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Spectral vs periodic orbit correlations

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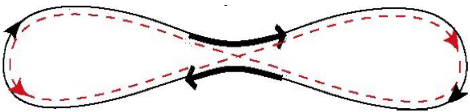
EMS-IAMP Summer school on Quantum Chaos

Sebastian Müller

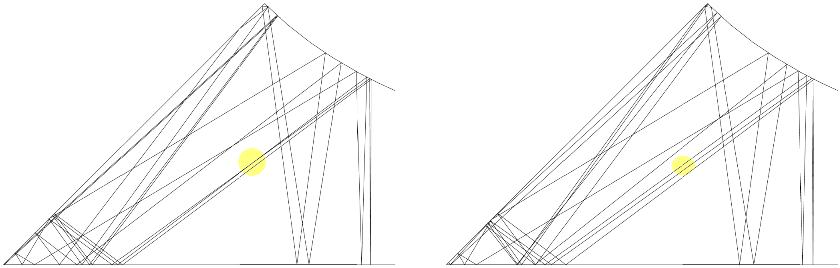
Orbit correlations

Orbit correlations

Example (Sieber & Richter 2001):



Realistic picture:



Orbit correlations

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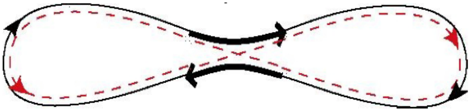


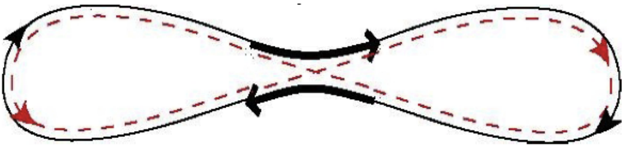
- encounters
  = regions where parts of an orbit come close to each other (up to time reversal)
- can switch connections to get different (but very similar) orbits
- present example requires time reversal invariance

Underlying mechanism

Phase space directions in hyperbolic systems:

- stable direction:
  deviations shrink asymptotically like $e^{-\lambda t}$
  (λ=Lyapunov exponent)
- unstable direction:
  deviations grow for $t \to \infty$ and shrink for $t \to -\infty$ like $e^{\lambda t}$
  $\Rightarrow$ sensitive dependence on initial conditions

Construction of partner orbit:



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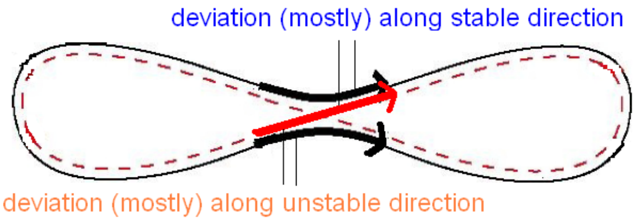
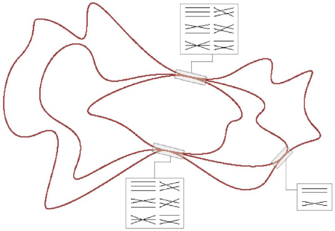
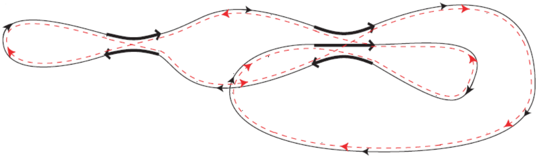


Generalisation

- orbits can differ in arbitrarily many encounters where arbitrarily many stretches come close



- for time reversal invariant systems: stretches may be almost mutually time reversed



Periodic orbits in chaotic systems come in bunches.

Spectral correlations

Spectral correlations

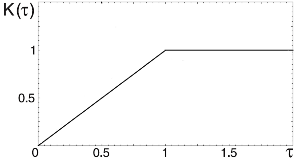
2-point correlation function $R_2(x)$:
how many pairs of levels with distance x?

here: spectral form factor= Fourier transform of $R_2(x) - 1$

$$
\begin{aligned}
d(E) &= \sum_j \delta(E - E_j) = \bar{d}(E) + d_{\mathrm{osc}}(E) \\
K(\tau) &= \frac{1}{\bar{d}^2} \left\langle \int_{-\infty}^{\infty} d_{\mathrm{osc}}\left(E + \frac{x}{2\bar{d}}\right) d_{\mathrm{osc}}\left(E - \frac{x}{2\bar{d}}\right) e^{2\pi i x \tau} dx \right\rangle
\end{aligned}
$$

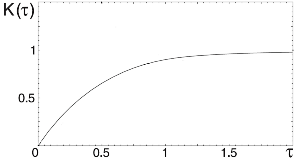
Predictions from Random Matrix Theory

- no symmetries: H Hermitian, Gaussian Unitary Ensemble



$$K(\tau) = \begin{cases} \tau & (\tau < 1) \\ 1 & (\tau > 1) \end{cases}$$

- systems with time reversal invariance:
  H real symmetric, Gaussian Orthogonal Ensemble



$$K(\tau) = \begin{cases} 2\tau - \tau \ln(1+2\tau) \\ = 2\tau - 2\tau^2 + 2\tau^3 - \frac{8}{3}\tau^4 + \ldots \\ \quad (\tau < 1) \\ 2 - \ln \frac{2\tau+1}{2\tau-1} \quad (\tau > 1) \end{cases}$$

$\tau > 1$ terms connected to oscillatory terms in

$$R_2(x) = \mathrm{Re} \sum_n (c_n + d_n e^{2\pi i x}) \frac{1}{x^n}$$

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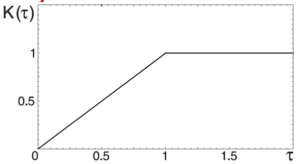

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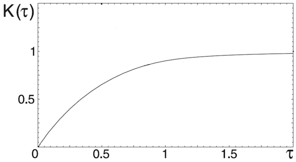

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Bohigas, Giannoni, Schmit: Spectral statistics of individual (generic) chaotic systems are faithful to these predictions for large energies.

Why?

Semiclassical approach

Weyl term

average level density approximated by

$$\bar{d}(E) \sim \frac{\Omega(E)}{(2\pi\hbar)^f}$$

$$\begin{aligned} \Omega(E) &= \text{volume of energy shell} \\ f &= \#\text{degrees of freedom (e.g. 2)} \end{aligned}$$

Gutzwiller trace formula

$$d_{\text{osc}}(E) \sim \frac{1}{\pi\hbar} \; \text{Re} \sum_{\text{periodic orbits } \gamma} A_\gamma e^{iS_\gamma/\hbar}$$

here:

$$\begin{aligned} S_\gamma &= \text{classical action} \\ A_\gamma &= \text{stability amplitude} \\ &\quad (\text{incorporates factor} e^{-i\mu_\gamma \frac{\pi}{2}}, \mu_\gamma \in \mathbb{N}) \end{aligned}$$

Valid if actions $\gg \hbar$, i.e., for large energies.

Spectral form factor

Spectral form factor

$$K(\tau) \sim \frac{1}{T_H} \sum_{\gamma,\gamma'} \left\langle A_\gamma A^*_{\gamma'} e^{i(S_\gamma - S_{\gamma'})/\hbar} \delta\left(\tau T_H - \frac{T_\gamma + T'_\gamma}{2}\right)\right\rangle$$

- relevant periods of order

$$\text{Heisenberg time}\, T_H = 2\pi\hbar\bar{d} \sim \frac{\Omega}{2\pi\hbar} \to \infty$$

- Need pairs of orbits with small action difference!

Diagonal approximation

(Berry, Hannay/Ozorio de Almeida)

- for systems without time reversal invariance: take $\gamma' = \gamma$

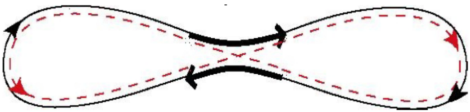
$$K_{\mathrm{diag}}(\tau) = \frac{1}{T_H} \left\langle \sum_{\gamma} |A_\gamma|^2 \delta(\tau T_H - T_\gamma) \right\rangle \sim \tau$$

sum over orbits evaluated using ergodicity

- time reversal invariant systems: $\gamma' = \gamma$ or time reversed of γ

$$K_{\mathrm{diag}}(\tau) = 2\tau$$

Sieber/Richter pairs



Decompose separation between encounter stretches into unstable component u and stable component s. These determine:

- action difference

$$S_\gamma - S_{\gamma'} \sim us$$

- duration of the encounter (defined by $|s|, |u| < c$)

$$t_{\mathrm{enc}}(u,s) \sim \frac{1}{\lambda} \ln \frac{c}{|u|} + \frac{1}{\lambda} \ln \frac{c}{|s|} = \frac{1}{\lambda} \ln \frac{c^2}{|us|}$$

relevant encounters of order

$$T_{\mathrm{Ehrenfest}} = \frac{1}{\lambda} \ln \frac{c^2}{\hbar} \ll T_{\mathrm{Heisenberg}} = \frac{\Omega}{2\pi\hbar}$$

(Spehner 2003; Turek/Richter 2003; Heusler, S.M., Braun, Haake 2003)

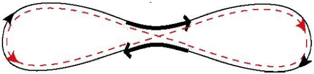
Sieber/Richter pairs

- probability of encounters with given separations

$$w_T(u,s) \sim \frac{T(T-2t_{\rm enc}(u,s))}{\Omega t_{\rm enc}(u,s)}$$

determined using

- ergodicity
- orbits must leave encounter before reentering



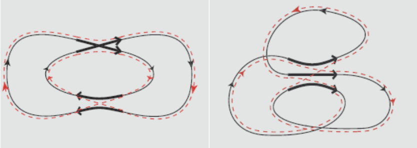
$\Rightarrow$ Contribution to form factor

$$K_{\rm SR}(\tau) \sim \left\langle \sum_\gamma |A_\gamma|^2 \delta(\tau T_H - T_\gamma) \int du \int ds\, w_{\tau T_H}(u,s) e^{ius/\hbar} \right\rangle = -2\tau^2$$

agrees with GOE ☺

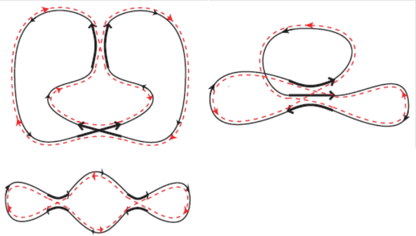
τ^3

- orbit pairs in systems without time reversal invariance



$\Rightarrow$ contributions cancel, agreement with GUE

- additional pairs requiring time reversal invariance

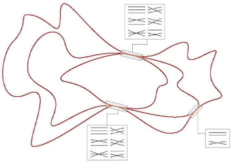


$\Rightarrow 2\tau^3$, agreement with GOE

(Heusler, S.M., Braun, Haake 2003)

All orders in τ

need arbitrarily many encounters with arbitrarily many stretches



contribution of each “diagram” proportional to

$$\tau^{\#\text{stretches}-\#\text{encounters}+1}$$

sum over infinitely many diagrams!

Result:

$$K(\tau) = \begin{cases} \tau & \text{without reversal invariance} \\ 2\tau - \tau \ln(1+2\tau) & \text{with reversal invariance} \end{cases}$$

(S.M., Heusler, Braun, Haake, Altland, PRL 2004 + PRE 2005)

$\tau > 1$

$\tau > 1$

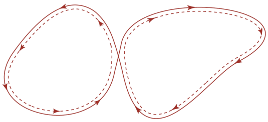
Need improved semiclassical approximation (Berry, Keating 1990)

$$d(E) = -\frac{1}{2\pi}\text{Im}\frac{\partial}{\partial E'}\frac{det(E-H)}{\det(E'-H)}\Big|_{E'=E}$$

$$\det(E-H) = e^{-i\pi\bar{d}E} \times \sum_\Gamma A_\Gamma e^{iS_\Gamma(E)/\hbar} + \text{c.c.}$$

sum over sets of classical periodic orbits shorter than $T_H/2$

This incorporates more QM ($\det(E-H) \in \mathbb{R}$).
Now orbits may decompose:



$\Rightarrow$ Full agreement with random matrix theory

Heusler, S.M., Altland, Braun, Haake, PRL 06; Keating, S.M., Proc. R. Soc. 07; S.M., Heusler, Altland, Braun, Haake, NJP 09

Conclusions

- periodic orbits of chaotic systems come in bunches
- bunching explains universal spectral statistics
- for $\tau > 1$ need improved semiclassical approximation
- examples for further applications:
  - symmetries: geometric, many particle, arithmetic
  - mesoscopic quantum transport

see also S.M. & Martin Sieber, Quantum Chaos and Quantum Graphs,
The Oxford Handbook of Random Matrix Theory (2011)

Appendix

All orders in τ

sum over infinitely many diagrams!

- describe diagrams by permutations,
  derive recursion between coefficients in $K(\tau)$
- similarity to Feynman diagrams:

establish 1-to-1 relation to diagrams in RMT

Conditions for universality

- existence of bunches requires hyperbolicity
- universal contribution obtained using
  - ergodicity, mixing
  - semiclassical limit
- no other orbit correlations