ESI Lectures on Semiclassical Analysis and Quantum Ergodicity 2012

Andrew Hassell

Department of Mathematics
Australian National University

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Outline

Lecture 1

Introduction

Quantum chaos is about the behaviour of eigenfunctions and eigenvalues of elliptic operators (such as the Laplacian on a compact Riemannian manifold) in the limit as the eigenvalue tends to infinity.

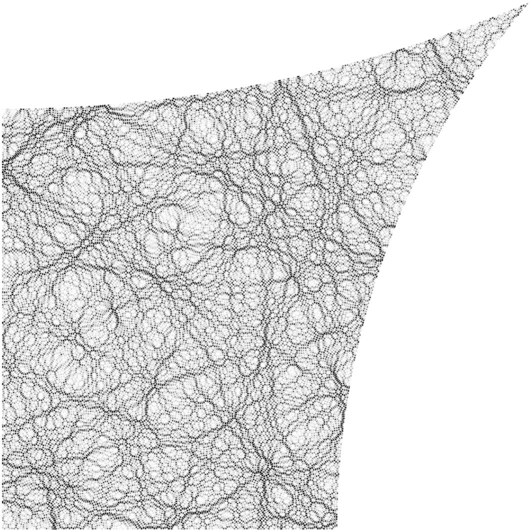

Figure: An eigenfunction on the Barnett stadium.

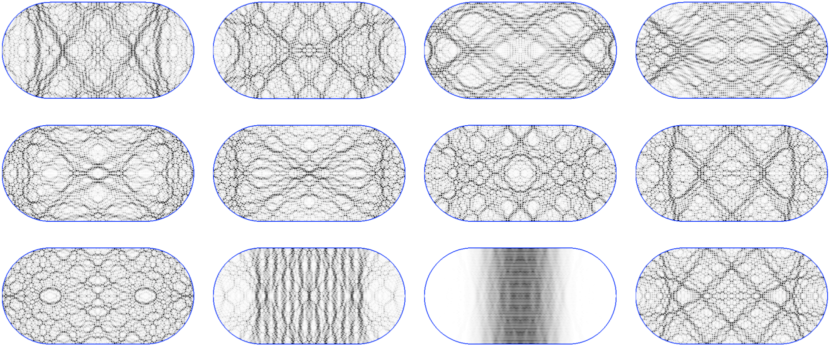

Figure: Eigenfunctions on the stadium; the second last is a 'bouncing ball mode' and is not equidistributed.

We will be interested in answers to the following sorts of questions:

- What is the typical behaviour of eigenfunctions at very high energies (eigenvalues)?
- Do they spread out evenly (equidistribute) over the manifold, or do some eigenfunctions concentrate in some regions?
- How are they related to classical dynamical properties of the Riemannian manifold, such as complete integrability, ergodicity, or chaos?

Examples to build intuition.

- On some special domains, such as spheres, tori, etc, we can write down explicit eigenfunctions in terms of special functions. However, such domains are completely integrable and give little or no insight into the ergodic or chaotic case.
- On hyperbolic manifolds there are methods using group theory or (in arithmetic settings) number theory.
- There are numerical studies of simple domains such as piecewise analytic plane domains.
- Toy models such as the ‘cat map’.

Semiclassical pseudodifferential operators

Two main reasons to introduce pseudodifferential operators:

- a very useful extension of the class of differential operators — big enough to include the inverses of positive elliptic operators.
- useful as a localization tool (this will hopefully become clearer soon).

Semiclassical differential operators look like the following. Let $h > 0$ be a small parameter; we will be interested in the limit $h \downarrow 0$. Given a differential operator

$$P = \sum_{\alpha} a_\alpha(x) D_x^\alpha, \quad D_i = \big(- i \frac{\partial}{\partial x_i} \big),$$

$$\alpha = (\alpha_1, \ldots, \alpha_n) \quad |\alpha| = \alpha_1 + \cdots + \alpha_n, \quad D_x^\alpha = \prod_{i=1}^{n} \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}},$$

its semiclassical version P_h is obtained by including a factor of h for every differentiation:

$$P_h = \sum_{\alpha} a_\alpha(x) (h D_x)^\alpha = \sum_{\alpha} a_\alpha(x) h^{|\alpha|} D_x^\alpha.$$

This is motivated by the form of the time-dependent Schrödinger equation in physics,

$$ih\frac{\partial\psi}{\partial t} = \frac{h^2}{2m}\Delta\psi + V(x)\psi, \quad \psi = \psi(x,t), \quad \Delta = -\sum_j \frac{\partial^2}{\partial x_j^2}$$

We can write a semiclassical differential operator, P_h say on $\mathbb{R}^n$, using the Fourier transform as follows:

$$(P_h f)(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} \sum a_\alpha(x)(h\xi)^\alpha f(y)\, dy\, d\xi$$

$$= (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi/h} \sum a_\alpha(x)\xi^\alpha f(y)\, dy\, d\xi.$$

The **symbol** of this differential operator, written $\sigma(P_h)$, is $\sum a_\alpha(x)\xi^\alpha$, a polynomial in ξ with smooth coefficients.

- If you compose two semiclassical differential operators P_h, Q_h, then

$$\sigma(P_h \circ Q_h) = \sigma(P_h)\sigma(Q_h) + O(h).$$

- Idea leading to pseudo differential operators: given P_h, look for an operator Q_h so that the product of the symbols of P_h and Q_h is 1, which is the symbol of the identity operator. Then Q_h would be an inverse of P_h, up to an $O(h)$ error, hopefully negligible as $h \to 0$.

The idea of pseudodifferential operators is to allow the symbol to range over a larger class of functions. We define the class of semiclassical symbols of order k, $S^{k,0}(\mathbb{R}^{2n})$, to be those smooth functions $a(x, \xi, h)$, depending parametrically on $h \geq 0$, satisfying

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi, h) \right| \leq C_{\alpha,\beta} \langle \xi \rangle^{k - |\beta|}. \tag{1}$$

- Here, k can be any real number.
- Note that the estimate in (1) is uniform as $h \to 0$.
- The gain in growth through differentiation is important to ensure invariance of $S^{k,m}(\mathbb{R}^{2n})$ w.r.t. coordinate changes (see next lecture).

More generally we define “symbols of semiclassical order m and differential order k” to be symbols of the form $h^{-m}a$, where $a \in S^{k,0}(\mathbb{R}^{2n})$. We will write this class $S^{k,m}(\mathbb{R}^{2n})$. Notice that the symbols get ‘more singular’ as either k or m increases.

Corresponding to each semiclassical symbol $a \in S^{k,m}(\mathbb{R}^{2n})$ is a semiclassical pseudodifferential operator, (really a family of operators parametrized by $h > 0$), with Schwartz kernel

$$\mathrm{Op}_h(a)(x,y) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi/h} a(x,\xi,h)\, d\xi. \tag{2}$$

That is, $\mathrm{Op}_h(a)$ (also denoted by $a(x, hD)$) acts on a function $f \in \mathcal{S}$ according to

$$\big(\mathrm{Op}_h(a) f\big)(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi/h} a(x,\xi,h) f(y)\, dy\, d\xi. \tag{3}$$

This is the "standard quantization". There is also the more symmetric Weyl quantization

$$\mathrm{Op}_h^w(a)(x,y) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi/h} a(\frac{x+y}{2},\xi,h)\, d\xi. \tag{4}$$

- This makes sense as a Lebesgue integral provided that the differential order of a is $< -n$.

• If the differential order is larger, then it is defined as follows: we choose $M > 0$ so that $a(1 + |\xi|^2)^{-M}$ has order $\leq -n$, and then we define

$$\big(\operatorname{Op}_h(a)f\big)(x) =$$
$$(2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi/h} a(x,\xi,h)(1+|\xi|^2)^{-M}\Big((1+h^2\Delta)^M f(y)\Big)\, dy\, d\xi$$

where $\Delta = \sum_i D_i^2$ is the (positive) Laplacian.

The class of pseudo differential operators with symbols in the class $S^{k,m}(\mathbb{R}^{2n})$ is denoted $\Psi_h^{k,m}(\mathbb{R}^n)$ (this is independent of quantization).

Basic properties:

- The Schwartz kernel of $\mathrm{Op}_h(a)$ is smooth and $O(h^\infty)$ away from the diagonal. On the diagonal, the kernel is not smooth (except for operators of differential order $-\infty$).
- $\mathrm{Op}_h(a)$ is a differential operator iff it is supported on the diagonal.
- If a is real, then $\mathrm{Op}_h^w(a)$ is formally self-adjoint.

Example: consider a frequency cutoff. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be smooth, with $\eta(\xi) = 1$ for $|\xi| \le 1$, $\eta(\xi) = 0$ for $|\xi| \ge 2$. Then the operator

$$f(x) \mapsto \mathcal{F}^{-1} \eta(\xi/R)(\mathcal{F} f)$$

cuts off f to frequencies $\le 2R$. (Here $\mathcal{F}$ is the Fourier transform.) Let $h = 1/R$. We can write the Schwartz kernel of the operator as

$$(2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\tilde{\xi}} \eta(\frac{\tilde{\xi}}{R})\, d\tilde{\xi}$$

$$= (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi/h} \eta(\xi)\, d\xi, \quad \xi = h\tilde{\xi}$$

which is manifestly a semiclassical pseudo differential operator of semiclassical order 0 and differential order $-\infty$. We see from this example that the semiclassical frequency ξ represents ‘true’ frequency ξ/h, which tends to infinity as $h \to 0$. For this reason the semiclassical calculus is the perfect tool for investigating high frequency phenomena, such as quantum chaos.

Some properties of pseudo differential operators:

- Pseudodifferential operators map Schwartz functions to Schwartz functions.
- The class of pseudo differential operators forms an algebra:

$$\Psi_h^{k,m}(\mathbb{R}^n) \circ \Psi_h^{k',m'}(\mathbb{R}^n) \subset \Psi_h^{k+k',m+m'}(\mathbb{R}^n).$$

Moreover, there is an asymptotic formula for the symbol of the product (see next page).

- Pseudos of order $(0,0)$ are bounded on $L^2(\mathbb{R}^n)$, uniformly in h.

Moreover, if $|a(x,\xi,h)| \leq C$, then

$$\|\operatorname{Op}_h(a)\|_{L^2\to L^2} \leq C + O(h). \tag{5}$$

- Elliptic pseudos are invertible modulo $\Psi_h^{-\infty,-\infty}(\mathbb{R}^n)$ (a symbol $a \in S^{k,0}(\mathbb{R}^{2n})$ is elliptic if $|a(x,\xi)| \geq C\langle\xi\rangle^k$). In fact, smooth *symbolic* functions of elliptic pseudos are also pseudodifferential.

The asymptotic formula for the symbol of a composition of PDOs is as follows:

$$\sigma^w(AB)(x,\xi,h) = e^{ih\sigma(D_x,D_\xi;D_y,D_\eta)/2}\sigma^w(A)(x,\xi,h)\sigma^w(B)(y,\eta,h)\Big|_{x=y,\xi=\eta}$$

$$= \sum_{j=0}^{N-1} \frac{\Big(ih\sigma(D_x,D_\xi;D_y,D_\eta)\Big)^j}{2^j j!}\sigma^w(A)(x,\xi,h)\sigma^w(B)(y,\eta,h)\Big|_{x=y,\xi=\eta}$$

$$\text{modulo } S^{k+k'-N,m+m'-N}(\mathbb{R}^{2n}),$$

$$\sigma(D_x,D_\xi;D_y,D_\eta) = D_y\cdot D_\xi - D_x\cdot D_\eta. \tag{6}$$

There are similar formulae for changing quantizations, and for the adjoint of a pseudo. They are proved using the stationary phase lemma (see section on FIOs).

Next we define the **Bargmann transform** or **FBI transform** to introduce the idea that pseudodifferential operators 'localize in phase space'. The Bargmann transform of a function $f \in L^2(\mathbb{R}^n)$ is a function $W_h f$ in $\mathbb{R}^{2n}$, defined by

$$W_h f(x,\xi) = 2^{-n/2}(\pi h)^{-3n/4} \int_{\mathbb{R}^n} e^{(i(x-y)\cdot\xi/h - |x-y|^2/2h)} f(y)\, dy$$

The kernel first localizes the function in a $\sqrt{h}$ neighbourhood around x, then takes the semiclassical Fourier transform at the (h-scaled) frequency ξ. Heuristically, the **Husimi density** $|W_h f(x,\xi)|^2$ measures the 'amount' of semiclassical frequency ξ present in the frequency decomposition of f near x.
Notice that $W_h f(x,\xi)$ is the L^2 scalar product between f and the Gaussian coherent state $\psi_{x,\xi}$.

Theorem

If $f \in L^2(\mathbb{R}^n)$ and $a \in S^{0,0}(\mathbb{R}^{2n})$, then

$$W_h\Big(\mathrm{Op}_h(a) f\Big)(x,\xi) = a \cdot W_h f + O_{L^2}(h). \tag{7}$$

This result allows us to interpret $\mathrm{Op}_h(a)$ as a localising operator. That is, if either $W_h f$ or a is (esssentially) supported on a small open set U, then $W_h(\mathrm{Op}_h(a)f)$ will also be (essentially) supported on U (up to a remainder $O_{L^2}(h)$).

- In the next lecture, we will introduce the related notion of semiclassical measure, using pseudodifferential operators directly. It is based on the intuition from (7) that a pseudodifferential operator 'localizes a function in phase space' to the support of its symbol.

Lecture 2

Pseudodifferential operators on manifolds

Our classes of pseudodifferential operators are **coordinate invariant**: if $A \in \Psi^{k,m}(\mathbb{R}^n)$, say with kernel supported in $U \times U$, and if $\kappa : U \to V$ is a diffeomorphism from the open set U to the open set V, then $(\kappa^{-1})^* \circ A \circ \kappa^* \in \Psi^{k,m}(\mathbb{R}^n)$. It follows that pseudodifferential operators can be defined on manifolds.

Definition

Let M be a closed manifold (i.e. compact, no boundary). We say a linear operator A from $C^\infty(M)$ to $C^\infty(M)$, depending on a parameter $h > 0$, is a pseudodifferential operator in $\Psi^{k,m}(M)$ if

- its Schwartz kernel is C^∞ on $M \times M$ away from the diagonal, such that its C^k-norm on any set $K_1 \times K_2$, where K_i are disjoint closed sets in M, is $O(h^\infty)$;
- for each $m \in M$ there are local coordinates x defined in a neighbourhood U of m and a smooth function ϕ supported in U with $\phi(m) \neq 0$, such that $\phi A \phi$ is a pseudodifferential operator on $\mathbb{R}^n_x$ of order (k, m).

The symbol of a PDO on a manifold is no longer canonically well-defined. However, the symbol of $A \in \Psi^{k,m}(M)$ is well-defined modulo symbols of order $(k-1, m-1)$; the representative of the symbol in the quotient space $S^{m,k}/S^{k-1,m-1}$ is called the principal symbol, denoted $\sigma_{pr}(A)$.

Also, the standard or Weyl quantizations are no longer canonically defined. One has to specify an open cover by coordinate charts U_i, a partition of unity ϕ_i subordinate to this open cover, etc; then one can define a Weyl quantization $a \mapsto \mathrm{Op}_h^w(a)$ depending on these choices. The principal symbol of $\mathrm{Op}_h^w(a)$ is independent of these choices.

A very important aspect of pseudodifferential equations on manifolds is that the symbol naturally takes values on the **cotangent bundle**. To see why this should be true, consider a differential operator of order 1; that is, a vector field, or a section of the tangent bundle. Any point in the tangent bundle $T_m M$ may be viewed as a linear function on the cotangent space $T^*_m M$. By taking products of such functions, we see that a differential operator of degree k on M determines a polynomial of degree k on each fibre of the cotangent space, modulo polynomials of degree $k-1$. We write $S^{k,m}(T^*M)$ for symbols of order (k,m) on T^*M.

Example: if (M,g) is a Riemannian manifold, and $\Delta = \Delta_g$ is the Laplacian associated to g, then the principal symbol of $h^2\Delta_g$ is $|\xi|^2_g$, the squared norm on the cotangent bundle determined by g (actually its dual metric on T^*M).

The importance of this is that the cotangent bundle is, in a canonical way, a symplectic manifold. The symplectic form plays a key role in the study of pseudodifferential operators, especially in the relationship between classical and quantum dynamics.
One way in which symplectic geometry shows up can be seen from (6); from this we see that if $A \in \Psi_h^{k,m}(\mathbb{R}^n)$, $B \in \Psi_h^{k',m'}(\mathbb{R}^n)$ then $[A, B] \in \Psi_h^{k+k'-1,m+m'-1}(\mathbb{R}^n)$ and

$$\sigma_{pr}\big(\frac{i}{h}[A, B]\big) = \{\sigma_{pr}(A), \sigma_{pr}(B)\} = H_{\sigma_{pr}(A)}(\sigma_{pr}(B)) \qquad (8)$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket, and H_a the Hamiltonian vector field associated with the function a.

- Therefore, the Hamilton flow of the Hamiltonian $|\xi|_g^2$, a.k.a. **geodesic flow**, will naturally arise in the study of the semiclasical Laplacian on Riemannian manifolds.

Local Weyl Law

Quantum chaos aims at studying the behaviour of individual eigenfunctions. But before we do this it is useful to have some results about average behaviour of eigenfunctions. In particular an important result for us is the local Weyl law (LWL). Our setting is now that (M, g) is a closed Riemannian manifold, and $u_j = u_{h_j}$ are the normalized eigenfunctions of the Laplacian Δ, i.e. satisfying $(h_j^2\Delta - 1)u_j = 0$. We start with the following result:

Lemma

Suppose that A_h is a PDO of order $(k, 0)$ where $k < -n$. Then A_h is trace class, and

$$\operatorname{tr} A_h = (2\pi h)^{-n} \int_{T^*M} \sigma_{pr}(A)(x,\xi)\, dx\, d\xi + O(h^{1-n}),\ h \to 0.$$

See for example Dimassi-Sjöstrand.

Next, using this we show that we have an asymptotic formula for the sum of expectation values $\langle A_{h_j} u_j, u_j \rangle$. Let $\chi(t)$ be a smooth function equal to 1 for $t \leq 1$ and 0 for $t \geq 2$. Then we have

Theorem (Local Weyl Law)

Let A_h be a semiclassical pseudo of semiclassical order 0. Then

$$\sum_{j=1}^{\infty} \langle A_{h_j} u_j, u_j \rangle \chi\left(\frac{h}{h_j}\right) = (2\pi h)^{-n} \int_{S^* M} \sigma_{pr}(A) \int_0^{\infty} t^{n-1} \chi(t)\, dt + O(h^{1-n}) \tag{9}$$

and

$$\lim_{j \to \infty} N(h)^{-1} \sum_{h_j \geq h} \langle A_{h_j} u_j, u_j \rangle = |S^* M|^{-1} \int_{S^* M} \sigma_{pr}(A) \tag{10}$$

where $N(h)$ is the number of j such that $h_j \geq h$ (eigenvalue counting function).

Proof: we only prove (9); (10) follows from this by taking a sequence of χ approximating the characteristic function of $[0, 1]$. We first claim that the result essentially only dependent of the symbol of A_h inside $\{|\xi|_g \le 2\}$. This follows from the proof of the support theorem for semiclassical measures, below.
Next, by density of polynomials in continuous functions on $\{|\xi|_g \le 2\}$, it suffices to prove this for polynomial symbols, i.e. for differential operators A_h.
By linearity it suffices to treat monomials. So suppose that $Q_h = a(x)(hD_x)^\alpha$ is a monomial, with $|\alpha| = k$. Then with $\tilde{\chi}(t^2) = \chi(t)$, we have

$$\langle Q_{h_j} u_j, u_j \rangle \chi(\frac{h}{h_j}) = (\frac{h}{h_j})^{-k} \langle Q_h u_j, \tilde{\chi}(h^2\Delta) u_j \rangle$$
$$= \langle (h^2\Delta)^{-k/2} Q_h u_j, \tilde{\chi}(h^2\Delta) u_j \rangle.$$

So

$$\sum_{j=1}^{\infty} \langle Q_{h_j} u_j, u_j \rangle \chi(\frac{h}{h_j}) = \operatorname{tr} \tilde{\chi}(h^2\Delta)(h^2\Delta)^{-k/2} Q_h.$$

Using our theorem on traces this is

$$(2\pi h)^{-n} \int_{T^*M} |\xi|_g^{-k} q(x,\xi)\chi(|\xi|_g)\, dx\, d\xi + O(h^{1-n})$$

and since q is homogeneous of degree k, this is equal to

$$(2\pi h)^{-n} \int_{S^*M} q(x,\xi) \cdot \int_0^\infty t^{n-1}\chi(t)\, dt + O(h^{1-n}).$$

- Note that (9) implies that $N(h)$ admits the asymptotics

$$N(h) \sim (2\pi h)^{-n}|B^*M| \qquad \text{(Weyl Law)}.$$

Further work provides an error estimate $O(h^{1-n})$. This shows that the eigenvalues $\lambda_j = h_j^{-1}$ of $\sqrt{\Delta}$ are quite evenly distributed in intervals $[\lambda, \lambda + C]$ for large enough C. The uncertainty principle suggests that going beyond this point is rather difficult, and drastically depends on the characteristics of the geodesic flow.

Semiclassical measures

From now on M always denotes a closed manifold. We can use pseudodifferential operators on M to discuss the distribution of functions on M 'in phase space'. (Recall the Bargmann transform.) Consider a sequence of functions $u_{h_j} \in L^2(M)$, defined for at least a sequence of $h_j \downarrow 0$. We shall assume that $u_j = u_{h_j}$ is normalized: $\|u_j\|_{L^2(M)} = 1$. The usual example is that u_j is a solution of

$$(h_j^2 \Delta - 1) u_h = 0, \quad \text{i.e. } u \text{ is an eigenfunction with eigenvalue } h_j^{-2}.$$

From this sequence we can produce a measure on T^*M as follows (following Zworski's book):

- We first choose a sequence of smooth functions $a_l \in C_c^\infty(T^*M)$, dense in the Banach space $C_0(T^*M)$.
- We quantize these to semiclassical PDOs $A_l \in \Psi_h^{-\infty,0}(M)$.

- We observe that for each fixed l, the sequence $\langle A_l(h_j)u_j, u_j\rangle$ is uniformly bounded in j (using uniform boundedness of $\Psi_h^{0,0}(M)$ on $L^2(M)$). Therefore, we can extract a subsequence h_j^1 such that

$$\langle A_1(h_j^1)u(h_j^1), u(h_j^1)\rangle \to \alpha_1.$$

- Iteratively, we select a subsequence h_j^l of h_j^{l-1} such that

$$\left\langle A_l(h_j^l)u(h_j^l), u(h_j^l)\right\rangle \to \alpha_l.$$

Then using the diagonal subsequence h_j^j, we find that

$$\left\langle A_l(h_j^j)u(h_j^j), u(h_j^j)\right\rangle \to \alpha_l \text{ for all } l.$$

Now we define a map $\Psi : C_0(T^*M) \to \mathbb{C}$ by

$$\Psi(a_l) = \alpha_l.$$

Notice that Ψ is uniformly continuous; in fact,

$$\left|\Psi(a_l) - \Psi(a_m)\right| \leq \|a_l - a_m\|_{L^\infty},$$

using (5). It follows that it extends to a continuous function on $C_0(T^*M)$, which is easily checked to be linear. Consequently, by the Riesz Representation theorem, Ψ is given by a Borel measure μ on T^*M:

$$\lim_{j\to\infty} \langle A_l(h_j^j)u(h_j^j), u(h_j^j)\rangle = \mu(a_l).$$

The measure μ is called a **semiclassical measure** associated to the sequence of functions $u(h_j^j)$. Our argument shows that there exists a semiclassical measure for every sequence u_j of L^2-normalized functions; be aware that it is only rarely unique.

- In effect, the semiclassical PDO is acting as a localizer in phase space (depending on the support of the symbol).
- For an arbitrary sequence u_j, the measure μ could be trivial, or have mass < 1. (escape of mass to infinity)
- Assume $\mu(T^*M) = 1$. Replacing $\mathrm{Op}_h(a)$ by multiplication operators, we see that if a subsequence of functions u_h is associated to the semiclassical measure μ, then the spatial probability measure

$$|u_h|^2(x)\,dx \to \pi_*\mu \text{ in the weak-}* \text{ topology of measures }, \qquad (11)$$

where $\pi : T^*M \to M$ is the natural projection.

Examples:

- $u_h(x) = (\pi h)^{-n/4} e^{-|x|^2/2h}$ is associated to a unique semiclassical measure which is the delta function $\delta(x)\delta(\xi)$.
- Consider eigenfunctions on the flat torus $S^1_{2\pi} \times S^1_{2\pi}$. These take the form $(2\pi)^{-1} e^{ix\cdot k}$, where $k = (k_1, k_2)$ is a wave vector with integer entries. Given $k \neq 0$ we write $\hat{k} = k/|k|$ and $h = 1/|k|$. Then the eigenfunctions can be written

$$u_k(x) = (2\pi)^{-1} e^{ix\cdot\hat{k}/h}.$$

Here $\hat{k}$ is a unit vector in $\mathbb{R}^2$, pointing in the direction of an integer lattice point. For any unit vector ω we can find a sequence of k_j such that $\hat{k}_j \to \omega$, as $h_j = 1/|k_j| \to 0$. For such a sequence, there is an associated semiclassical measure which is

$$(2\pi)^{-2}\delta_\omega(\xi) = (2\pi)^{-2} 1(x)\delta_\omega(\xi).$$

- On S^2, the sectoral harmonics $Y^l_l(\theta, \varphi)$ are given by

$$Y^l_{\pm l}(\theta, \varphi) = c_l e^{\pm il\varphi}(\sin\theta)^l, \quad c_l \sim cl^{1/4}.$$

There is a unique semiclassical measure associated with these spherical harmonics, namely

$$1(\varphi)\delta(\theta - \pi/2)\delta(\sigma)\delta(\tau \mp 1)$$

where (σ, τ) are the dual variables to (θ, φ).
In fact, the geodesic flow on S^2 is completely integrable. For each of the two Lagrangian tori in T^*S^2 projecting to the region $\{\alpha \leq \theta \leq \pi - \alpha\}$, there is a semiclassical measure supported on that torus, associated to any sequence of spherical harmonics $Y^l_m(\theta, \varphi)$ such that $m/l \to \pm \sin\alpha$.

Now consider the case where u_h satisfies

$$\|u_h\|_{L^2(M)} = 1, \quad \|P_h u_h\|_{L^2(M)} = o(1), \quad h \to 0, \tag{12}$$

for some semiclassical PDO $P_h \in \Psi_h^{k,0}(M)$ which is elliptic for $|\xi|$ large, say $|\xi| \geq R$, that is, its principal symbol $p = \sigma_{pr}(P_h)$ satisfies

$$|p(x,\xi)| \geq C\langle\xi\rangle^k \text{ for } |\xi| \geq R.$$

For example, $P_h = h^2\Delta_g - 1$, where Δ_g is the positive Laplacian with respect to a Riemannian metric g, and u_h is an approximate eigenfunction with eigenvalue h^{-2}.
Then we have the following result:

Theorem (Support of semiclassical measure)

Let P_h be as above, and suppose that u_{h_j} is an L^2-normalized family of functions satisfying (12)*. Then any semiclassical measure μ associated to the u_{h_j} is a probability measure supported in the set $\{(x,\xi),\ p(x,\xi)=0\}$ ("classical energy shell").*

Proof.

It suffices to show that if a is a smooth, compactly supported function supported where $p \geq \epsilon > 0$, and $A_h = \mathrm{Op}_h(a)$, then $A_h u_h = o_{L^2}(1)$. Let $b = a/p$. Then b is a symbol in $S^{-\infty,0}(T^*M)$; let $B_h = \mathrm{Op}_h(b)$. Then $A_h = B_h P_h + E_h$, where $E_h \in \Psi^{-\infty,-1}(M)$ and hence $E_h = O_{L^2(M)\to L^2(M)}(h)$. Hence

$$\|A_h u_h\|_{L^2(M)} = \|B_h P_h u_h\|_{L^2(M)} + \|E_h u_h\|_{L^2(M)} = o(1),$$

using (12) and the uniform $L^2(M) \to L^2(M)$ boundedness of B_h. □

The importance of semiclassical measures is that the measure lives on phase space, which is the setting for the classical dynamics, making it easier to formulate (potential) connections between high-energy behaviour of eigenfunctions and the classical dynamics. Moreover, localization statements in phase space are much finer than localization in physical space only, and in particular imply such statements by integrating in the fibre directions (cf. (11)).

We can now reformulate the main questions of quantum chaos:

- What is the typical behaviour of semiclassical measures associated to sequences of eigenfunctions?
- Does their support spread out evenly (equidistribute) over $\{p = 0\}$, or can their support be concentrated in small regions?
- How are they related to classical dynamical properties of the Hamiltonian flow generated by p, such as complete integrability, ergodicity, or chaos?

Lecture 3

Fourier Integral Operators and WKB expansions

In the previous lecture we defined semiclassical PDOs and recalled their main properties. Here we introduce a more general class of operators, semiclassical Fourier Integral Operators or FIOs, which are similar inasmuch as they are given by oscillatory integrals, but with a more general class of phase functions.
In fact, their Schwartz kernels will take the form

$$(2\pi h)^{-k+N/2-n/2} \int_{\mathbb{R}^N} e^{i\psi(x,y,\theta)/h} a(x,y,\theta,h)\, d\theta$$

where the phase function ψ has the following property:

- The functions $\partial\psi/\partial\theta_i$ have linearly independent differentials on the set

$$C_\psi := \{(x,y,\theta) \mid d_\theta\psi(x,y,\theta) = 0\} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^N.$$

- $a(x, y, \theta, h)$ satisfies symbol estimates for some $k \in \mathbb{R}$:

$$\left| D_x^\alpha D_y^\beta D_\theta^\gamma a(x, y, \theta, h) \right| \leq C \langle \theta \rangle^{k - |\gamma|}.$$

In fact, for the purposes of these lectures, I will always assume that a is compactly supported in θ, for simplicity. Thus the condition is that a is smooth in (x, y, θ), uniformly in h.

Remark: the Schwartz kernel of an FIO is a so-called Lagrangian distribution. We will not define these in generality here, but remark that the set

$$\{(x, y, \xi, \eta) \mid \exists \theta \text{ such that } d_\theta \psi(x, y, \theta) = 0,\ d_{x,y} \psi(x, y, \theta) = (\xi, \eta)\}$$

is a Lagrangian submanifold of $T^*(\mathbb{R}^{2n}_{x,y})$.
Like PDOs, FIOs have a principal symbol which is a function on the Lagrangian (more precisely, a half-density with values in the Maslov bundle). An FIO is determined to leading order by
(i) the Lagrangian submanifold
(ii) its principal symbol.

An example of an FIO is the operator $e^{-it\sqrt{\Delta}}$ on $\mathbb{R}^n$. This can be understood as a Fourier multiplier

$$f \mapsto \mathcal{F}^{-1} e^{-it|\xi|} \mathcal{F} f.$$

It has a Schwartz kernel given by an oscillatory integral

$$(2\pi)^{-n} \int e^{i(x-y)\cdot\xi} e^{-it|\xi|}\, d\xi = (2\pi h)^{-n} \int e^{i\left((x-y)\cdot\xi - t|\xi|\right)/h}\, d\xi$$

where the second expression uses semiclassical frequency. The associated Lagrangian submanifold is

$$\{(x, y, \xi, \eta) \mid x - y = t\hat{\xi}, \eta = -\xi\}.$$

In other words, (x, ξ) is the image of $(y, -\eta)$ through the geodesic flow at time t.

More generally, consider the following situation. Suppose we have a pseudodifferential operator A_h of order $(-\infty, 0)$ on M. The solution to the Schrödinger equation

$$hD_t u(x,t,h) + A_h u(x,t,h) = 0, \quad u(x,0) = u_0 \in L^2(M)$$

is

$$u(\cdot,t,h) = U_h(t)u_0 := e^{-itA_h/h}u_0.$$

It is thus of interest to have an expression for the Schwartz kernel of the 'propagator' $e^{-itA_h/h}$. We can find a **WKB expansion** for this Schwartz kernel, from which we get that $U_h(t)$ is a semiclassical FIO.

First, we recall the stationary phase lemma.

Lemma (Stationary phase lemma)

Let Q be a nondegenerate real quadratic form on $\mathbb{R}^n$, and let $f(x) \in \mathcal{S}(\mathbb{R}^n)$. Then there is an asymptotic expansion of

$$\int_{\mathbb{R}^n} e^{iQ(x)/2h} f(x)\, dx$$

of the form

$$|\det Q|^{-1/2} e^{i\pi \operatorname{sgn} Q/4} (2\pi h)^{n/2} \sum_{j=0}^{\infty} h^j a_j, \tag{13}$$

where

$$a_j = \frac{1}{j!}\Big(\frac{-i\sum_{kl} {Q^{-1}}_{kl} D_k D_l}{2}\Big)^j f(0) = \frac{1}{j!}\Big(\frac{-i\langle D, Q^{-1} D\rangle}{2}\Big)^j f(0)$$

In particular, $a_0 = f(0)$, $a_1 = \frac{i}{2}\langle \partial, Q^{-1}\partial\rangle f(0)$.

Now we show that $U_h(t)$ is a semiclassical FIO. To do this, we write an operator equation for $U_h(t)$:

$$(hD_t + A_h)U_h(t) = 0, \quad U_h(0) = \mathrm{Id}\,. \tag{14}$$

We write (locally) a coordinate expression for $U_h(0)$:

$$(2\pi h)^{-n} \int e^{i(x-z)\cdot\eta/h}\phi_i(z)\, d\eta = (2\pi h)^{-n} \int e^{ix\cdot\eta/h} e^{-iz\cdot\eta/h}\phi_i(z)\, d\eta.$$

Here I have taken a partition of unity ϕ_i subordinate to a covering of M by coordinate charts, and decomposed $\mathrm{Id} = U_h(0) = \sum_i U_h(0)\phi_i = \sum_i \phi_i$. The term I wrote above is just the ith term in this sum (which I do not indicate in further notation).

The key to the WKB construction is the following Ansatz:

$$U_h(t)(x,z) = (2\pi h)^{-n} \int e^{i\psi(x,\eta,t)/h} e^{-iz\cdot\eta/h} \sum_{j=0}^{\infty} h^j b_j(x,\eta,t)\phi_i(z)\, d\eta.$$

That is, we allow the phase function $x \cdot \eta$ to evolve in time, as well as allowing an expansion in powers of h for the symbol. Now we apply the operator equation $(A_h + hD_t)U_t = 0$ (for simplicity we assume that the symbol of A_h is independent of h):

$$\begin{aligned} 0 = (2\pi h)^{-2n} \int e^{i(x-y)\cdot\xi/h} e^{i\psi(y,\eta,t)/h} e^{-iz\cdot\eta/h} \\ \times \sum_{j=0}^{\infty} h^j a((x+y)/2,\xi) b_j(y,\eta,t)\phi_i(z)\, dy\, d\xi\, d\eta \\ + (2\pi h)^{-n} \int e^{i\psi(x,\eta,t)/h} e^{-iz\cdot\eta/h} \sum_{j=0}^{\infty} h^j \left(\psi_t + hD_t\right) b_j(x,\eta,t)\phi_i(z)\, d\eta. \end{aligned} \tag{15}$$

We will equate the left and right hand sides of this expression pointwise in η. Thus we require that

$$\frac{1}{(2\pi h)^n}\int e^{i(x-y)\cdot\xi/h}a(\frac{x+y}{2},\xi)\, e^{i\psi(y,\eta,t)/h}\sum_{j=0}^{\infty}h^j b_j(y,\eta,t)\,dy\,d\xi$$
$$+\, e^{i\psi(x,\eta,t)/h}\sum_{j=0}^{\infty}h^j\big(\psi_t+hD_t\big)b_j(x,\eta,t)=0. \tag{16}$$

I thus want to write the first term as

$$e^{i\psi(x,\eta,t)/h}\sum_{j=0}^{\infty}h^j c_j(x,\eta,t).$$

This requires

$$(2\pi h)^n \sum_{j=0}^{\infty} h^j c_j =$$

$$\sum_{j=0}^{\infty} h^j \int e^{-i\psi(x,\eta,t)/h} e^{i(x-y)\cdot\xi/h} e^{i\psi(y,\eta,t)/h} a(\frac{x+y}{2}, \xi) b_j(y, \eta, t)\, dy\, d\xi. \tag{17}$$

We use the stationary phase lemma. The phase function here is stationary where $\xi = \partial_y \psi$ and $x = y$. To use our version of stationary phase with a quadratic phase function, we shift the integration variables to $\overline{y} = y - x$, $\overline{\xi} = \xi - \psi_x$, where $\psi_x = \partial_x \psi(x, \eta, t)$. We Taylor expand ψ to second order:

$$\psi(y, \eta, t) - \psi(x, \eta, t) = \langle \overline{y}, \psi_x \rangle + \frac{1}{2} \langle \overline{y}, \psi_{xx}\, \overline{y} \rangle + C(\overline{y}), \quad C = O(|\overline{y}|^3)\,.$$

Each term in the right hand side of (17) can be written

$$\int e^{iQ(\overline{y},\overline{\xi})/2h}\Big(e^{iC(\overline{y})/h}a(x+\overline{y}/2,\psi_x+\overline{\xi})b_j(x+\overline{y},\eta,t)\Big)\,d\overline{y}\,d\overline{\xi}\,,$$

with the quadratic phase $Q(\overline{y},\overline{\xi}) = -2\langle\overline{y},\overline{\xi}\rangle + \langle\overline{y},\psi_{xx}\overline{y}\rangle$.
The leading term in the stationary phase expansion (13) gives

$$(2\pi h)^n\, a(x,\psi_x(x,\eta,t))b_j(x,\eta,t).$$

The next term takes the form

$$(2\pi h)^n(ih)\Big(\langle\partial_{\overline{\xi}},\partial_{\overline{y}}\rangle+1/2\langle\partial_{\overline{\xi}},\psi_{xx}\partial_{\overline{\xi}}\rangle\Big)\,a(x+\overline{y}/2,\psi_x+\overline{\xi})\,b_j(x+\overline{y})\Big|_{\overline{y}=\overline{\xi}=0}$$

(note that the cubic phase $e^{iC(\overline{y})/h}$ does not contribute until $O((2\pi h)^n h^2)$).

Now we can write the equality (16) in a more amenable form:

$$e^{i\psi(x,\eta,t)/h}\Bigg(\sum_{j=0}^{\infty} h^j\Big\{a\,b_j$$

$$+ih\Big(\langle\partial_\xi a,\partial_x b_j\rangle+\frac{1}{2}\,b_j\,(\langle\partial_x,\partial_\xi\rangle+\langle\partial_\xi,\psi_{xx}\partial_\xi\rangle)a\Big)+O(h^2)(a,b_j)$$

$$+\big(\psi_t+hD_t\big)b_j\Big\}\Bigg)=0, \tag{18}$$

where the function a (resp. b_j) and its derivatives are taken at the point $(x,\psi_x(x,\eta,t))$ (resp. (x,η,t)).

We rewrite this expression as

$$
\begin{aligned}
&\sum_{j=0}^{\infty} h^j \Big(\psi_t(x,\eta,t) - a(x,\psi_x(x,\eta,t))\Big) b_j(x,\eta,t) \\
&\quad + \sum_{j=0}^{\infty} i h^{j+1} \Big(-\partial_t + \sum_k a_{\xi_k}(x,\psi_x(x,\eta,t))\partial_{x_k} \\
&\qquad + \frac{1}{2}\partial_{x_k}[a_{\xi_k}(x,\psi_x(x,\eta,t))]\Big) b_j(x,\eta,t) \\
&\qquad\quad = \sum_{j=1}^{\infty} h^{j+1} F_j(\psi, a, b_0, \dots, b_{j-1}),
\end{aligned}
\tag{19}
$$

with initial conditions at $t = 0$

$$
\psi(x,\eta,0) = x\cdot\eta, \quad b_0(x,\eta,0) = 1, \quad b_j(x,\eta,0) = 0,\ j \geq 1. \tag{20}
$$

Now we iteratively set each coefficient of h^j to zero, for $j = 0, 1, \dots$. If we look at the $j = 0$ term, we must have (due to the nonvanishing of b_0, at least for small t)

$$\psi_t(x, \eta, t) - a(x, \psi_x(x, \eta, t)) = 0.$$

This is called the **eikonal equation.** It is a first order nonlinear equation for ψ, of a type known as a **Hamilton-Jacobi equation.** For the given initial condition at $t = 0$, it has a unique solution (at least for small t).

Once the eikonal equation has been solved, the first term vanishes identically and we next have the h^1 equation:

$$\Big(-\partial_t+\sum_k a_{\xi_k}(x,\psi_x(x,\eta,t))\partial_{x_k}+\frac{1}{2}\partial_{x_k}a_{\xi_k}(x,\psi_x(x,\eta,t))\Big)b_0(x,\eta,t)=0$$

This is a linear first order **transport equation** for b_0 with known coefficients (since ψ is determined by the eikonal equation), and therefore has a unique solution given an initial condition at $t=0$. We notice that a_{ξ_k} is the x-component of the Hamiltonian vector field (taken on the Lagrangian submanifold generated by $\psi(\cdot,\eta,t)$). The last term can be interpreted by the fact that we are transporting a half-density $b_0(x)\,|dx|^{1/2}$.

Assuming inductively that the equations for $h^1,\dots,h^j$ determine $b_0,\dots,b_{j-1}$ uniquely, the h^{j+1} equation is an inhomogeneous transport equation for b_j with known inhomogeneous term (it depends only on $b_0,\dots,b_{j-1}$ and ψ).

Remark: this structure (eikonal equation, then a sequence of transport equations) is common to all WKB constructions.

Egorov theorem

Our next major result is the Egorov theorem. This tells us that the conjugation of a pseudodifferential operator by certain sorts of FIOs is another pseudodifferential operator, but with symbol obtained from the original pseudo by a symplectic transformation determined by the FIO. It is a key result enabling us to link quantum phenomena (high-energy eigenfunctions) with classical phenomena (geodesic flow).

Theorem (Egorov Theorem)

Let B_h be a semiclassical pseudo on M of order (k, m), and let $F_h = e^{-iA_h/h}$ where A_h is a pseudo of order $(1, 0)$ on M with real principal symbol. Then $C_h := F_h^{-1} B_h F_h = e^{iA_h/h} B_h e^{-iA_h/h}$ is a pseudo of order (k, m) and the principal symbol of C_h is equal to

$$\sigma_{pr}(C) = (\Phi)^* \sigma_{pr}(B) = \sigma_{pr}(B) \circ \Phi,$$

where Φ is the time-one Hamiltonian flow on T^*M generated by $\sigma_{pr}(A)$.

Proof: For simplicity we assume that $(k, m) = (0, 0)$. Let $B(t) = e^{itA_h/h} B_h e^{-itA_h/h}$. Then $B(0) = B_h$ and $B(1) = C_h$. If we differentiate in t we find that

$$B'(t) = \frac{i}{h} e^{itA/h}(AB - BA)e^{-itA/h} = \frac{i}{h}[A, B(t)].$$

So we are seeking the solution to the operator equation

$$B'(t) = \frac{i}{h}[A, B(t)], \quad B(0) = B. \tag{21}$$

Now **assume** that $B(t)$ is a pseudo, with symbol $b(t)$. By our formula for the commutator of pseudos, (21) would imply

$$b'(t) = \{a, b(t)\} = H_a(b(t)), \quad b(0) = b$$

where $\{\cdot, \cdot\}$ denotes Poisson bracket and H_a is the Hamilton vector field of a. The solution to this first order ODE is $b(t) = b \circ \Phi_t = \Phi_t^* b$ where Φ_t is the time t flow generated by H_a.

Now define $B_0(t)$ to be a quantization of $b(t)$. We have

$$\sigma_{pr}\Big(B_0'(t) - \frac{i}{h}[A, B_0(t)]\Big) = 0 \implies B_0'(t) - \frac{i}{h}[A, B_0(t)] = hE_1$$

where E_1 has semiclassical order 0. Now we try to solve away the error E_1. We try a solution of the form $B_1(t) = B_0(t) + hT_1(t)$. We want

$$B_1'(t) = \frac{i}{h}[A, B_1(t)] + h^2 E_2(t)$$

where $E_2(t)$ has semiclassical order 0. This gives an equation for the principal symbol $t_1(t)$ of $T_1(t)$:

$$t_1'(t) - H_a(t_1(t)) = e_1(t), \quad t_1(0) = 0,$$

which has a unique solution

$$t_1(t) = \int_0^t \Phi_{t-s}^* e_1(s)\, ds.$$

By iterating in this way, we find a solution to

$$\tilde{B}'(t) = \frac{i}{h}[\tilde{B}(t), A] + R$$

where R is a 'semiclassically trivial' operator, i.e. is of order $O(h^\infty)$ with a smooth kernel. The solution is then

$$B(t) = \tilde{B}(t) + \int_0^t e^{-i(t-s)A_h/h} R(s) e^{i(t-s)A_h/h} \, ds,$$

where the second term is $O(h^\infty)$ with a smooth kernel, hence a 'trivial' pseudodifferential operator of order $(-\infty, -\infty)$.

Remark: one can give an estimate of the form

Theorem

Let $A_h(t)$ and Φ_t be as above, and let $b \in S_h^{0,0}(M)$. Then we have an estimate

$$\left\| e^{itA_h/h} \operatorname{Op}_h^w(b) e^{-itA_h/h} - \operatorname{Op}_h^w(b \circ \Phi_t) \right\|_{L^2(M) \to L^2(M)} \leq C_1 \, h \, e^{C_2 t}$$

where C_1, C_2 do not depend on t.

The best constant C_2 in this estimate is often an important quantity, directly related to the dynamics on T^*M. This theorem shows that the Egorov theorem is a useful result for a time interval of length $\sim C_2^{-1} \log(1/h)$, a upper limit known as the **Ehrenfest time.**

Lecture 4

Invariance of semiclassical measures

Suppose that we have a closed Riemannian manifold M, consider the operator $P_h = h^2\Delta - 1$ and approximate eigenfunctions u_h for some sequence $h_j \downarrow 0$, satisfying

$$\|u_h\|_{L^2(M)} = 1, \quad P_h u_h = o_{L^2}(h), \quad h \to 0. \tag{22}$$

Let μ be a semiclassical measure associated to (u_h). Recall that we have already shown, just under the assumption that $P_h u_h = o_{L^2}(1)$, that μ is supported on the set $\{\sigma_{pr}(P_h) = 0\}$, that is, on the unit cosphere bundle S^*X. Under the stronger assumption (22) we have:

Theorem (Invariance of semiclassical measures)

Let P_h be as above, and suppose that u_{h_j} satisfy (22). *Then any semiclassical measure μ associated to the u_{h_j} is invariant under the Hamilton flow generated by $\sigma_{pr}(P_h)$ (i.e. geodesic flow). That is, if $\Phi_t : T^*M \to T^*M$ is the geodesic flow, then $(\Phi_t)_*\mu = \mu$.*

Proof: we prove under the assumption that the u_{h_j} are exact eigenfunctions. We need to show that

$$\langle \mu, a \rangle = \langle \mu, \Phi_t^* a \rangle \tag{23}$$

for all continuous functions a supported near S^*M. By density it is enough to show it for smooth a. Choose a quantization A_h of a; the left hand side of (23) is

$$\lim_{h_j^1 \to 0} \left\langle A(h_j^1) u(h_j^1), u(h_j^1) \right\rangle.$$

Here h_j^1 is a subsequence of the h_j leading to μ. Next we use a result mentioned in the previous lecture: $S_h = \sqrt{h^2 \Delta}$ is a semiclassical pseudo of order $(1, 0)$, and $e^{-itS_h/h} = e^{-it\sqrt{\Delta}}$; therefore, we have by Egorov's theorem

$$e^{it\sqrt{\Delta}} A e^{-it\sqrt{\Delta}} = A(t)$$

where $A(t)$ is a pseudo with principal symbol $\Phi_t^* a$.

Therefore, the RHS of (23) is

$$\lim_{h_j \to 0} \left\langle e^{it\sqrt{\Delta}} A(h_j) e^{-it\sqrt{\Delta}} u(h_j), u(h_j) \right\rangle = \lim_{h_j \to 0} \left\langle A(h_j) e^{-it\sqrt{\Delta}} u(h_j), e^{-it\sqrt{\Delta}} u(h_j) \right\rangle$$

(writing h_j from now on, for brevity). Now if $u(h_j)$ are exact eigenfunctions, we have $e^{-it\sqrt{\Delta}} u(h_j) = e^{it/h_j} u_j$, the factors e^{it/h_j} cancel, and we find that this is equal to the left hand side of (23), as desired. In fact this remains so if u_j are only $o(h_j)$-quasimodes of P_h (exercise).

Quantum ergodicity theorem

We now come to the main theorem of these lectures: the quantum ergodicity theorem (QET) of Shnirelman-Zelditch-Colin de Verdière. The setting is as above: we have a closed Riemannian manifold (M, g) with Laplacian $\Delta = \Delta_g$ and we consider the eigenfunction equation on M

$$(h^2\Delta - 1)u_h = 0,$$

for which we know there is a discrete sequence of eigenvalues h_j^{-2} and L^2-normalized eigenfunctions u_j, $j = 1, 2, \ldots$. We consider semiclassical measures associated to the u_j.
We begin with two definitions:

Definition

We say that Δ is quantum unique ergodic (QUE) if there is a unique semiclassical measure μ.

This is an extremely strong condition: it means that the full sequence of eigenfunctions equidistributes in phase space. Example: the unit circle. Eigenfunctions are $u_n = (2\pi)^{-1/2} e^{in\theta}$, for $n \in \mathbb{Z}$. Let A_h be a pseudo on the circle; we may assume without loss of generality that its kernel is supported where $|\theta - \theta'|$ is small. Let $\chi(t)$ be a function equal to 1 for $|t| < \epsilon$ and 0 for $|t| \geq 2\epsilon$. Then A_h has a representation

$$(2\pi h)^{-1} \int_{\mathbb{R}} e^{i(\theta-\theta')\xi/h} \chi(\theta - \theta') a(\theta, \xi) \, d\xi.$$

For $h = 1/n$ we can take a limit along the full sequence of eigenfunctions with $n > 0$:

$$\lim_{n \to +\infty} \langle A_{1/n} u_n, u_n \rangle =$$

$$\frac{1}{2\pi h} \int_{S^1 \times S^1 \times \mathbb{R}} e^{in(\theta-\theta')\xi} \chi(\theta - \theta') a(\theta, \xi) \frac{e^{in\theta'} e^{-in\theta}}{2\pi} \, d\xi \, d\theta \, d\theta'.$$

The phase function is quadratic in (θ', ξ); applying stationary phase, we find that the limit is equal to

$$\frac{1}{2\pi} \int_{S^1} a(\theta, 1)\, d\theta.$$

Similarly, taking the limit $n \to -\infty$, we get the limit

$$\frac{1}{2\pi} \int_{S^1} a(\theta, -1)\, d\theta.$$

Thus in this case there are two semiclassical measures, the uniform distribution on the sets $\{\xi = \pm 1\}$, so the unit circle is not QUE (though it almost is!)
Exercise: show that the Laplacian on the interval $[0, 2\pi]$, with Dirichlet boundary conditions, is QUE.

Rudnick-Sarnak (1995) conjectured that compact hyperbolic manifolds are QUE. This has been verified in the case of arithmetic surfaces, but is otherwise wide open.

Since the QUE condition is rarely satisfied, we introduce a weaker notion.

Definition

We say that Δ is quantum ergodic (QE) if there exists a density one subset J of natural numbers of density one such that the subsequence $(u_j) : j \in J$ has a unique semiclassical measure.

Here, to say J has density one means that

$$\lim_{N\to\infty} \frac{\# J \cap \{1, 2, \ldots, N\}}{N} = 1.$$

That is, J fails to contain only a negligible proportion of the natural numbers.

Remark: if the Laplacian on (M, g) is either QE or QUE, then the distinguished semiclassical measure is necessarily Liouville measure on S^*M. This follows from the Local Weyl Law (10).

Finally, we come to the main theorem of these lectures:

Theorem (Quantum Ergodicity Theorem (QET))

Suppose that the geodesic flow on S^*M is ergodic. Then Δ is quantum ergodic.

Proof of the QE theorem:
Step 1. It suffices to show that, for each pseudo A_h of order $(0,0)$ with symbol supported in $\{|\xi|_g \leq 2\}$, we have

$$\lim_{j\to\infty} N(h)^{-1} \sum_{h_j \geq h} \left| \langle A_{h_j} u_j, u_j \rangle - \omega(A_h) \right|^2 = 0 \qquad (24)$$

where

$$\omega(A_h) := |S^*M|^{-1} \int_{S^*M} \sigma_{pr}(A) = \mu_L(\sigma_{pr}(A))$$

is the action of Liouville measure μ_L on the symbol of A. Using this it is not hard to show that for each A_h there is a density one subset J of natural numbers **depending on** A_h for which we have

$$\lim_{j\in J} \langle A_{h_j} u_j, u_j \rangle \to \omega(A).$$

A density $+$ diagonal argument, shows that we can find a density one subsequence that works for all A_h simultaneously.

Step 2. We write $U(t) = e^{-it\sqrt{\Delta}}$ and define

$$\langle A_h \rangle_T = \frac{1}{2T} \int_{-T}^{T} U(-t) A_h U(t)\, dt.$$

Notice that for every $T \in \mathbb{R}$,

$$\big\langle \langle A_{h_j} \rangle_T u_j, u_j \big\rangle = \big\langle A_{h_j} u_j, u_j \big\rangle.$$

Therefore we can write

$$\limsup_{j \to \infty} N(h)^{-1} \sum_{h_j \geq h} \Big| \big\langle A_{h_j} u_j, u_j \big\rangle - \omega(A_h) \Big|^2$$
$$= \limsup_{j \to \infty} N(h)^{-1} \sum_{h_j \geq h} \Big| \Big\langle \big(\langle A_{h_j} \rangle_T - \omega(A_h) \big) u_j, u_j \Big\rangle \Big|^2.$$

Step 3. Using Cauchy-Schwarz we then bound the RHS by

$$\limsup_{j\to\infty} N(h)^{-1} \sum_{h_j \geq h} \left\langle (\langle A_{h_j} \rangle_T - \omega(A))^* (\langle A_{h_j} \rangle_T - \omega(A)) u_j, u_j \right\rangle. \quad (25)$$

Notice that the absolute values have disappeared, as the expectation value is automatically positive. We can now apply the Local Weyl Law, which tells us that the limit of (25) exists and is equal to

$$\int_{S^*M} \left| \sigma_{pr} \big(\langle A_{h_j} \rangle_T - \omega(A) \big) \right|^2. \quad (26)$$

Step 4. Next we use Egorov's theorem. This tells us that the principal symbol of $\langle A_{h_j}\rangle_T - \omega(A)$ is equal to

$$\frac{1}{2T}\int_{-T}^{T} a(\Phi_t(x,\xi)) - \omega(A)\, dt$$

where $a = \sigma_{pr}(A)$ and Φ_t is geodesic flow. Finally we use the ergodicity hypothesis, in the form that almost every trajectory is equidistributed. In particular this implies that

$$\lim_{T\to\infty} \sigma_{pr}\big(\langle A_{h_j}\rangle_T - \omega(A)\big)(x,\xi) = 0 \text{ almost everywhere.}$$

Moreover, for each T, the principal symbol of $\langle A_{h_j}\rangle_T - \omega(A)$ is bounded by the sup norm of a. The dominated convergence theorem implies that the limit of (26) is zero as $T \to \infty$, proving (24).

Extensions of the Quantum Ergodicity theorem

The quantum ergodicity theorem is the foundation of quantum chaos. For this reason it has been generalized and extended in many different settings, for example

- QE for manifolds with boundary;
- QE for boundary values of eigenfunctions;
- QER: restrictions of eigenfunctions to hypersurfaces.

Manifolds with boundary: Here the theorem works as stated, say for either Dirichlet or Neumann boundary conditions. The theorem is a little more delicate to prove on account of the complicated nature of geodesic flow near rays tangent to the boundary. As this occurs on a set of measure zero, it ends up not affecting the QE statement.

Boundary values of eigenfunctions: Here there are a couple of new features. First is the dependence on the boundary condition, and second is the fact that one gets equidistribution according to certain measures on the ball bundle $B^*(\partial M)$ at the boundary. This theorem is due to Gérard-Leichtnam, Hassell-Zelditch, Burq:

Theorem

Let (M, g) be a compact Riemannian manifold with piecewise smooth boundary and ergodic geodesic flow. Let u_j^D, resp. u_j^N denote the normalized eigenfunctions of the Laplacian on M with Dirichlet, resp. Neumann boundary conditions. Let ψ_j, resp. v_j denote the restriction of $h_j d_\nu u_j^D$, resp. u_j^N to ∂M. Then there is a density one subset J of $\mathbb{N}$ such that for every $A_h \in \Psi^{0,0}(\partial M)$,

$$\lim_{j \in J \to \infty} \langle A_{h_j} \psi_j, \psi_j \rangle = \frac{4}{|S^* M|} \int_{B^* \partial M} \sigma_{pr}(A)(y, \eta)(1 - |\eta|_g^2)^{+1/2} \, dy \, d\eta$$

$$\lim_{j \in J \to \infty} \langle A_{h_j} v_j, v_j \rangle = \frac{4}{|S^* M|} \int_{B^* \partial M} \sigma_{pr}(A)(y, \eta)(1 - |\eta|_g^2)^{-1/2} \, dy \, d\eta,$$

Restrictions of eigenfunctions to a hypersurface: The following result was proved recently by Toth and Zelditch (see also Dyatlov-Zworski):

Theorem

Let (M, g) be a compact Riemannian manifold with ergodic geodesic flow, and let $H \subset M$ be a hypersurface. Let w_j be the restriction of normalized eigenfunctions of Δ to H. If H is 'asymmetric' with respect to the geodesic flow, then there is a density one subset J of $\mathbb{N}$ such that for every $A_h \in \Psi^{0,0}(H)$,

$$\lim_{j \in J \to \infty} \langle A_{h_j} w_j, w_j \rangle = \frac{2}{|S^* M|} \int_{B^* \partial M} \sigma_{pr}(A)(y, \eta)(1 - |\eta|_g^2)^{-1/2} \, dy \, d\eta.$$

References

Good general references for semiclassical analysis include

• Dimassi-Sjöstrand, Spectral asymptotics in the semi-classical limit, volume 268 of London Mathematical Society Lecture Note Series.

• A. Martinez, An introduction to semiclassical analysis and microlocal analysis, Universitext, Springer-Verlag, 2002.

• M. Zworski, Semiclassical Analysis, to be published by the American Mathematical Society, 2012.

See also useful lecture notes

• S. Zelditch, Recent developments in mathematical quantum chaos, arXiv:0911.4312v1

• N. Anantharaman, Eigenfunctions of the Laplacian on negatively curved manifolds: a semiclassical approach, http://www.math.u-psud.fr/ anantharaman/Clay-notes.pdf