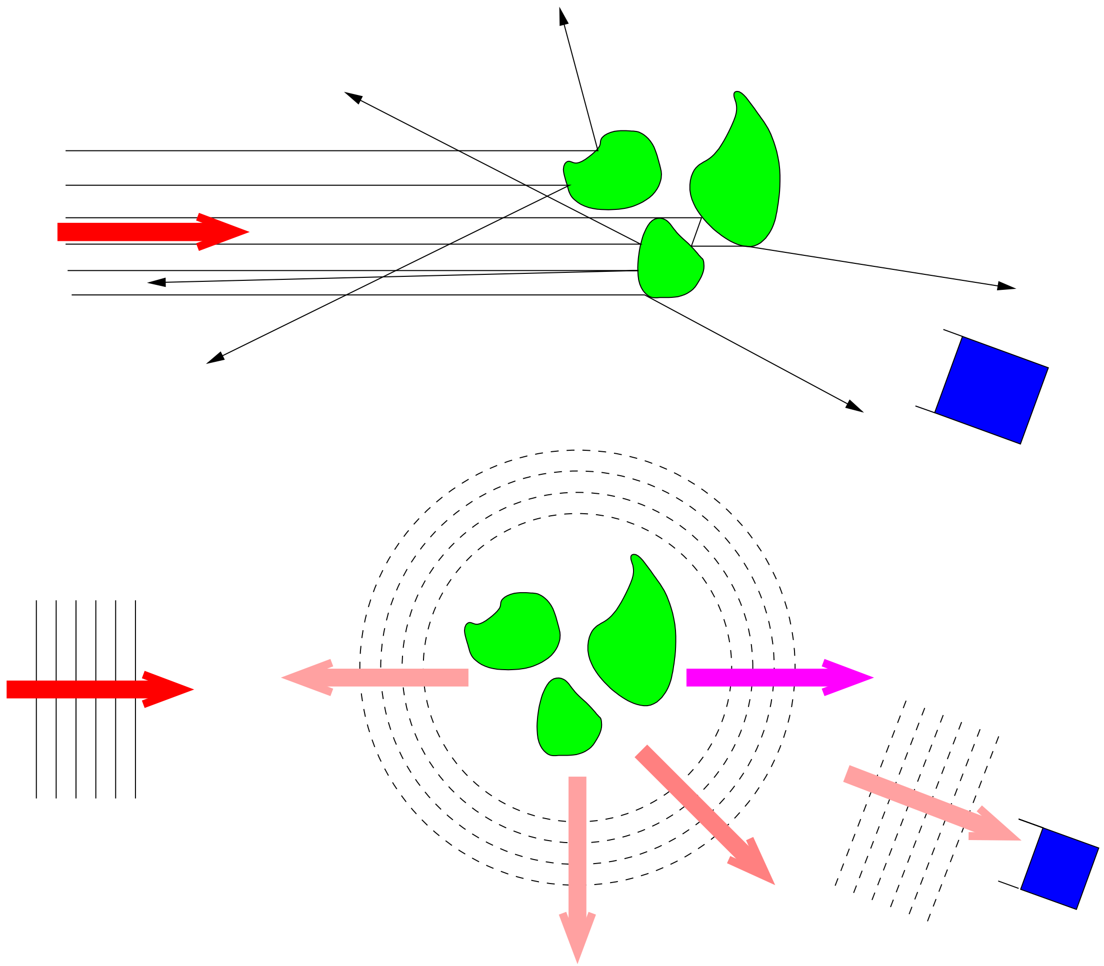
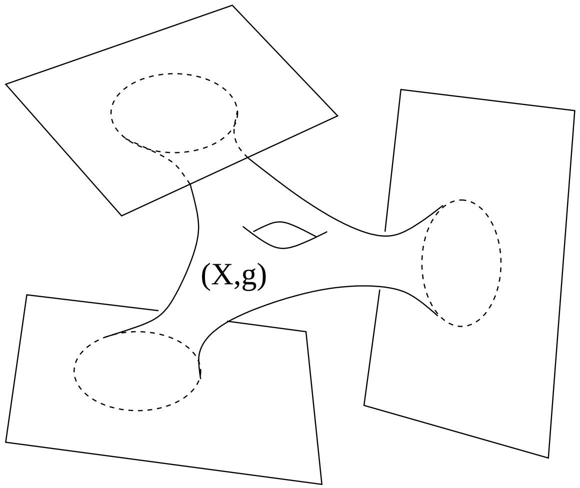
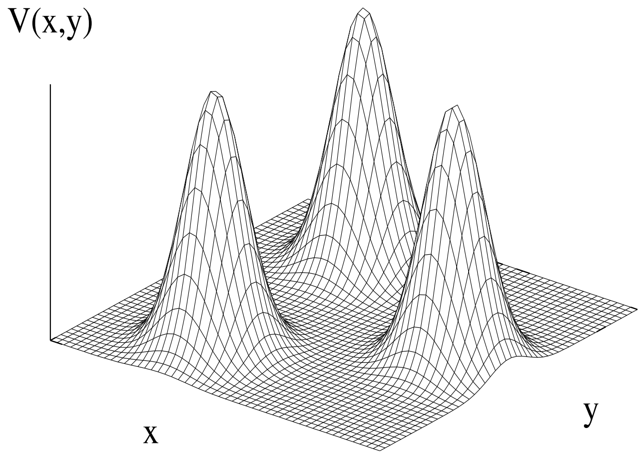
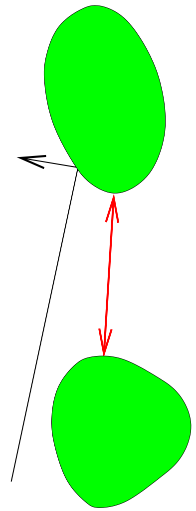


Classical vs. quantum scattering







(X, g) Riemannian mfold of infinite volume, "nice geometry" near infinity. Possible "internal boundaries" (obstacles).

Classical scattering:

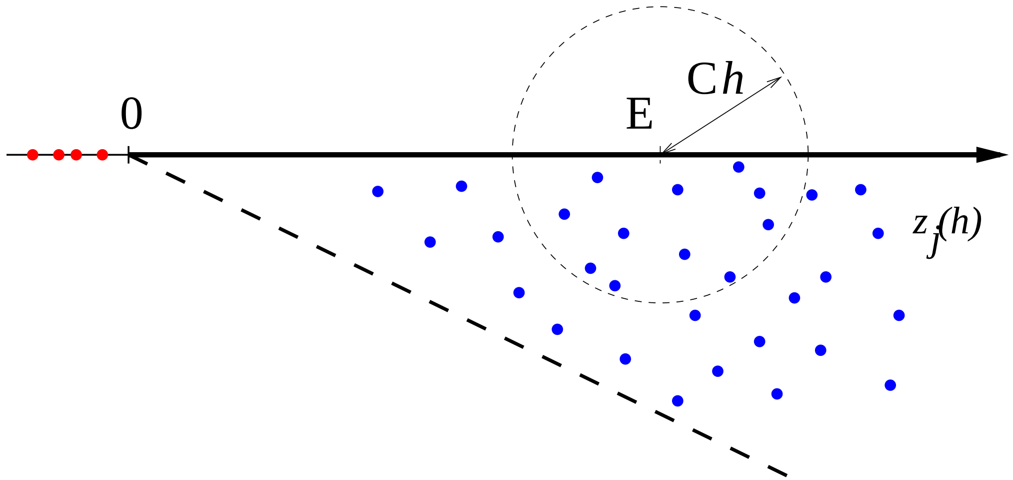
- geodesic flow on $S^*X \equiv$ Hamiltonian flow generated by $p(x,\xi) = \frac{|\xi^2|_g}{2}$ on T^*X
- Hamiltonian flow, $p(x,\xi) = \frac{|\xi^2|_g}{2} + V(x)$, with $V \in C_c^\infty(X)$.

Quantum scattering: Schrödinger eq. $ih\partial_t\psi = P(h)\psi$

- semiclassical Laplace-Beltrami operator $P(h) = -\frac{h^2}{2}\Delta_X$
- semiclassical Schrödinger operator $P(h) = -\frac{h^2}{2}\Delta_X + V(x)$

High frequencies for $\Delta_X \iff P(h) \approx E$ fixed, semiclassical régime $h \to 0$.

Quantum resonances



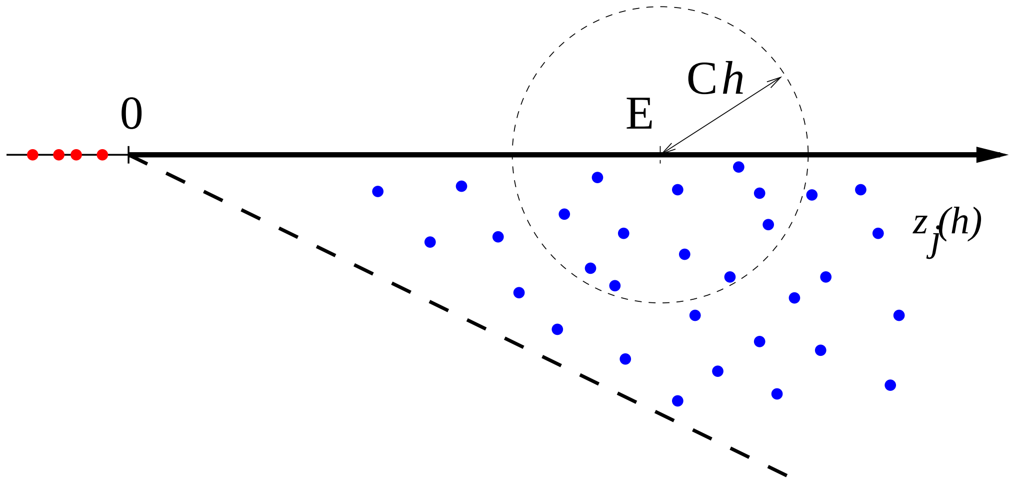
For an energy $E > 0$, the energy shell $p^{-1}(E)$ is unbounded $\Longrightarrow \operatorname{Spec} P(h)$ **absol. continuous** on $\mathbb{R}^+$.

Still, the (cutoff) resolvent $\chi(P - z)^{-1}\chi$ can be meromorphically continued from $\{\operatorname{Im} z > 0\}$ to $\{\operatorname{Im} z < 0\}$. In general it admits a discrete set of poles $\{z_j(h)\}$: **quantum resonances**.

$z_j(h) \leftrightarrow$ **metastable state** $u_j(h) \notin L^2$, with **lifetime** $\tau_j(h) = h(2|\operatorname{Im} z_j|)^{-1}$

$\Longrightarrow$ (semiclassically) long living if $\operatorname{Im} z_j(h) \geq -C\,h$.

Quantum resonances: a nonselfadjoint spectral problem



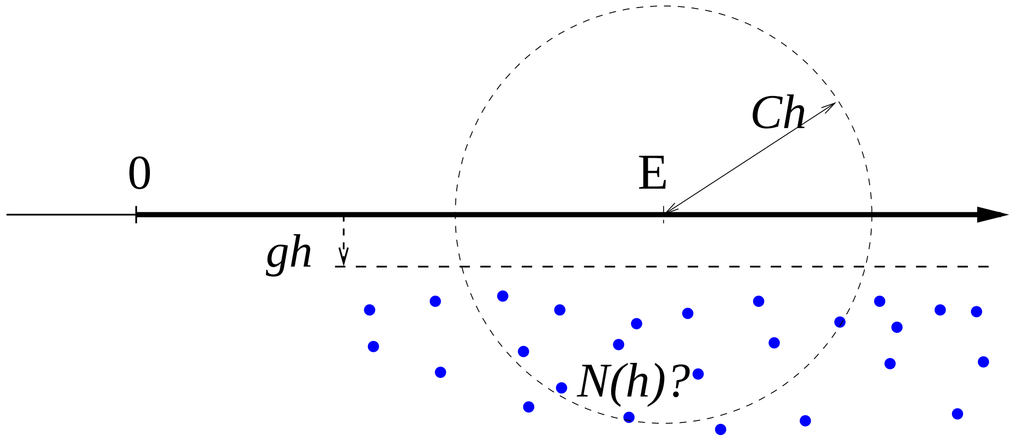
To uncover the resonances, one may apply a complex deformation to $P(h)$ near infinity (where (X, g) is analytic) [Aguilar-Balslev-Combes,Simon,Helffer-Sjöstrand...]

$P(h) \rightsquigarrow P_{\theta}(h)$, $P_{\theta}(h) = -e^{-2i\theta}\frac{h^2\Delta}{2}$ near infinity

$\Rightarrow$ discrete L^2 spectrum in $\{0 \geq \arg z > -2\theta\}$, equivalent with the resonances $\{z_j(h)\}$.
The metastable states $u_j \rightsquigarrow u_{j,\theta} \in L^2$.

We are now facing a nonselfadjoint semiclassical spectral problem for $P_{\theta}(h)$.

Relevant questions in the semiclassical limit



- fixing $E > 0$, what is the distribution of **long-living** resonances $z_j(h) \in D(E, Ch)$ when $h \to 0$?
  How dense are they? Is there a resonance free strip?

- uniform estimates for the cutoff resolvent for $z \approx E$?

- spatial structure of the metastable states? (semiclassical measures)

$\rightsquigarrow$ PDE applications: resonance expansion for $e^{-itP(h)/h}u$, local energy decay for $e^{it\sqrt{\Delta_X}}u$

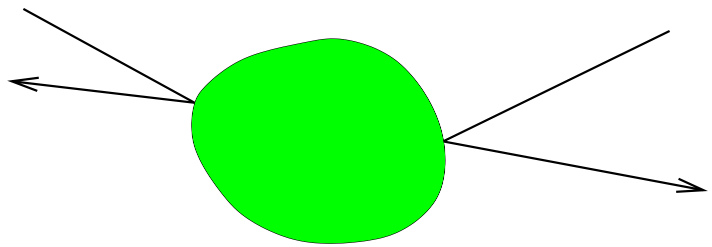
Semiclassical distribution of resonances - Trapped set

Main idea: the distribution of resonances in $D(E, Ch)$ and of the corresp. metastable states is guided by the structure of the **classical trapped set**

$$K_E = K_{E,+} \cap K_{E,-}, \quad K_{E,\pm} = \{\rho \in p^{-1}(E),\, \Phi^t(\rho) \not\to \infty,\ t \to \mp\infty\}$$

K_E compact subset of $p^{-1}(E)$, invariant through the Hamiltonian flow Φ^t.

- $K_E = \emptyset$: all $\operatorname{Im} z_j \leq -Ch \log h^{-1} \Longrightarrow$ no long-living state [Martinez'02].



- K_E contains an elliptic periodic orbit. $\Rightarrow$ resonances with $\operatorname{Im} z = \mathcal{O}(h^\infty)$ (quasi-modes).

  $\#\{\operatorname{Res}(P(h)) \cap D(E, \gamma h)\} \sim C\, h^{-n+1}$, like for a closed system.

  [Popov,Vodev,Stefanov]

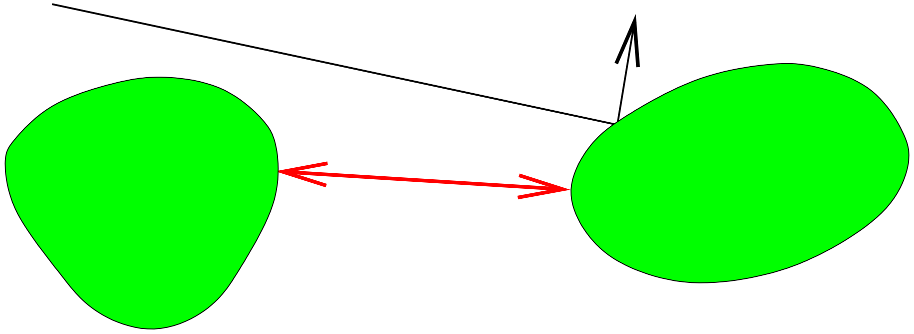
Semiclassical distribution of resonances - 1 hyperb. orbit

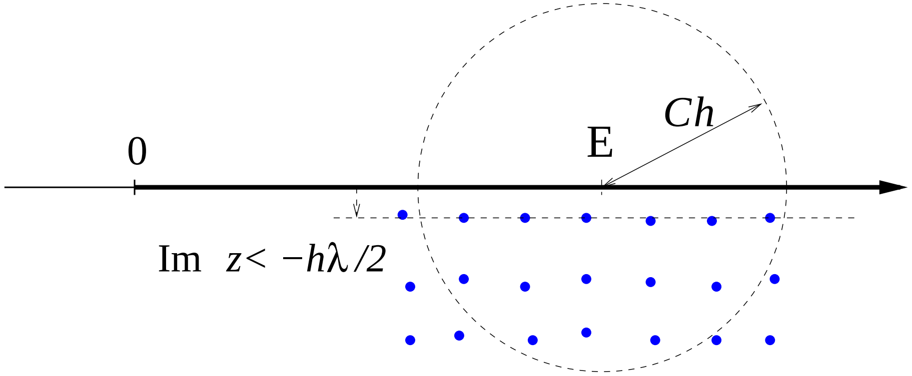
- $d = 2$, $K_E =$ single hyperbolic periodic orbit.

  Resonances form a deformed half-lattice, with $\operatorname{Im} z_j = -h\lambda(1/2 + n) + \mathcal{O}(h^2)$.

  $\#\{\operatorname{Res}(P(h)) \cap D(E, \gamma\, h)\} = \mathcal{O}(1)$.

  [Ikawa'85, Gérard-Sjöstrand'87, Gérard'88,...]



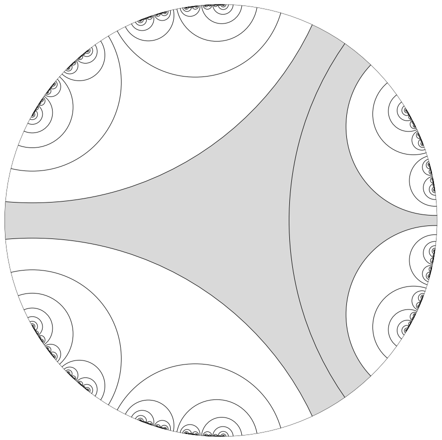


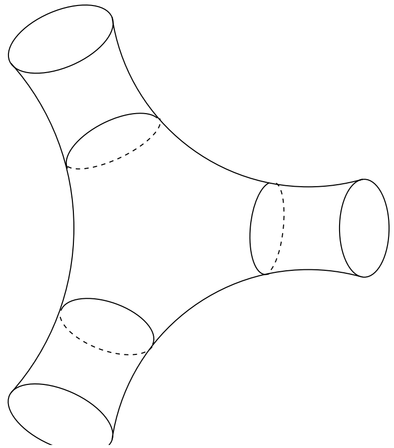
Chaotic scattering

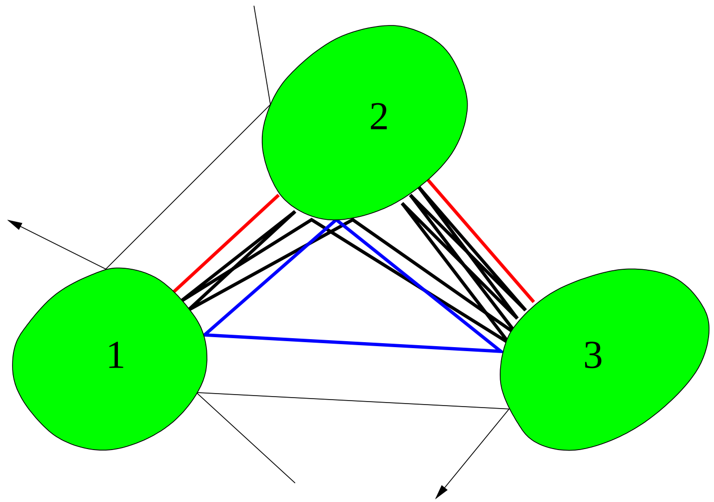
Chaotic situation: K_E a **fractal hyperbolic set**.

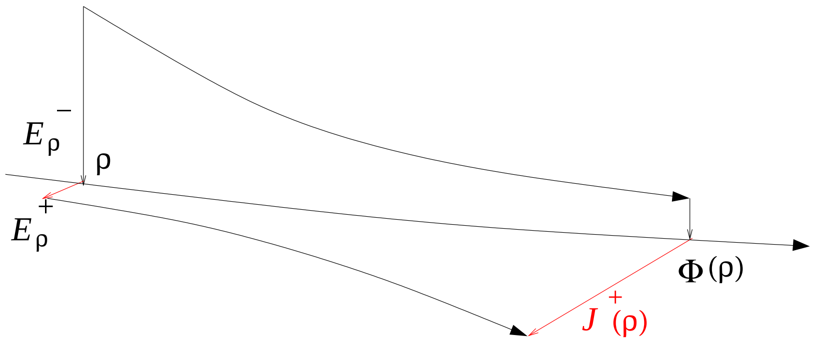
Examples: $X_0 = \Gamma\backslash\mathbb{H}^{\not\models}$ hyperbolic surface of infinite volume. 3 convex obstacles in $\mathbb{R}^d$

[Ikawa'88,Gaspard-Rice'89,...]





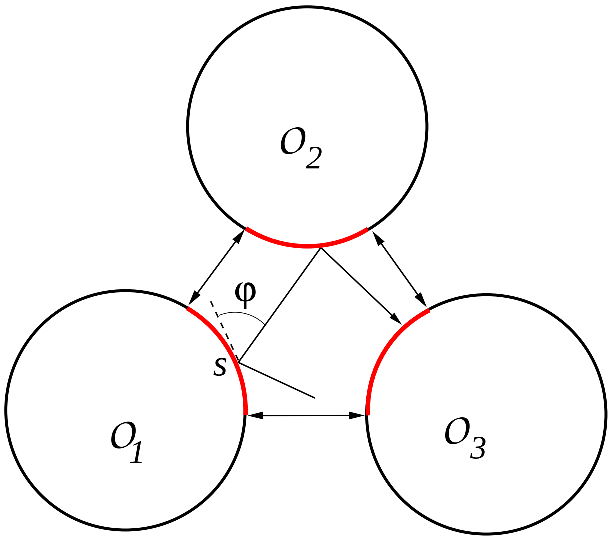


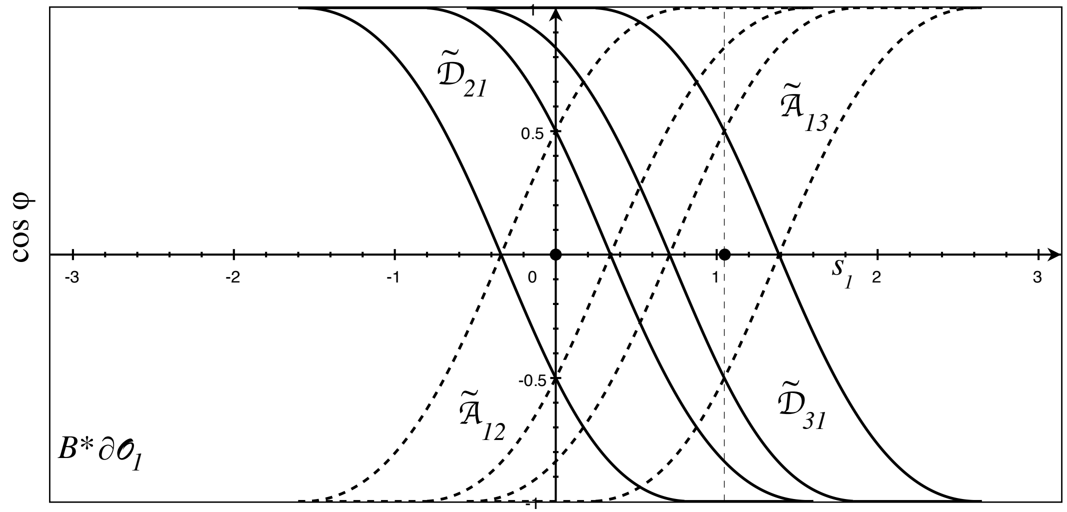


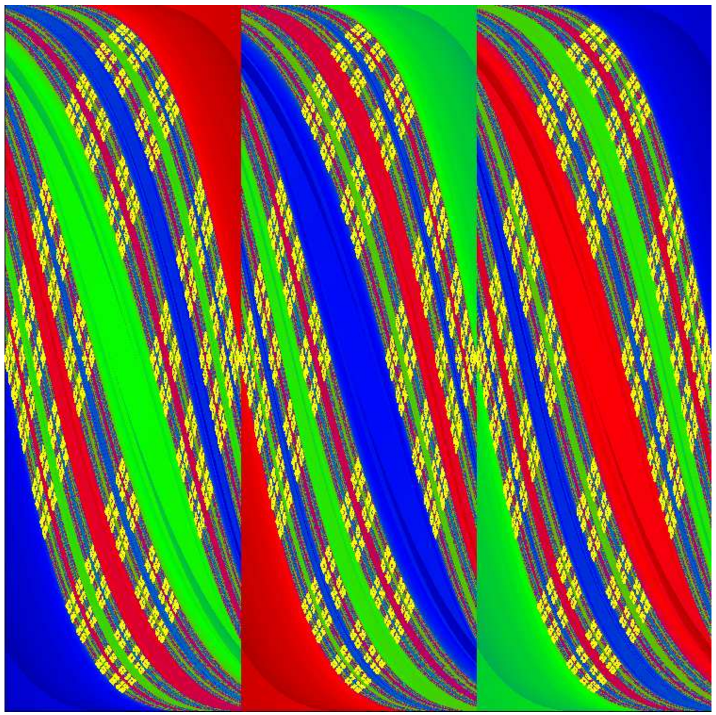
Hyperbolicity: $\forall \rho \in K_E$, $T_\rho p^{-1}(E) = H_p(\rho) \oplus E_\rho^+ \oplus E_\rho^-$ unstable/stable subspaces

The unstable Jacobian $J^+(\rho) = |\det(d\Phi^1_{|E_\rho^+})|$ measures the degree of hyperbolicity.

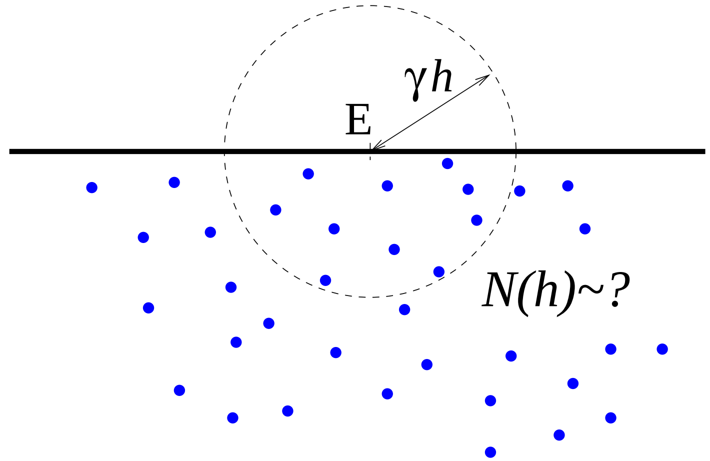
Ex: 3 circular obstacles in $\mathbb{R}^2$.







Counting long-living resonances: Fractal Weyl upper bound



Theorem. • $P(h) = -\frac{h^2 \Delta_{\mathbb{R}^d}}{2} + V(x)$ [Sjöstrand'90, Sjöstrand-Zworski'07]
• $X = \Gamma \backslash \mathbb{H}^d$ *Schottky quotient* [Zworski'99, Guillopé-Lin-Zworski'04]
• $J \geq 3$ *convex obstacles (no-eclipse condition)* [N-Sjöstrand-Zworski'11]

$$\forall \gamma > 0,\ \exists C_\gamma, \qquad \# \{\mathrm{Res}(P(h)) \cap D(E, \gamma\, h)\} \leq C_\gamma\, h^{-\nu+0}\,.$$

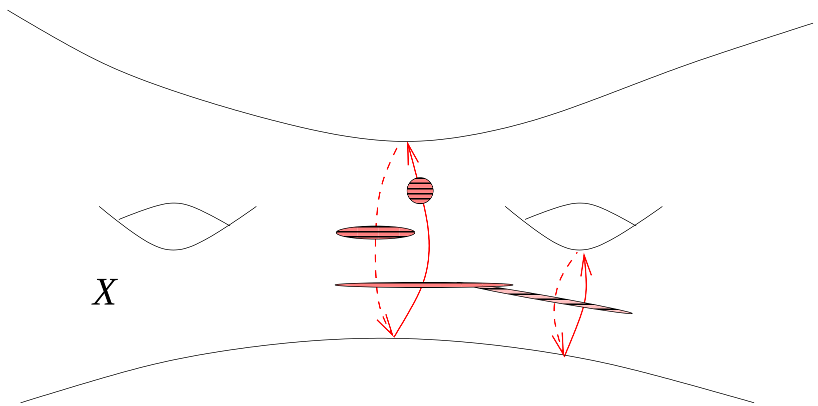
Here $\nu = \frac{\dim(K_E)-1}{2}$ *(upper Minkowski dimension).*

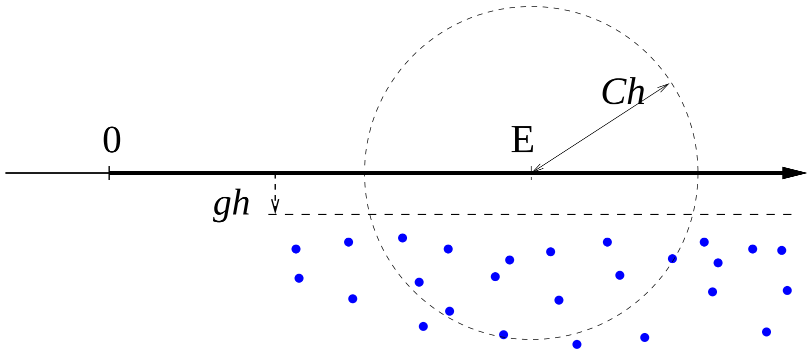
Main idea: the long-living metastables "live" in an $h^{1/2}$-neighbourhood of K_E.
$\rightsquigarrow$ count the number of "quantum boxes" (of volume h^{d-1}) in this nbhood.

Conjecture: the upper bound is sharp (at least at the level of the power ν):
Fractal Weyl law [Guillopé-Zworski'99, Lin-Zworksi'02...]

Nonselfadjoint spectral problem ⇒ lower bounds difficult to obtain.

A classical criterion for a resonance gap





Hyperbolicity of $\Phi^t \restriction_{K_E} \Longrightarrow$ a wavepacket will disperse fast through $e^{-itP(h)/h}$.
On the other hand, possible relocalization through constructive interferences.
What criterion for a global decay?

Topological pressure of $\Phi^t \restriction_{K_E}$: generalization of the topological entropy.

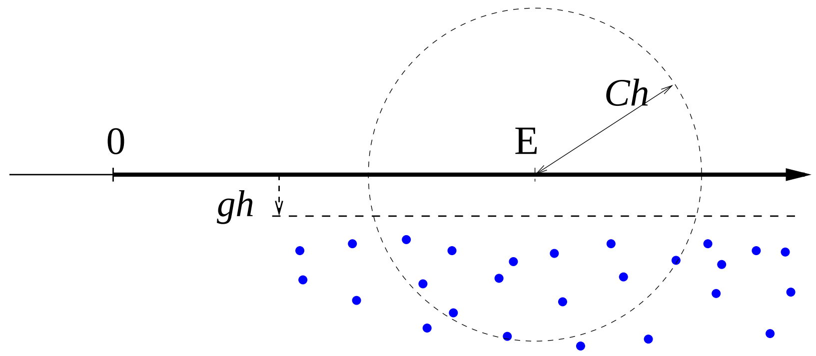
Choose a test function $f \in C(K_E)$. The pressure is obtained by summing over Bowen balls $B(x, \epsilon, T)$ weighted by $e^{f_T(x)}$, $f_T(x) = \int_0^T f \circ \Phi^t(x)\, dt$.

Equivalently, sum over weighted T-periodic orbits:

$$\mathcal{P}_E(f) \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T} \log \sum_{\gamma: T-1 \le T_\gamma \le T} e^{f_T(\gamma)} \qquad (\gamma = \text{ periodic orbits on } K_E)$$

$f \equiv 0$ leads to the topological entropy.

"Thin" trapped set and resonance gap (2)



Choose the test function $f = -s \log J^u$, $s \geq 0$, to test the hyperbolicity of the trajectories.

$\rightsquigarrow$ balance between complexity and hyperbolicity of $\Phi^t \restriction_{K_E}$:

$$\mathcal{P}_E(-s \log J^u) \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T} \log \sum_{\gamma : T-1 \leq T_\gamma \leq T} J^u(\gamma)^{-s}$$

Theorem. [Ikawa'88,Gaspard-Rice'89,N-Zworski'09]
Suppose the trapped set is such that $\mathcal{P}_E(-1/2 \log J^u) < 0$.
Then, for any $0 < g < |\mathcal{P}_E(-1/2 \log J^u)|$ *and* h *small enough, the strip* $[E - Ch, E + Ch] - i[0, gh]$ *is free of resonances.*

Remark: $\mathcal{P}(0) = H_{top}(\Phi \restriction_{K_E}) > 0$. $\mathcal{P}(-\log J^u) = -\gamma_{cl} < 0$ (classical escape rate).

$d = 2$: $\mathcal{P}_E(-1/2 \log J^u) < 0 \Longleftrightarrow \dim K_E < 2$ ("thin" trapped set).

Phase space distribution of metastable states

Theorem. [Bony-Michel'04,Keating *et al.*'06,N-Rubin'07,N-Zworski '09]
Consider a sequence of metastable states $(u_h)_{h\to 0}$ associated with $z_h = E + \mathcal{O}(h)$, normalized by $\|u_h\|_{L^2(\Omega)} = 1$ for Ω a neighbourhood of $\pi(K_E)$.

Up to extracting a subsequence, we can assume that a semiclassical measure μ is associated with $(u_h)_{h\to 0}$:

$$\forall f \in C_c^\infty(T^*X),\ \forall \chi \in C_c^\infty(X), \qquad \langle \chi u_{h_k}, \mathrm{Op}_h(f)\chi u_{h_k}\rangle \to \int_{T^*X} f(\rho)\, d\mu(\rho)\,.$$

Then μ is supported on the outgoing set of K_E (unstable manifold), and there exists $\gamma \geq 0$ s.t.

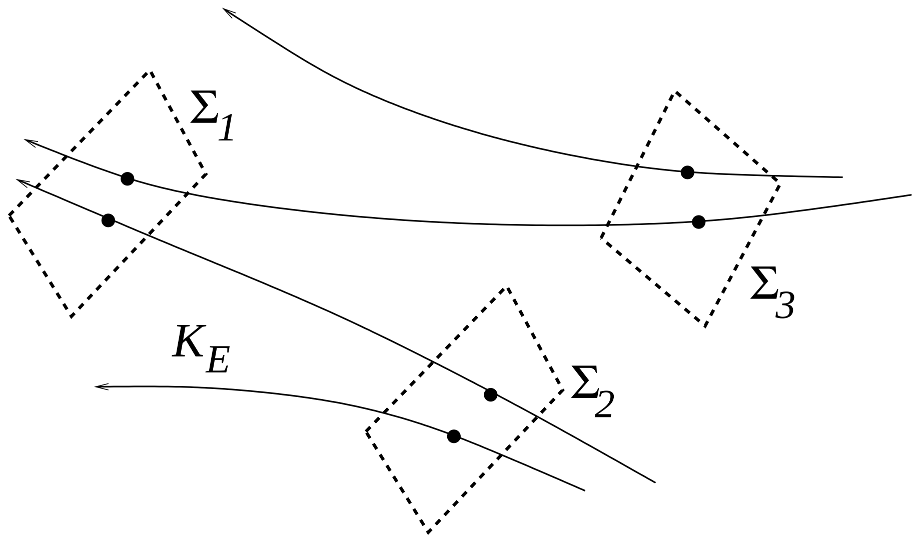
$$\frac{\mathrm{Im}\, z_{h_k}}{h_k} \to -\gamma/2 \qquad \textit{and} \qquad \mathcal{L}_{H_p}\,\mu = \gamma\mu\,.$$

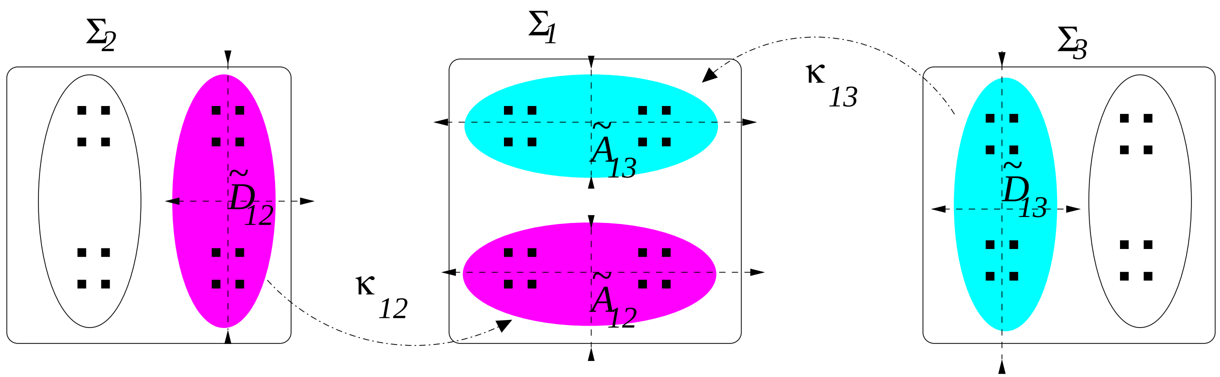
μ is a Conditionally Invariant Measure for the flow. The proof mimics the proof of invariance of μ for closed systems.

Questions:
– which CIM can appear as semiclassical measures?
– is there a form of *quantum ergodicity*?

Partial answers for a *solvable* open quantum baker's map [Keating *et al.*'08].

Poincaré section: reduction of the Hamiltonian flow

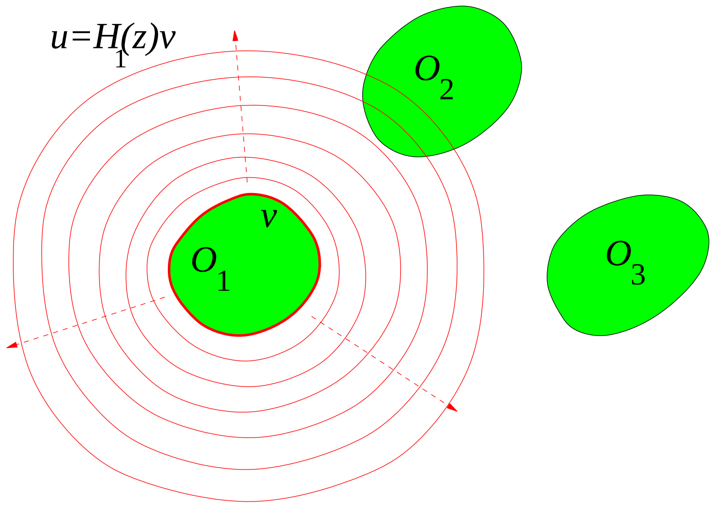




• $\mathbf{\Sigma} = \sqcup_{j=1}^{J} \Sigma_j$ hypersurfaces in $p^{-1}(E)$ transverse to the flow near K_E (dim $= 2d-2$).
$\rightsquigarrow$ Φ^t replaced by the Poincaré map $\kappa : \Sigma \to \Sigma$ and return time $\tau : \Sigma \to \mathbb{R}^+$.

• Can one **quantize** this reduction, namely study $P(h)$ or $e^{-itP(h)/h}$ through a **quantum propagator** assoc. with κ, and depending on τ)?

Ex. of reduction: Euclidean obstacle scattering



J convex obstacles on $\mathbb{R}^d$, $P(h) = -\frac{h^2 \Delta_D}{2}$.

Poisson operator $H_j(z) : C^\infty(\partial O_j) \to C^\infty(\mathbb{R}^n \setminus O_j)$, for each obstacle $j = 1, \ldots, J$.

Definition: $u = H_j(z)v$ satisfies $(P(h) - z)u = 0$, $u \restriction_{\partial O_j} = v$, u outgoing

$\rightsquigarrow$ $J \times J$ matrix of boundary operators $\mathcal{M}_{ij}(z,h) : C^\infty(\partial O_j) \to C^\infty(\partial O_i)$:

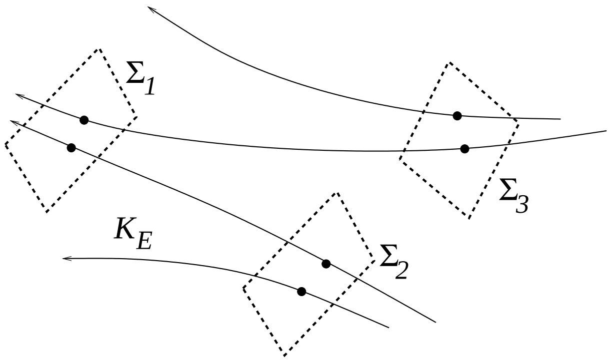
$$\mathcal{M}_{ij}(z,h) v_j \stackrel{\text{def}}{=} \Big(H_j(z) v_j \Big) \restriction_{\partial O_i} \quad \text{for } i \neq j, \qquad \mathcal{M}_{jj}(z,h) = 0\,.$$

FIO assoc.w. the boundary map $\kappa_{ij} : B^*\partial O_j \to B^*\partial O_i$.

Reduction: z resonance de $P(h)$ $\iff$ z pole of $(I - \mathcal{M}(z,h))^{-1}$

Monodromy operator for smooth potential scattering

Case $P(h) = -\frac{h^2\Delta}{2} + V(x)$, with K_E hyperbolic repeller.



Theorem. [N-Sjöstrand-Zworski'10] *Consider a Poincaré section* $\mathbf{\Sigma} = \sqcup_{j=1}^{J} \Sigma_j \subset p^{-1}(E)$, *assume the reduced trapped set* $\mathcal{K} \stackrel{\text{def}}{=} K_E \cap \mathbf{\Sigma}$ *doesn't touch* $\partial\mathbf{\Sigma}$.
Then one can construct a quantum monodromy operator $M(z,h)$*:*

- $M(z,h) = (M(z,h)_{ij}) : L^2(\mathbb{R}^{d-1})^J \to L^2(\mathbb{R}^{d-1})^J$ *matrix of FIO assoc. w.* κ_{ij}.
  Holomorphic in $z \in D(E, Ch\log(1/h))$, $M(z,h) \approx M(E,h)\,\mathrm{Op}_h(e^{i\frac{z-E}{h}\tau})$

- $M(z,h)$ *is microlocally supported near* $K_E \cap \mathbf{\Sigma}$, *rank* $\asymp h^{-(d-1)}$.

- z *resonance of* $P(h) \iff$ *pole of* $(I - M(z,h))^{-1} \iff \det(I - M(z,h)) = 0$.

A similar reduction appeared in the physics literature [Bogomolny'92,Doron-Smilansky'92,Prosen'95].

Proof of fractal Weyl upper bound from monodromy op.

Aim: prove a fractal Weyl upper bound for the solutions of $\det(I - M(z,h)) = 0$.

Trick: $M(z,h)$ is conjugate with $\tilde{M}(z,h)$ microlocalized in $h^{1/2}$-nbhd of $\mathcal{K}$.

• use an appropriate escape function $G(x,\xi)$:

$$M(z) \rightsquigarrow M_G(z) = e^{-\operatorname{Op}_h(G)} M(z) e^{\operatorname{Op}_h(G)} \overset{\text{Egorov}}{\approx} M(z) e^{\operatorname{Op}_h(G - G\circ\kappa)^w}$$

We construct $G(x,\xi)$ s.th. $G\circ\kappa - G \geq C_1 \gg 1$ outside this $h^{1/2}$-nbhd (uses the hyperbolicity of K_E).

$$\Longrightarrow \text{symbol}(M_G(z)) \leq e^{-C_1} \ll 1 \quad \text{outside the } h^{1/2}\text{-nbhd.}$$

• $\rightsquigarrow$ effective monodromy operator $\tilde{M}(z,h)$ microlocalized in this nbhd, with $\text{rank}(\tilde{M}(z,h)) \asymp h^{-\nu+0}$.

$\tilde{M}(z,h)$ a "minimal matrix" encoding the long-living quantum dynamics near energy E.

z-holomorphic $\overset{\text{Jensen}}{\rightsquigarrow}$ $\#\{z \in D(E, Ch),\ \det(1 - \tilde{M}(z,h)) = 0\} \leq C' \,\text{rank}(\tilde{M})$. $\square$

Proof of resonance gap from monodromy op.

Strategy: long time iteration in order to bound the spectral radius of $M(z,h)$:

$$[M(z,h)^N]_{i_N i_0} = \sum_{i_{N-1},\dots,i_1} M_{i_N i_{N-1}} M_{i_{N-1} i_{N-2}} \cdots M_{i_1 i_0} \stackrel{\text{def}}{=} \sum_{\vec{i}} M_{\vec{i}}$$

Hyperbolic dispersion estimate [ANANTHARAMAN'06,N-ZWORSKI'09]: for each "path" $\vec{i}$,

$$\|M_{\vec{i}}(z)\| \leq h^{-(d-1)/2}\, J^u_{\kappa,N}(\vec{i})^{-1/2}\, e^{-\zeta \tau_N(\vec{i})}\,, \quad \zeta \stackrel{\text{def}}{=} \operatorname{Im} z/h\,,$$

valid for times $N \sim C \log(1/h)$, $C \gg 1$. Triangle $\leq$ implies

$$\|M(z,h)^N\| \lesssim \sum_{\vec{i}\ \text{admis.}} e^{N\epsilon}\, J^u_{\kappa,N}(\vec{i})^{-1/2}\, e^{-\zeta \tau_N(\vec{i})} \lesssim \exp\left\{N\big(\mathcal{P}_\kappa(-1/2 \log J^u_\kappa - \zeta\tau) + \epsilon\big)\right\}$$

$$\Longrightarrow r_{sp}(M(z,h)) \leq e^{\mathcal{P}_\kappa(-1/2 \log J^u_\kappa - \zeta\tau) + \epsilon}.$$

Relation between the topological pressures of $\kappa \restriction_{\mathcal{K}}$ and $\Phi^t \restriction_{K_E}$:

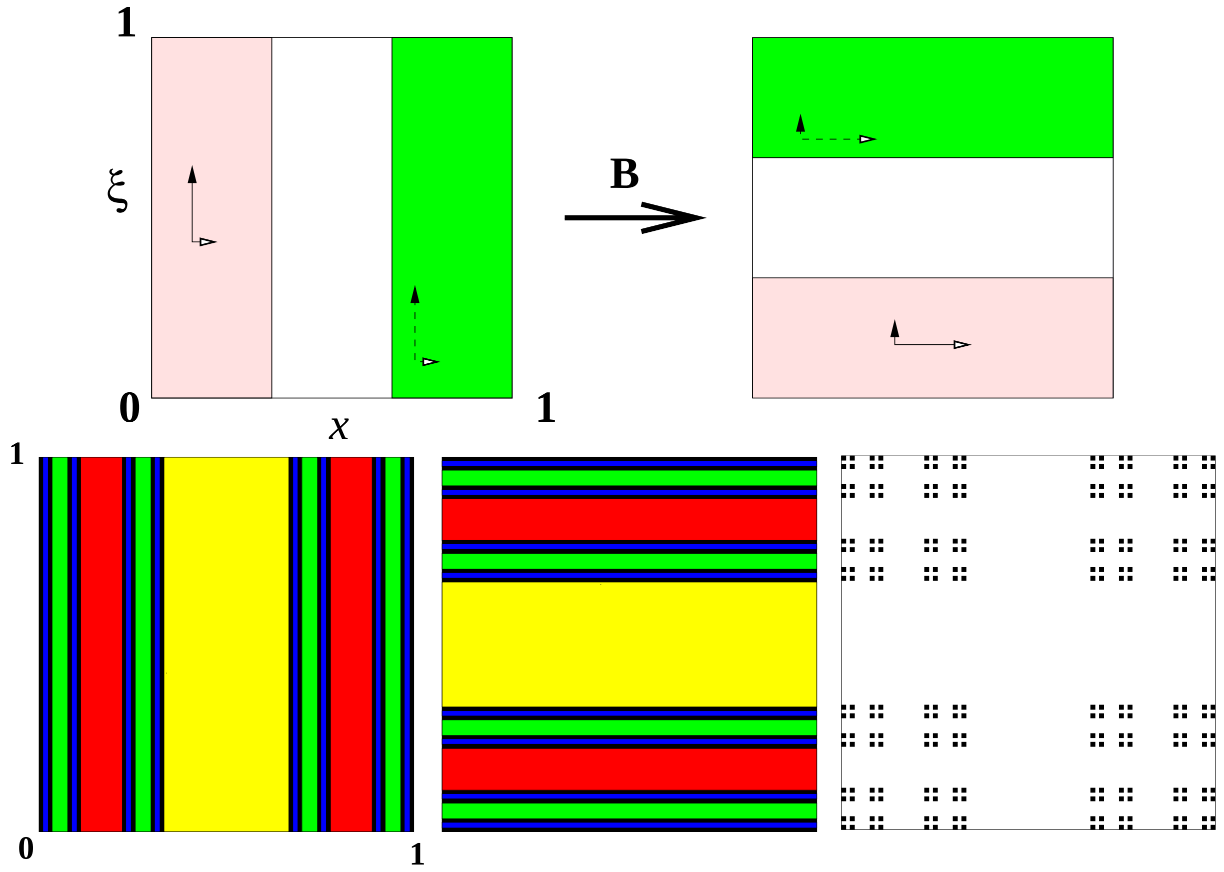
$$\mathcal{P}_\Phi(-1/2 \log J^u) < \zeta \iff \mathcal{P}_\kappa(-1/2 \log J^u_\kappa - \zeta\tau) < 0\,,$$

in which case $r_{sp}(M(z,h)) < 1 \Longrightarrow \det(I - M(z,h)) \neq 0$. □

A toy model: open quantum maps

Toy model for Poincaré maps: open chaotic map.
Symplectic diffeom $\kappa : V \mapsto \kappa(V)$, $V \Subset \mathbb{R}^{2(d-1)}$ with chaotic trapped set $\mathcal{K}$.

Ex: open baker's map (on $\mathbb{T}^2$). (Piecewise) smooth, simple dynamics.

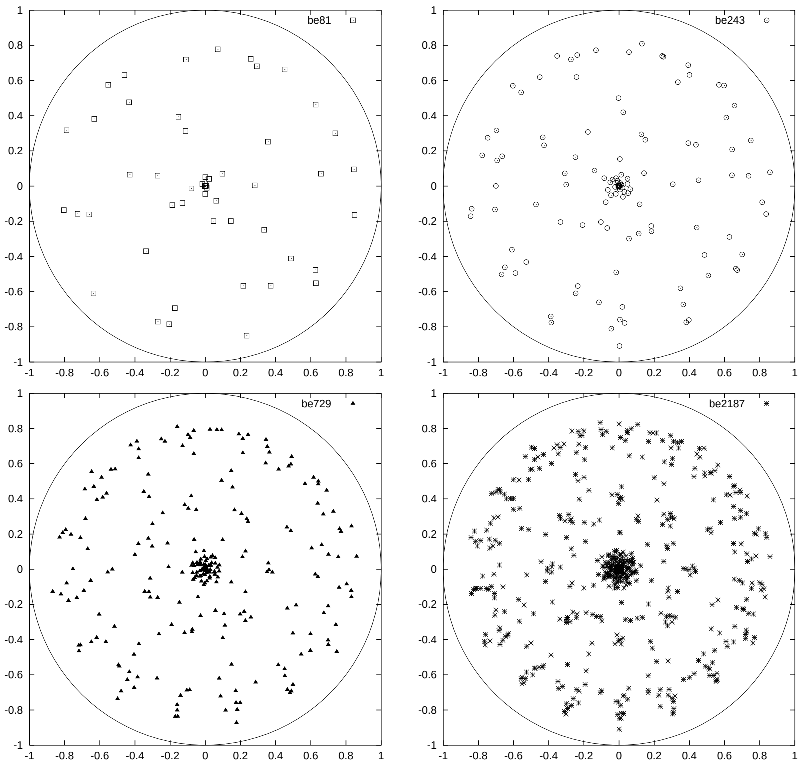


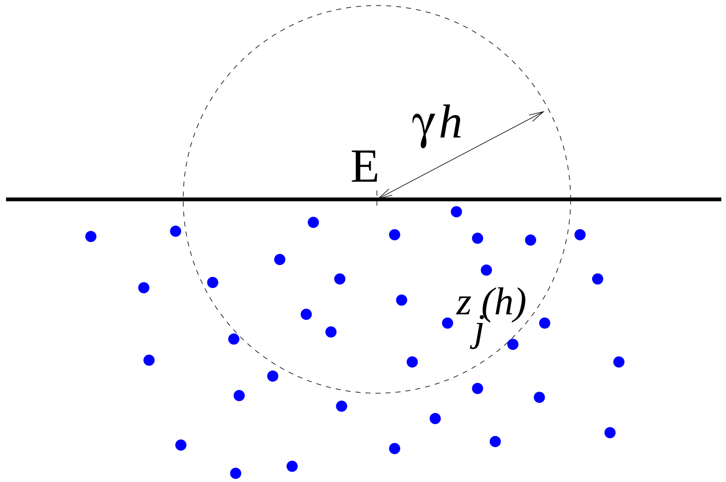
Quantization of κ: family of **subunitary** matrices $(M(h))_{h\to 0}$ of ranks $\sim h^{-(d-1)}$.
$M(h) \approx$ FIO associated with κ. Open quantum map. (baker's map: very explicit).

Open quantum (chaotic) map

$(M(h))_{h\to 0}$ subunitary propagagors assoc.w. κ, of ranks $\sim h^{-(d-1)}$.

<u>Heuristics</u>: $M(z,h) \stackrel{\text{def}}{=} M(h)e^{iz/h}$ resembles a quantum monodromy operator.
Zeros of $\det(I - M(z,h))$ give the nonzero spectrum $\{\lambda_j = e^{-iz_j/h}\}$ of $M(h)$.
$\Longrightarrow$ long-living spectrum of $M(h)$ inside some annulus $\{|\lambda| \geq r > 0\}$.
Easy to implement numerically.





Spectra of the quantum open baker's map for increasing values of h^{-1}.

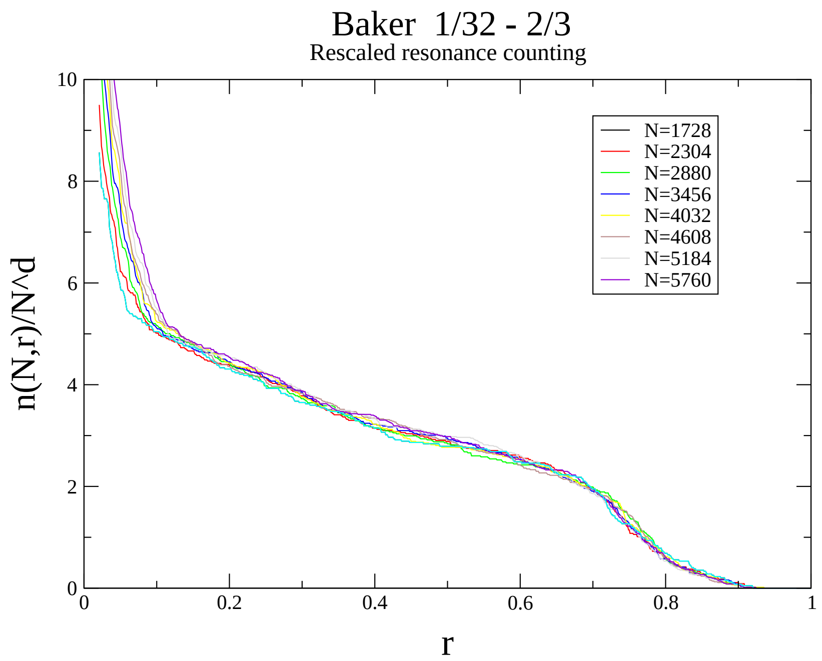
Numerical tests of the Fractal Weyl law

Do we have $\#\{\mathrm{Res}(P(h)) \cap D(E, \gamma h)\} \geq c\, h^{-\nu}$ for $\gamma > 0$ large enough?

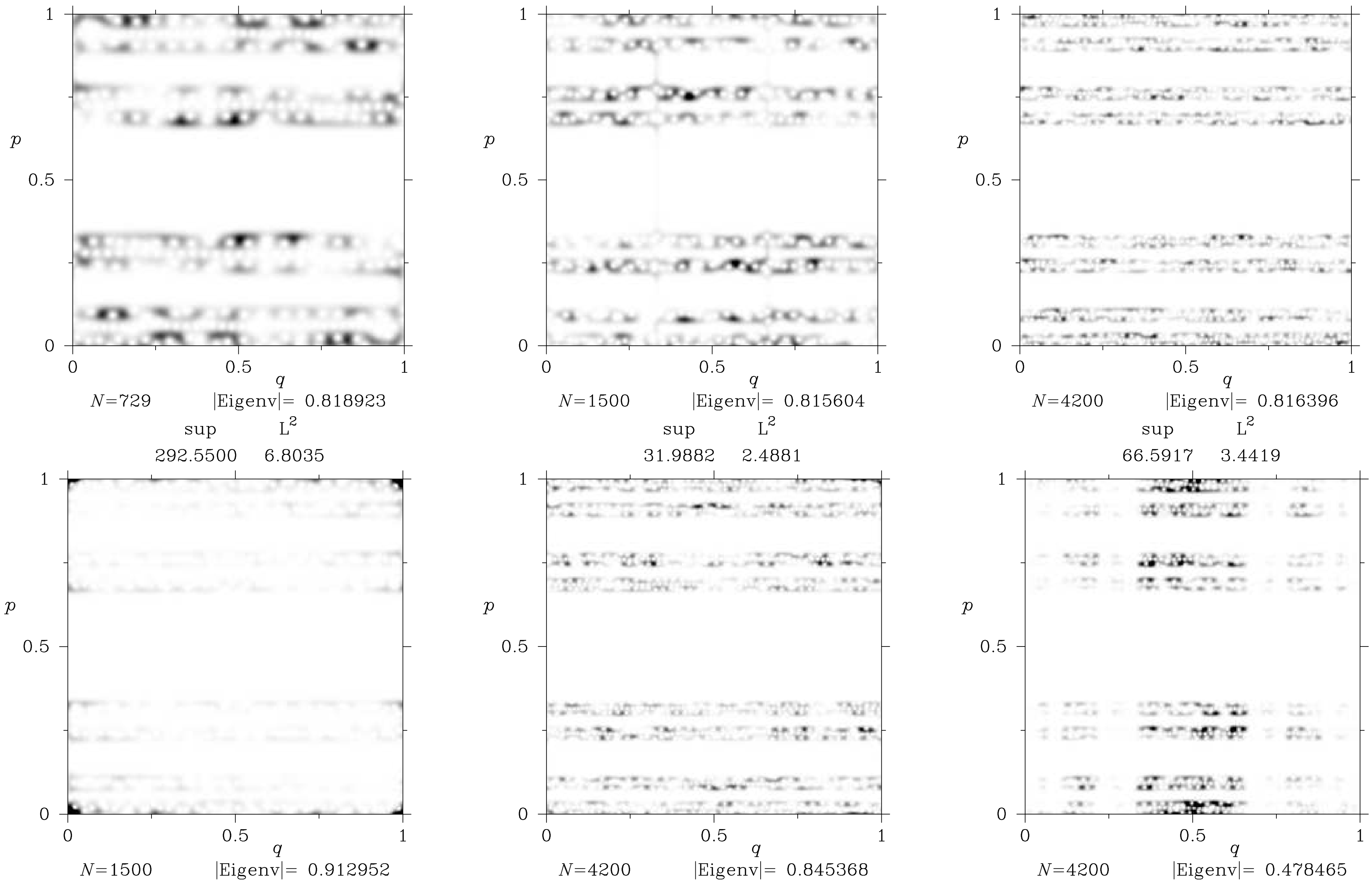
• Numerics for 3 differents $P(h)$ seem to confirm the fractal Weyl law [LIN'01, LU-SRIDHAR-ZWORSKI'03, GUILLOPÉ-LIN-ZWORSKI'04].

• Easier numerics for open quantum maps hint at a more precise scaling [SCHOMERUS-TWORZYDŁO'04, N-ZWORSKI'05, N-RUBIN'07...]

For an asymmetric open baker, we plot $\frac{\#\{\lambda_j \in \mathrm{Spec}(M(h)), |\lambda_j| \geq r\}}{h^{-\nu}} \approx F(r)$ for different values of h^{-1}.



Phase space distribution of metastable states



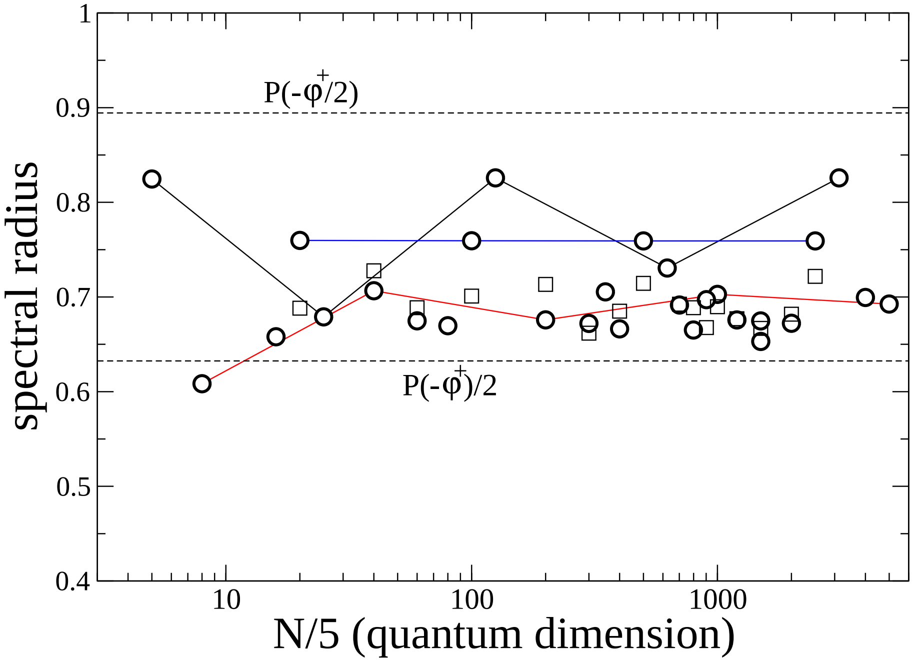
A few long-living metastable states for the open baker's map (Husimi density).

Numerical tests of the resonance gap

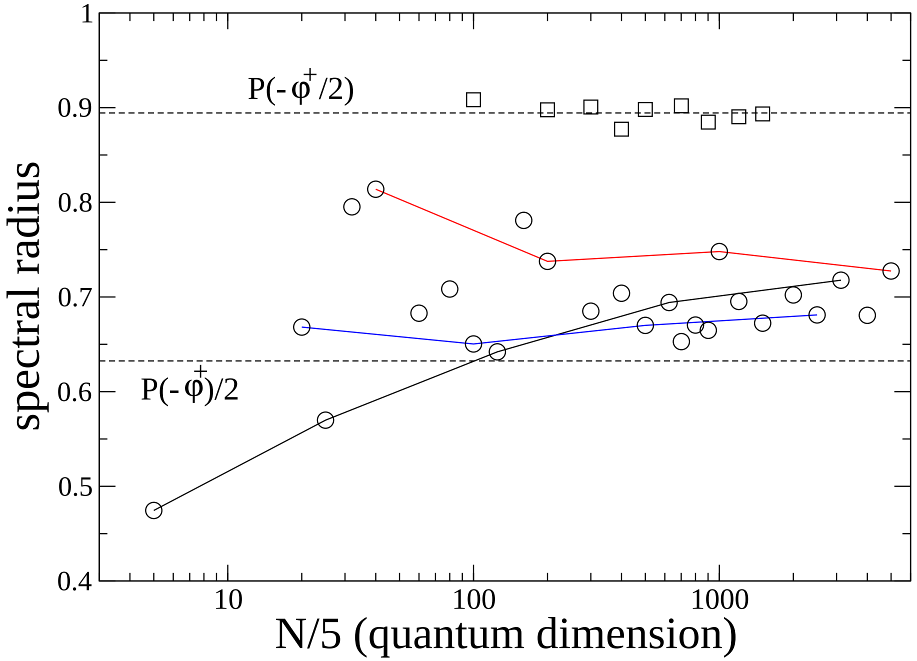
Spectral radii for two baker's maps (with same topological pressures).

Horizontal lines: $\mathcal{P}(-1/2 \log J^u)$ and $\mathcal{P}(-\log J^u)/2$.

Quantum open baker (D=5, columns 1-3)



Quantum open baker (D=5, columns 0-4)



$\mathcal{K}$ away from discontinuities.

$\mathcal{K}$ touches the discontinuities.

A solvable toy-of-the-toy model

One can quantize the open baker's maps in a nonstandard way (discrete Fourier transform on $\rightsquigarrow$ Walsh-Fourier transform), s.th. the quantum map $M(h)$ can be *analytically diagonalized*.

- fractal Weyl upper bound OK. Fractal Weyl law generally OK, but possibility of "accidental" degeneracies of $\tilde{M}(h)$, such that

$$\forall r < 1, \quad \#\{\lambda_j \in \mathrm{Spec}(M(h)),\, |\lambda_j| \geq r\} = \mathcal{O}(h^{-\tilde{\nu}}) \quad \text{for some} \quad \tilde{\nu} < \nu.$$

- spectral radius can take values in the range

$$0 \leq r_{sp}(M(h)) \leq e^{\min(0, \mathcal{P}(-\log J^u/2))}.$$

One can add some *randomness* in the model to ensure fractal Weyl law.

$\rightsquigarrow$ for a general system, does the fractal Weyl law only hold under some *genericity* condition?