Strings in $AdS_5 \times S^5$ and AdS/CFT

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- Review
- Spinning string in AdS_5
- Recent work: large spin and small spin limits
  long and short strings/operators
- Structure of $AdS_5 \times S^5$ superstring

AdS/CFT

$\mathcal{N} = 4$ SYM at $N = \infty$

dual to type IIB superstrings in $AdS_5 \times S^5$

$\lambda = g^2_{YM} N$ related to string tension

$2\pi T = \frac{R^2}{\alpha'} = \sqrt{\lambda}$

$g_s = \frac{\lambda}{4\pi N} \to 0$

String theory: beyond BPS states and
"supergravity + classical probes" approximation

Problems:

- spectrum of states (exact energies in λ)
- construction of vertex opeartors (closed and open string ones)
- computation of their correlation functions (graviton scattering, application to DIS in QCD ?)
- expectation values of various Wilson loops
- gluon scattering amplitudes
- generalizations to simplest less supersymmetric cases

  – orbifolds, exactly marginal deformations, ...
- strings at finite temperature in $AdS_5 \times S^5$ (without black hole and with it ...)
- solution of type 0 theory in $AdS_5 \times S^5$ ...
- non-critical superstrings: $AdS_5 \times S^1$, ...

$AdS_5 \times S^5$

Recent remarkable progress in quantitative understanding
interpolation from weak to strong ‘t Hooft coupling
based on/checked by perturbative gauge theory (4-loop in λ)
and perturbative string theory (2-loop in $\frac{1}{\sqrt{\lambda}}$) “data”
and assumption of exact integrability
string energies = dimensions of gauge-invariant operators

$$E(\sqrt{\lambda}, J, m, ...) = \Delta(\lambda, J, m, ...)$$

J - charges of $SO(2,4) \times SO(6)$: $S_1, S_2;\ J_1, J_2, J_3$
m - windings, folds, cusps, oscillation numbers, ...
Operators: $\mathrm{Tr}(\Phi_1^{J_1} \Phi_2^{J_2} \Phi_3^{J_3} D_+^{S_1} D_\perp^{S_2} ... F_{mn} ... \Psi ...)$

Solve susy 4-d CFT = Solve string in R-R background:
compute $E = \Delta$ for any λ (and J,m)

Perturbative expansions are opposite:

$\lambda \gg 1$ in perturbative string theory

$\lambda \ll 1$ in perturbative planar gauge theory

use perturbative results on both sides

and other properties (integrability, susy,...)

to come up with an exact answer – Bethe ansatz

Last 6 years: remarkable progress:

"semiclassical" string states with large quantum numbers

dual to "long" gauge operators (BMN, GKP, ...)

$E = \Delta$ – same dependence on $J, m, ...$

coefficients = interpolating functions of λ

SYM: dilatation operator that determines Δ

is same as an integrable spin chain Hamiltonian

integrability at both perturbative gauge ($\lambda \ll 1$)

and string ($\lambda \gg 1$) sides

suggests Bethe ansatz for the spectrum at any λ

But only asymptotic BA known:
restriction to very long operators –
strings with large quantum numbers (e.g. fast-moving, large spin)
Dimensions of “short” operators –
energies of small/slow quantum strings?

Problems:
1. Solve string theory in $AdS_5 \times S^5$ on an infinite line
$\rightarrow$ determine the magnon (BMN excitation) scattering S-matrix
$\rightarrow$ derive BA for the spectrum with BHL/BES phase

2. Generalize to finite-energy closed strings on S^1:
$\rightarrow$ TBA as for standard sigma models ?

Importance of better understanding the structure
of $AdS_5 \times S^5$ GS string theory

String Theory in $AdS_5 \times S^5$

bosonic coset $\frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

generalized to supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ (Metsaev, AT 98)

$$S = T \int d^2\sigma \Big[G_{mn}(x)\partial x^m \partial x^n + \bar{\theta}(D + F_5)\theta\partial x + \bar{\theta}\theta\bar{\theta}\theta\partial x \partial x + ... \Big]$$

tension $T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$

Conformal invariance: $\beta_{mn} = R_{mn} - (F_5)^2_{mn} = 0$

Classical integrability of coset σ-model (Luscher-Pohlmeyer 76)

also for $AdS_5 \times S^5$ superstring (Bena, Polchinski, Roiban 02)

Progress in understanding of implications of (semi)classical integrability (Kazakov, Marshakov, Minahan, Zarembo 04; Beisert et al 05; Dorey, Vicedo 06,...)

Computation of 1-loop quantum superstring corrections to classical string energies (Frolov, AT 02-4, ...)
results were used as input for 1-loop term
in strong-coupling expansion of the phase θ in BA
(Beisert, AT05; Hernandez, Lopez 06)

Tree-level S-matrix of BMN states from $AdS_5 \times S^5$ GS string agrees with limit of elementary magnon S-matrix
(Klose, McLoughlin, Roiban, Zarembo 06)

Semiclassical S-matrix in different limits:
string solitons on an infinite line – Giant magnons
(Hofman, Maldacena 06; Dorey 06, ...)

Last year:

2-loop string corrections (Roiban, Tirziu, AT; Roiban, AT 07)

2-loop check of finiteness of the GS superstring;

agreement with BA

– implicit check of integrability of quantum string theory

– non-trivial confirmation of BES exact phase in BA

– comparison to strong-coupling expansion

of BES equation (Basso, Korchemsky, Kotansky 07)

should extend to higher loop level

Key example of weak-strong coupling interpolation:

Spinning string in AdS_5

Folded spinning string in flat space:

$X_1 = \epsilon \ \sin\sigma \ \cos\tau, \quad X_2 = \epsilon \ \sin\sigma \ \sin\tau$

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2$$

$$t = \epsilon\tau \ , \quad \rho = \epsilon \sin\sigma \ , \quad \phi = \tau$$

If tension $T = \frac{1}{2\pi\alpha'} \equiv \frac{\sqrt{\lambda}}{2\pi}$

energy $E = \epsilon\sqrt{\lambda}$ and spin $S = \frac{\epsilon^2}{2}\sqrt{\lambda}$ satisfy Regge relation:

$$\mathcal{E} = \lambda^{1/4}\sqrt{2S}$$

Analogous solution in AdS_5:

(de Vega, Egusquiza 96; Gubser, Klebanov, Polyakov 02)

$$ds^2 = -\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \, d\phi^2$$

in conformal gauge

$$t = \kappa\tau, \quad \phi = w\tau, \quad \rho = \rho(\sigma)$$

$$\rho'^2 = \kappa^2 \cosh^2 \rho - w^2 \sinh^2 \rho, \qquad 0 < \rho < \rho_{\rm max}$$

$$\coth \rho_{\rm max} = \frac{w}{\kappa} \equiv \sqrt{1 + \frac{1}{\epsilon^2}}$$

ϵ measures length of the string

$$\sinh \rho = \epsilon \, {\rm sn}(\kappa \epsilon^{-1} \sigma, \, -\epsilon^2)$$

periodicity in $0 \leqslant \sigma < 2\pi$

$$\kappa = \epsilon \; {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; -\epsilon^2)$$

classical energy $E_0 = \sqrt{\lambda} \mathcal{E}_0$ and spin $S = \sqrt{\lambda} \mathcal{S}$

$$\mathcal{E}_0 = \epsilon \; {}_2F_1(-\frac{1}{2}, \frac{1}{2}; 1; -\epsilon^2), \qquad \mathcal{S} = \frac{\epsilon^2 \sqrt{1 + \epsilon^2}}{2} \; {}_2F_1(\frac{1}{2}, \frac{3}{2}; 2; -\epsilon^2)$$

solve for ϵ as in flat space – get analog of Regge relation

$$\mathcal{E}_0 = \mathcal{E}_0(\mathcal{S}) \, , \qquad E_0 = \sqrt{\lambda} \, \mathcal{E}_0(\frac{S}{\sqrt{\lambda}})$$

Flat space – AdS interpolation:
$\mathcal{E}_0 \sim \sqrt{\mathcal{S}}$ at $\mathcal{S} \ll 1$, $\mathcal{E}_0 \sim \mathcal{S}$ at $\mathcal{S} \gg 1$

Novel AdS "Long string" limit: $\epsilon \gg 1$, i.e. $\mathcal{S} \gg 1$

$$\mathcal{E}_0 = \mathcal{S} + \frac{1}{\pi} \ln \mathcal{S} + \ldots$$

$\mathcal{S} \to \infty$: ends of string reach the boundary ($\rho = \infty$)
$E = S$ from massless end points at AdS boundary (null geodesic)
$E - S = \frac{\sqrt{\lambda}}{\pi} \ln S$ from tension/stretching of the string
[Open string connection: Wilson loop interpretaion (Kruczenski)
Same world surface as for open string ending on null cusp
at boundary of Poincare patch (Kruczenski, Roiban, Tirziu, AT)]

$\mathcal{S} \to \infty$: solution drastically simplifies

$$t = \kappa\tau, \quad \phi \approx \kappa\tau, \quad \rho \approx \kappa\sigma\ , \quad \kappa \sim \epsilon \sim \ln \mathcal{S} \to \infty$$

string length is infinite, $R \times R$ effective world sheet

Can now compute quantum superstring corrections to E
remarkably, they respect the $S + \ln S$ structure:
string solution is homogeneous $\rightarrow$ const coeffs
in fluctuation Lagrangian, $\kappa \sim \ln \mathcal{S} \rightarrow \infty$ is "volume"

Semiclassical string theory limit

$$1.\ \lambda \gg 1\ , \qquad \mathcal{S} = \frac{S}{\sqrt{\lambda}} = \text{fixed}, \qquad 2.\ \mathcal{S} \gg 1$$

$$E = S + f(\lambda) \ln S + \dots\ ,$$

$$f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi} [1 + \frac{a_1}{\sqrt{\lambda}} + \frac{a_2}{(\sqrt{\lambda})^2} + \dots]$$

a_n–Feynmann graphs of 2d CFT – $AdS_5 \times S^5$ superstring
$a_1 = -3 \ln 2$: Frolov, AT 02
$a_2 = -K$: Roiban, AT 07
$K = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ = 0.915 from 2-loop σ-model integrals

Gauge theory: dual operators – minimal twist ones

$\mathrm{Tr}(\Phi D_+^S \Phi)$, $\quad \Delta - S - 2 = O(\lambda)$

Remarkably, same $\ln S$ asymptotics of anomalous dimensions on gauge theory side [symmetry argument: Alday, Maldacena]

Perturbative gauge theory limit:

$$1.\ \lambda \ll 1\,, \quad S = \text{fixed}, \quad 2.\ S \gg 1$$

$$\Delta - S - 2 = f(\lambda) \ln S + ...$$

$$f(\lambda \ll 1) = c_1 \lambda + c_2 \lambda^2 + c_3 \lambda^3 + c_4 \lambda^4 + ...$$

$$= \frac{1}{2\pi^2} [\lambda - \frac{\lambda^2}{48} + \frac{11\lambda^3}{2^8 \times 45} - (\frac{73}{630} + \frac{4(\zeta(3))^2}{\pi^6}) \frac{\lambda^4}{2^7} + ...]$$

c_n are given by Feynmann graphs of 4d CFT – N=4 SYM

c_3: Kotikov, Lipatov, et al 03;

c_4: Bern, Czakon, Dixon, Kosower, Smirnov 06;
Cachazo, Spradlin, Volovich

The two limits are formally different
but for leading $\ln S$ term that does not appear to matter $\rightarrow$
single $f(\lambda)$ provides smooth interpolation
from weak to strong coupling

remarkably, both expansions are reproduced from one
Beisert-Eden-Staudacher integral equation for $f(\lambda)$
[strong coupling expansion: numerical – Benna et al; Alday et al;
analytic – Basso, Korchemsky, Kotansky 07;
Kostov, Serban, Volin 08]
exact expression for $f(\lambda)$ from BES equation?

Generalization: add momentum J in S^5

related to sl(2) sector of gauge theory: $\mathrm{Tr}(D_+^S \Phi^J)$

string side: $\lambda \gg 1,\ \mathcal{S} = \frac{S}{\sqrt{\lambda}} \gg 1,\ \mathcal{J} = \frac{J}{\sqrt{\lambda}} \gg 1,$

$\ell \equiv \frac{\mathcal{J}}{\ln \mathcal{S}}$=fixed $\ll 1$

$$E = S + [f(\lambda) + h(\ell, \lambda)] \ln S + ...$$

limit on gauge theory side: $\lambda \ll 1$, S, J=fixed,

then $S \gg 1$, $j = \frac{J}{\ln S}$=fixed $\ll 1$

[Frolov, AT 02; Beisert, Frolov, Staudacher, AT 03;

Belitsky, Gorsky, Korchemsky 06; Frolov, Tirziu, AT 06;

Alday, Maldacena 07; Roiban, AT 07;

Freyhult, Rej, Staudacher 07;

Gromov 08; Basso, Korchemsky 08; Fioravanti et al, ...]

structure of h as function of $\ell = \frac{j}{\sqrt{\lambda}}$ or j

is different at weak and strong coupling:

string-theory and gauge-theory limits are different – need to resum

Connection to $O(6)$ sigma model (AM; BK)

Another extension: spiky strings vs higher twist operators
(Belitsky, Gorsky, Korchemsky 03; Kruczenski 04)

large S, large number of spikes limit:
all string near the boundary, pp-wave in AdS and at the boundary
ends ofstring move along null lines at the boundary:
Penrose limit $R \times S^3 \ \to\ $ 4d pp-wave
cusp anomaly as pp-wave anomaly (Kruczenski, AT 08)
$P_+ = \frac{1}{4} f(\lambda) \ln P_-$

Subleading terms in large S expansion

(Park, AT 05, unpublished; Forini, AT, to appear)

string has large but finite length: does not reach boundary

no longer expect connection to open strings –

to gauge theory amplitudes/Wilson lines

Classical string solution gives:

$E_0 = \sqrt{\lambda}\, \mathcal{E}(\mathcal{S})$: expand in large $\mathcal{S}$

$$\mathcal{E}_0(\mathcal{S} \gg 1) = S + c_0 \ln \mathcal{S} + c_1 + \frac{1}{\mathcal{S}}(c_2 \ln \mathcal{S} + c_3)$$

$$+\frac{1}{\mathcal{S}^2}(c_4 \ln^2 \mathcal{S} + c_5 \ln \mathcal{S} + c_6) + O(\frac{\ln^3 \mathcal{S}}{\mathcal{S}^3})$$

$c_0 = \frac{1}{\pi}$, $c_1 = \frac{1}{\pi} \ln(8\pi) - 1$, ...

$$E(\mathcal{S} \gg 1) = S + f_0 \ln(aS) + \frac{1}{S}[f_{11} \ln(aS) + f_{10}]$$

$$+\frac{1}{S^2}[f_{22} \ln^2(aS) + f_{21} \ln(aS) + f_{20}] + O(\frac{\ln^3 S}{S^3})$$

$f_0 = \frac{\sqrt{\lambda}}{\pi}$, $a = \frac{8\pi}{e\sqrt{\lambda}}$, $f_{11} = \frac{\lambda}{2\pi^2}$, $f_{10} = 0$,

$f_{22} = -\frac{\lambda^{3/2}}{8\pi^3}$, $f_{21} = \frac{5\,\lambda^{3/2}}{16\pi^3}$, $f_{20} = \frac{\lambda^{3/2}}{8\pi^3}$,

expect same structure including string loop corrections –

coefficients will have $\frac{1}{(\sqrt{\lambda})^n}$ corrections

$\rightarrow$ new “interpolating functions” like $f(\lambda)$?

Surprisingly, large spin expansion on the gauge side

has same structure in the “gauge-theory” limit:

$\lambda \ll 1, \quad S$=fixed and then $S \gg 1$

look at known anomalous dimensions of (scalar, vector, spinor)

operators with small $J = 2, 3$ twist

$\Delta(S)$ given in terms of harmonic sums (...,Kotikov et al, ...)

$\Delta - S - J = \gamma_1 \lambda + \gamma_2 \lambda^2 + \gamma_3 \lambda^3 + ...$
$\gamma_1 = \frac{1}{2\pi^2} \mathrm{S}_1$
$\gamma_2 = -\frac{1}{16\pi^4} \left[\mathrm{S}_3 + \mathrm{S}_{-3} - 2\mathrm{S}_{-2,1} + 2\mathrm{S}_1 \left(\mathrm{S}_2 + \mathrm{S}_{-2} \right) \right]$
$\gamma_3 = -\frac{1}{64\pi^6} \left[2\mathrm{S}_{-3}\mathrm{S}_2 - \mathrm{S}_5 - 2\mathrm{S}_{-2}\mathrm{S}_3 - 3\mathrm{S}_{-5} + 24\mathrm{S}_{-2,1,1,1} \right.$
$+6 \left(\mathrm{S}_{-4,1} + \mathrm{S}_{-3,2} + \mathrm{S}_{-2,3} \right) - 12 \left(\mathrm{S}_{-3,1,1} + \mathrm{S}_{-2,1,2} + \mathrm{S}_{-2,2,1} \right)$
$-\left(\mathrm{S}_2 + 2\mathrm{S}_1^2 \right) \left(3\mathrm{S}_{-3} + \mathrm{S}_3 - 2\mathrm{S}_{-2,1} \right) - \mathrm{S}_1 \left(8\mathrm{S}_{-4} + \mathrm{S}_{-2}^2 \right]$
$+4\mathrm{S}_2\mathrm{S}_{-2} + 2\mathrm{S}_2^2 + 3\mathrm{S}_4 - 12\mathrm{S}_{-3,1} - 10\mathrm{S}_{-2,2} + 16\mathrm{S}_{-2,1,1} \Big)$
$\mathrm{S}_n(S) \equiv \sum_{k=1}^{S} \frac{1}{k^n}, \quad \mathrm{S}_{n,m}(S) \equiv \sum_{k=1}^{S} \frac{1}{k^n} \mathrm{S}_m(k) \; , ...$
all harmonic sums evaluated at argument S
twist two operators in the same supermultiplet
$\Delta_{scalar} = \Delta(S), \quad \Delta_{spinor} = \Delta(S+1), \quad \Delta_{vector} = \Delta(S+2)$
expand at large S

$$\Delta - S - J = S + f_0 \ln(aS) + \frac{1}{S}[f_{11} \ln(aS) + f_{10}]$$

$$+\frac{1}{S^2}[f_{22} \ln^2(aS) + f_{21} \ln(aS) + f_{20}] + O(\frac{\ln^3 S}{S^3})$$

coeffs of $\frac{\ln^k S}{S^k}$ appear to be universal (in flavor and twist)

$$f_0 = \frac{1}{2\pi^2}[\lambda - \frac{1}{48}\lambda^2 + \frac{11}{11520}\lambda^3 - (\frac{73}{1290240} + \frac{\zeta_3^2}{512\pi^6})\lambda^4]$$
$$f_{11} = \frac{1}{8\pi^4}[\lambda^2 - \frac{1}{24}\lambda^3 + \frac{3}{1280}\lambda^4]$$
$$f_{22} = \frac{1}{64\pi^6}[-\lambda^3 + \frac{1}{16}\lambda^4]\,, \quad f_{33} = \frac{1}{384\pi^8}\lambda^4$$

same for twist 3 up to 4 loops in sl(2) sector (using Beccaria 07)

For twist two scalar operators Tr$(\Phi D_+^S \Phi)$

$$a = \frac{e^{\gamma_E}}{16\pi^2}[1 + 3\zeta_3\lambda + (\frac{\zeta_3}{48} - \frac{5\zeta_5}{4\pi^2})\lambda^2 - (\frac{7\zeta_3}{11520} + \frac{5\zeta_5}{384\pi^2} - \frac{175\zeta_7}{256\pi^4})\lambda^3 + (\frac{59\zeta_3}{430080} + \frac{3\zeta_3^3}{512\pi^6} + \frac{17\zeta_5}{18432\pi^2} + \frac{175\zeta_7}{12288\pi^4})\lambda^4]$$
$$f_{10} = \frac{1}{4\pi^2}[\lambda - \frac{1}{48}\lambda^2 + \frac{11}{11520}\lambda^3 + (-\frac{73}{1290240} - \frac{\zeta_3^2}{512\pi^6})\lambda^4]$$
$$f_{21} = \frac{1}{16\pi^4}[-\lambda^2 + (\frac{1}{48} + \frac{1}{2\pi^2})\lambda^3 - (\frac{1}{1920} + \frac{1}{32\pi^2} - \frac{\zeta_3}{64\pi^4})\lambda^4]$$
$$f_{20} = \frac{1}{24\pi}[-\lambda + (\frac{1}{48} + \frac{9}{4\pi^2})\lambda^2 - (\frac{11}{11520} + \frac{1}{16\pi^2})\lambda^3 + (\frac{73}{1290240} + \frac{17}{5120\pi^2} + \frac{\zeta_3^2}{512\pi^6})\lambda^4]$$

Connection to gauge theory amplitudes?
only for first two coefficients ?
$E - S = f_0(\lambda) \ln S + b(\lambda) + O(\frac{\ln S}{S})$, $\qquad b = f_0 \ln a$
(Dixon, Magnea, Sterman 07)

sl(2) sector:
(S, J) states, new limit $S \gg 1, \ \mathcal{J} \gg 1$
above expansion applies if

$$\frac{\mathcal{J}^2}{\ln \mathcal{S}} \ll \frac{\ln \mathcal{S}}{\mathcal{S}}$$

Expansion in $\frac{1}{S}$ cannot be convergent:
behavior at small $\mathcal{S}$ is different
generically $E(S, \lambda)$ non-trivial function of both arguments
nontrivial orders of limits; need resummation

Short string or small $\mathcal{S}$ limit of folded spinning string

(Tirziu, AT, to appear)

Spinning folded string at the center of AdS_5:

slowly rotating string = short string ≈ as in flat space

expansion in small $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$ – near flat space expansion

Motivation: understand strong-coupling coupling limit of Δ

for short operators – beyond asymptotic BA on string side

classical string solution: $\epsilon \to 0$ expansion

$\rho = \epsilon \sin \sigma \, [1 - \frac{\epsilon^2}{24}(5 + \cos 2\sigma) + O(\epsilon^3)]$

$\rho_{\rm max} = \epsilon - \frac{1}{6}\epsilon^3 + O(\epsilon^3)$

$\epsilon = \sqrt{2\mathcal{S}} \, [1 - \frac{1}{8}\mathcal{S} + O(\mathcal{S}^2)]$

$\mathcal{E}_0 = \sqrt{2\mathcal{S}} \, [1 + \frac{3}{8}\mathcal{S} + O(\mathcal{S}^2)]$

Quantum string ($\frac{1}{\sqrt{\lambda}}$) corrections respect this structure

in string limit: $\lambda \gg 1, \ \ \frac{S}{\sqrt{\lambda}} = \text{fixed} \ll 1$

$$E(S,\lambda) = \lambda^{1/4}\sqrt{2S}\ \Big[h_0(\lambda) + h_1(\lambda) S + h_2(\lambda) S^2 + ...\Big] \ ,$$

$$h_n = \frac{1}{(\sqrt{\lambda})^n}\Big(a_{n0} + \frac{a_{n1}}{\sqrt{\lambda}} + \frac{a_{n2}}{(\sqrt{\lambda})^2} + ...\Big)$$

classical string: $a_{00} = 1 \ , \ \ a_{10} = \frac{3}{8} \ , \quad a_{20} = -\frac{21}{128} \ , ...$

1-loop superstring:

$$a_{01} = 1 \ , \qquad a_{11} = -\frac{\zeta(3)}{2} + \frac{13}{64} \ , \ \ ...$$

UV finiteness of $AdS_5 \times S^5$ superstring implies

$$h_0(\lambda) = 1$$

(cf. non-renormalization of Regge $\sqrt{2S}$ term in flat space case)

$$E(S,\lambda) = \lambda^{1/4}\sqrt{2S}\ \Big[1 + (\frac{3}{8} - \frac{32\zeta(3) - 13}{64\sqrt{\lambda}} + ...)\frac{S}{\sqrt{\lambda}} + O(S^2)\Big]$$

Here one cannot interpolate to weak λ coupling for fixed small S
Indeed, $\Delta(S,\lambda)$ for $\mathrm{Tr}(\Phi D_+^S \Phi)$ computed for
$\lambda \ll 1$ and S=fixed and then formally expanded in $S \to 0$

$$E(\lambda, S) = q_1(\lambda) S + q_2(\lambda) S^2 + O(S^3)$$

$$q_0(\lambda) = 1 + d_{01}\lambda + d_{02}\lambda^2 + ... , \quad q_2(\lambda) = d_{21}\lambda + d_{22}\lambda^2 + ...$$

Both series in S in string and in gauge theory should not converge:
formal sum $E(S,\lambda)$ at large S should give $\ln S$ behaviour
To relate "small spin" string theory and gauge theory expansions
would need to sum up the gauge expansion and then
re-expand the result first in large λ for fixed $\frac{S}{\sqrt{\lambda}}$
and then in small $\frac{S}{\sqrt{\lambda}}$

$\zeta(3)$ also in dimensions of short operators at weak coupling...

Some details:

Expanding $AdS_5 \times S^5$ superstring action in conformal gauge to quadratic order in fluctuations

$$
\begin{aligned}
\widetilde{L}_B &= -\partial_a \widetilde{t} \partial^a \widetilde{t} - \mu_t^2 \widetilde{t}^2 + \partial_a \widetilde{\phi} \partial^a \widetilde{\phi} + \mu_\phi^2 \widetilde{\phi}^2 \\
&+ 4\widetilde{\rho}(\kappa \sinh\rho\, \partial_0 \widetilde{t} - w \cosh\rho\, \partial_0 \widetilde{\phi}) + \partial_a \widetilde{\rho} \partial^a \widetilde{\rho} + \mu_\rho^2 \widetilde{\rho}^2 \\
&+ \partial_a \beta_u \partial^a \beta_u + \mu_\beta^2 \beta_u^2 + \partial_a \varphi \partial^a \varphi + \partial_a \chi_s \partial^a \chi_s
\end{aligned}
$$

$$
\mu_t^2 = 2\rho'^2 - \kappa^2, \quad \mu_\phi^2 = 2\rho'^2 - w^2, \quad \mu_\rho^2 = 2\rho'^2 - w^2 - \kappa^2, \quad \mu_\beta^2 = 2\rho'^2
$$

β_u $(u = 1, 2)$ AdS_5 fluctuations transverse to AdS_3

φ, χ_s $(s = 1, 2, 3, 4)$ fluctuations in S^5

Fermionic part of quadratic fluctuation Lagrangian

$$
\widetilde{L}_F = 2i(\bar{\Psi}\gamma^a \partial_a \Psi - \mu_F \bar{\Psi}\Gamma_{234}\Psi) \,, \qquad \mu_F^2 = \rho'^2
$$

same as 4+4 2d Majorana fermions with σ-dependent mass μ_F

expand the coefficients in $\epsilon \to 0$

$\mu_t^2 = \epsilon^2 \cos 2\sigma + ...,\qquad \mu_\phi^2 = -1 + \epsilon^2(\cos 2\sigma + \frac{1}{2}) + ...$

$\mu_\rho^2 = -1 + \epsilon^2(\cos 2\sigma - \frac{1}{2}) + ...,\qquad \mu_\beta^2 = 2\mu_F^2 = 2\epsilon^2 \cos^2 \sigma + ...$

$\epsilon = 0$ is flat-space case: free action after a rotation

Leading ϵ^2 part of 1-loop correction to string energy vanishes

1-loop correction to string energy

$E_1 = \frac{\Gamma_1}{\kappa \mathcal{T}}, \quad \mathcal{T} \equiv \int d\tau \to \infty, \quad \kappa = \epsilon + ...$

$$E_1 = \frac{1}{2} \sum_i (-1)^{n_i} \ln \frac{\det[\partial_0^2 - \partial_1^2 + \epsilon^2 M_i^2]}{\det[\partial_0^2 - \partial_1^2]}$$

$$\sim \epsilon^2 \int d\tau \int_0^{2\pi} d\sigma \, \mathrm{Tr} \sum_i (-1)^{n_i} M_i^2 + O(\epsilon^4)$$

1-loop UV divergencies in the $AdS_5 \times S^5$ superstring action expanded near arbitrary string solution cancel (Frolov, AT 02)

mass sum rule $\to$ no ϵ^2 term in Γ_1: no $\epsilon \sim \sqrt{2\mathcal{S}}$ term in E_1

Should generalize to higher string loops

To find first non-trivial correction – 2nd order perturbation theory for determinants with σ dependent masses

$$\ln \frac{\det[A+\epsilon^2 B+\epsilon^4 C]}{\det A}$$
$$= \epsilon^2 \mathrm{Tr}[A^{-1}B] - \frac{\epsilon^4}{2}\mathrm{Tr}[A^{-1}BA^{-1}B] + \epsilon^4 \mathrm{Tr}[A^{-1}C] + ...$$

e.g.

$$\ln \frac{\det[-\partial_1^2+\omega^2+\epsilon^2\cos^2\sigma]}{\det[-\partial_1^2+\omega^2]}$$
$$\approx \epsilon^2 \sum_n \frac{1}{n^2+\omega^2}\int_0^{2\pi}\frac{d\sigma}{2\pi}\cos^2\sigma = \frac{\epsilon^2}{2}\sum_n \frac{1}{n^2+\omega^2}$$

$$\Gamma_1(\epsilon^4) = -\frac{\mathcal{T}\epsilon^4}{4\pi}\int_{-\infty}^{\infty} d\omega \Big\{ \sum_n \Big[-\frac{7}{8}\frac{1}{n^2+w^2} - \frac{1}{32}\frac{1-8i\omega}{n^2+(\omega+i)^2} + c.c. \Big]$$
$$+ \sum_n \Big[-\frac{\omega^2}{[n^2+(\omega+i)^2]^2} + \frac{1}{[n^2+w^2][(n-2)^2+\omega^2]} + \frac{1}{[n^2+(\omega+i)^2][n^2+(\omega-i)^2]}$$
$$+\omega^2\Big(\frac{1}{(n+1)^2+\omega^2} + \frac{1}{(n-1)^2+\omega^2}\Big)\Big(\frac{1}{n^2+(\omega+i)^2} + \frac{1}{n^2+(\omega-i)^2}\Big)$$
$$+\frac{(1+\frac{i\omega}{2})^2}{4}\frac{1}{n^2+(\omega-i)^2}\Big(\frac{1}{(n-2)^2+(\omega-i)^2} + \frac{1}{(n+2)^2+(\omega-i)^2}\Big)$$
$$+\frac{(1-\frac{i\omega}{2})^2}{4}\frac{1}{n^2+(\omega+i)^2}\Big(\frac{1}{(n-2)^2+(\omega+i)^2} + \frac{1}{(n+2)^2+(\omega+i)^2}\Big)\Big]\Big\}$$

$\sum_{n=3}^{\infty} \frac{n^4-7n^3+3n^2+16n-16}{n^3(n^2-4)(n-1)} = \frac{149}{32} - 4\zeta(3)$

$E_1 = \frac{\Gamma_1}{\mathcal{T}\kappa} = \frac{1}{4}\left[\frac{13}{32} - \zeta(3)\right] \epsilon^3 \ + \ O(\epsilon^5)$

$$E_1 = \frac{1}{\sqrt{2}}\Big[\frac{13}{32} - \zeta(3)\Big]\, \mathcal{S}^{3/2} + O(\mathcal{S}^{5/2})$$

Generalisation to J in S^5:

related to sl(2) sector in gauge theory
start with $(S = \sqrt{\lambda}\mathcal{S}, J = \sqrt{\lambda}\mathcal{J})$ solution
expand in short string limit: $\epsilon \ll 1$, i.e. $\rho_{\rm max} \to 0$
If $\mathcal{J} \gg 1$ – fast (BMN-like) short string limit
$\mathcal{E}_0 = \nu + \mathcal{S} + \frac{\mathcal{S}}{2\nu^2} + ... \,, \qquad \mathcal{J} \gg 1, \quad \frac{\mathcal{S}}{\mathcal{J}} \ll 1$
Instead, in slow short string limit
$\mathcal{S} \ll 1, \ \ \mathcal{J} \ll \sqrt{\mathcal{S}} \ll 1$
classical energy has "near flat space" expansion form

$$\mathcal{E}_0 = \sqrt{2\mathcal{S}} \, \big(1 + \frac{\nu^2}{4\mathcal{S}} + ...\big) + \frac{3}{4\sqrt{2}}\mathcal{S}^{3/2}\big(1 + \frac{5\nu^2}{12\mathcal{S}} + ...\big) + ...$$

Again coeff. of $\sqrt{2\mathcal{S}}$ term is not changed by quantum corrections
Interpretation on gauge theory side?

Moral: $E(\lambda, S, J)$ function of 3 arguments ...

Green-Schwarz superstring in $AdS_5 \times S^5$

Superstring in curved type II supergravity background

$\int d^2\sigma \, G_{MN}(Z)\partial Z^M \partial Z^N + \dots , \qquad Z^M = (x^m, \theta^I_\alpha)$

$m = 0, 1, ...9, \quad \alpha = 1, 2..., 16, \quad I = 1, 2$

Explicit form of action is generally hard to find

$AdS_5 \times S^5$: coset space symmetry facilitates explicit construction

Algebraic construction of unique κ-invariant action as in flat space

GS superstring in flat space:

$R^{1,9} = \frac{G}{H} = \frac{\text{Poincare}}{\text{Lorentz}}$

Flat superspace = $\frac{\widehat{G}}{H} = \frac{\text{SuperPoincare}}{\text{Lorentz}}$

structure of action is fixed by superPoincare algebra (P, M, Q)

$[P, M] \sim P, \ \ [M, M] \sim M, \ \ [M, Q] \sim Q, \ \ \{Q, Q\} \sim P$

$g^{-1}dg = J^m P_m + J^I_\alpha Q^\alpha_I + J^{mn} M_{mn}$

Supercoset action= $\int \text{Tr}(g^{-1}dg)^2_{G/H}$ + fermionic WZ-term

$I = \int d^2\sigma (J^m J^m + a\bar{J}^I J^I) + b \int \ J^m \wedge \bar{J}^I \Gamma_m J^J s_{IJ}$

$s_{IJ} = (1, -1)$

$J^m = dx^m - i\bar{\theta}^I \Gamma^m \theta^I, \qquad J^I_\alpha = d\theta^I_\alpha$

Manifest superPoincare symmetry, but

unitarity and right fermionic spectrum iff $a = 0,\ b = \pm 1$:

κ-invariance $\rightarrow$ Green-Schwarz action:

$$L = -\tfrac{1}{2}(\partial_a x^m - i\bar{\theta}^I \Gamma^m \partial_a \theta^I)^2$$

$$+\, i\epsilon^{ab} s_{IJ} \bar{\theta}^I \Gamma_m \partial_a \theta^J (\partial_b x^m - \tfrac{i}{2}\bar{\theta}^K \Gamma^m \partial_b \theta^K)$$

peculiar "degenerate" Lagrangian: no $\partial\bar{\theta}\partial\theta$ term

$L \sim \partial x \partial x + \partial x \bar{\theta} \partial \theta + (\bar{\theta}\partial\theta)^2$

perturbative expansion is well-defined

near $\bar{x}$ background, e.g., $x^m = N^m_a \sigma^a$

$x = \bar{x} + \xi, \ \ \theta' = \sqrt{\partial \bar{x}}\ \theta$

$L \sim \partial\xi\partial\xi + \bar{\theta}'\partial\theta' + \frac{1}{\sqrt{\partial\bar{x}}}\partial\xi\bar{\theta}'\partial\theta' + ...$

non-renormalizable by power counting

but κ-symmetry (uniqueness of action) implies finiteness direct check of cancellation of 2-loop logarithmic UV divergences and trivial partition function (Roiban, Tirziu, AT 07)

preservation of κ-symmetry implies that semiclassical loop (α') expansion must be finite also in curved space

but regularization issues are non-trivial starting with 2 loops

$AdS_5 \times S^5 = \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

Killing vectors and Killing spinors of $AdS_5 \times S^5$:

$PSU(2,2|4)$ symmetry

replace G/H=SuperPoincare/Lorentz in flat GS case by

$$\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$$

generators: $(P_q, M_{pq}); (P'_r, M'_{rs}); Q^I_\alpha, \quad m = (q,r)$

$$[P,P] \sim M, \;\; [P,M] \sim P, \;\; [M,M] \sim M,$$

$$[Q,P_q] \sim \gamma_q Q, \;\; [Q,M_{pq}] \sim \gamma_{pq} Q$$

$$\{Q^I, Q^J\} \sim \delta^{IJ}(\gamma \cdot P + \gamma' \cdot P') + \epsilon^{IJ}(\gamma \cdot M + \gamma' \cdot M')$$

$PSU(2,2|4)$ invariant action:

$\int \text{Tr}(g^{-1}dg)^2_{G/H}$ + WZ-term

$J = g^{-1}dg = J^m P_m + J^I_\alpha Q^\alpha_I + J^{mn} M_{mn}$

$$I = \frac{\sqrt{\lambda}}{2\pi}\Big[\int d^2\sigma (J^m J^m + a\bar{J}^I J^I) + b\int \ J^m \wedge \bar{J}^I \Gamma_m J^J s_{IJ}\Big]$$

as in flat space $a = 0, \ \ b = \pm 1$ required by κ-symmetry
unique action with right symmetry and right flat-space limit

Formal argument for UV finiteness (2d conformal invariance):

1. global symmetry – only overall coefficient
   of J^2 term (radius) can run
2. non-renormalization of WZ term (homogeneous 3-form)
3. preservation of κ-symmetry at the quantum level
   – relating coefficients of J^2 and WZ terms

Component form:

coset representative $g(x,\theta) = f(x) e^{\theta Q}$

$J^m = e^m(x) - i\bar\theta^I \Gamma^m D\theta^I + O(\theta^4), \quad J^I = D\theta^I + O(\theta^3)$

solving Maurer-Cartan eqs:

$$J_a^A \;=\; \partial_a x^m e_m^A - 4i\bar\theta^I \Gamma^A \Big[\frac{\sinh^2(\frac{s}{2}\mathcal{M})}{\mathcal{M}^2}\Big]_{IJ} D_a\theta^J \ , \qquad J_a^I = \Big[\frac{\sinh}{}$$

$$D\theta^I = \mathcal{D}\theta^I - \frac{i}{2}\epsilon^{IJ} e^A(x)\Gamma_*\Gamma_A\theta^J \ , \qquad \mathcal{D}\theta^I = d\theta^I + \frac{1}{4}\omega^{AB}(x)\Gamma_{AB}$$

$$(\mathcal{M}^2)^{IL} = -\epsilon^{IJ}\Gamma_*\Gamma^A\theta^J\bar\theta^L\Gamma_A + \frac{1}{2}\epsilon^{LK}(\Gamma^{pq}\theta^I\bar\theta^K\Gamma_{pq}\Gamma_* - \Gamma^{rs}\theta^I\bar\theta^K\Gamma$$

$e^A(x) = dx^m\, e_m^A(x), \quad A = (p, r)$

$\Gamma_* = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4, \quad \Gamma'_* = i\Gamma_5\Gamma_6\Gamma_7\Gamma_8\Gamma_9$

RR coupling: “mass term” in D

D in IIB Killing spinor eq. $D^{IJ}\epsilon^J = 0, \ [D_M, D_N] = 0$

Expansion near string soliton solution $x = \bar{x}$:
conformal gauge and κ-symmetry gauge $\theta^1 = \theta^2$
$I = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \, (L_{\rm kin} + L_{\rm WZ})$

$$L_{\rm kin} = -\frac{1}{2}\partial_a x^\mu \partial^a x^\nu G_{\mu\nu}(x) + 2i e_a^A \bar{\theta}\Gamma_A \mathcal{D}^a \theta + 2\bar{\theta}\Gamma^A \mathcal{D}_a \theta \bar{\theta}\Gamma_A \mathcal{D}$$

$$+\frac{1}{12} e_a^A e^{aB} \, \bar{\theta}\Gamma_A(\Gamma^{pq}\theta\bar{\theta}\Gamma_{pq} - \Gamma^{rs}\theta\bar{\theta}\Gamma_{rs})\Gamma_B\theta + O(\theta^6)$$

$$L_{\rm WZ} = \epsilon^{ab}\Big[- e_a^A e_b^B \bar{\theta}\Gamma_A\Gamma_*\Gamma_B\theta + \frac{4i}{3} e_a^A \bar{\theta}\Gamma_A\Gamma_*\Gamma_B\theta\bar{\theta}\Gamma^B\mathcal{D}_b\theta\Big] + O$$

Expansion: $x \to x+\xi, \quad L = \xi D^2 \xi + \bar{\theta} D \theta + \xi^3 + \xi^4 + \xi\theta^2 + \theta^4 + \ldots$

1-loop results:

- check of finiteness of GS action for generic $\bar{x}$ solution
- computation of 1-loop quantum string corrections to energies of rigid rotating string solutions (Frolov, AT 02,03; Park, AT 05)

Simple form of the $AdS_5 \times S^5$ action

special choice of coordinates (Poincar e)

and special κ-symmetry gauge: $\theta^1 = \Gamma_{0123}\theta^2$

plus "Killing spinor" redefn of fermions (Kallosh, Rajaraman 98)

$$I = \frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma \big[z^2(\partial_a x^m - i\bar{\theta}\Gamma^m\partial_a\theta)^2 + \frac{1}{z^2}\partial^a z^s \partial_a z^s + 4\epsilon^{ab}\bar{\theta}\partial_a z^s \Gamma_s \partial$$

$m = 0, 1, 2, 3;\ s = 4, ..., 9,\ \ z^2 = z^s z^s,\ a, b = 0, 1$

after formal T-duality: $x^m \to \widetilde{x}^m$

action becomes exactly quadratic in θ (Kallosh, AT 98)

$$I = \frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma \big[\frac{1}{z^2}(\partial^a x^m \partial_a x_m + \partial^a z^s \partial_a z^s) + 4\epsilon^{ab}\bar{\theta}(\partial_a x^m \Gamma_m + \partial_a z^s$$

starting point of computation of 2-loop string correction

to cusp anomalous dimension (Roiban, AT 07)

check of 2-loop finiteness of $AdS_5 \times S^5$ GS string

relation to dual superconformal symmetry

How to solve quantum string theory in $AdS_5 \times S^5$?

GS string on supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$
not of known solvable type (cf. free oscillators; WZW)
analogy with exact solution of $O(n)$ model (Zamolodchikovs) or
principal chiral model (Polyakov-Wiegmann; KWZ; ...) ?
– 2d CFT – no mass generation

Try as in flat space –
light-cone gauge: analog of $x^+ = p^+\tau,\ p^+ = \text{const},\ \Gamma^+\theta = 0$
Two natural options:
(i) null geodesic parallel to the boundary in Poincare patch –
action/Hamiltonian quartic in fermions (Metsaev, Thorn, AT, 01)
(ii) null geodesic wrapping S^5:
hidden $su(2|2) \times su(2|2)$ symmetry
but complicated action (Callan et al, 03;
Arutyunov, Frolov, Plefka, Zamaklar, 05-06)

Common problem:
lack of manifest 2d Lorentz symmetry
hard to apply known 2d integrable field theory methods –
S-matrix depends on two rapidities, not on their difference only
constraints on it are unclear, etc.

An alternative approach: “Pohlmeyer reduction”
use conf. gauge, solve Virasoro conditions in terms of currents,
find “reduced” action for physical number of d.o.f.,
use it as a starting point for quantization

Aim: construct PR version for $AdS_5 \times S^5$ superstring

(i) introduce new fields locally related to supercoset currents

(ii) solve conformal gauge (Virasoro) condition explicitly

(iii) find local 2d Lorentz-invariant

action for independent (8B+8F) d.o.f

– fermionic generalization of non-abelian Toda theory

PR: a nonlocal map that preserves integrable structure

1. gauge-equivalent Lax pairs; map between soliton solutions

gives integrable massive local field theory

2. quantum equivalence to original GS model ?

may expect for full $AdS_5 \times S^5$ string model = CFT

3. integrable theory: semiclassical solitonic spectrum

may essentially determine quantum spectrum

the two solitonic S-matrices should be closely related:

Lorentz-invariant S-matrix of PR-model should effectively

give complicated magnon S-matrix

Pohlmeyer reduction: bosonic coset models

Prototypical example: S^2-sigma model $\to$ Sine-Gordon theory

$$L = \partial_+ X^m \partial_- X^m - \Lambda(X^m X^m - 1)\,, \qquad m = 1,2,3$$

Equations of motion:

$$\partial_+\partial_- X^m + \Lambda X^m = 0\,, \quad \Lambda = \partial_+ X^m \partial_- X^m\,, \quad X^m X^m = 1$$

Stress tensor: $\mathrm{T}_{\pm\pm} = \partial_\pm X^m \partial_\pm X^m$

$$\mathrm{T}_{+-} = 0\,, \quad \partial_+ \mathrm{T}_{--} = 0\,, \qquad \partial_- \mathrm{T}_{++} = 0$$

implies $\mathrm{T}_{++} = f(\sigma_+)$, $\mathrm{T}_{--} = h(\sigma_-)$

using the conformal transformations $\sigma_\pm \to F_\pm(\sigma_\pm)$ can set

$$\partial_+ X^m \partial_+ X^m = \mu^2\,, \qquad \partial_- X^m \partial_- X^m = \mu^2\,, \qquad \mu = \mathrm{const}\ .$$

3 unit vectors in 3-dimensional Euclidean space:

$$X^m\,, \qquad X^m_+ = \mu^{-1}\partial_+ X^m\,, \qquad X^m_- = \mu^{-1}\partial_- X^m\,,$$

X^m is orthogonal ($X^m\partial_\pm X^m = 0$) to both X^m_+ and X^m_-
remaining $SO(3)$ invariant quantity is scalar product

$$\partial_+ X^m \partial_- X^m = \mu^2 \cos 2\varphi$$

then $\partial_+\partial_-\varphi + \frac{\mu^2}{2}\sin 2\varphi = 0$
following from sine-Gordon action (Pohlmeyer, 1976)

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \frac{\mu^2}{2}\cos 2\varphi$$

2d Lorentz invariant despite explicit constraints
Classical solutions and integrable structures
(Lax pair, Backlund transformations, etc) are directly related
e.g., SG soliton mapped into rotating folded string on S^2
"giant magnon" in the $J = \infty$ limit (Hofman, Maldacena 06)
other examples for CSG (Chen, Dorey, Okamura 06;
Okamura, Suzuki, Hayashi, Vicedo 07;
Jevicki, Spradlin, Volovich, et al 07)

Analogous construction for S^3 model gives

Complex sine-Gordon model (Pohlmeyer; Lund, Regge 76)

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \cot^2\varphi\, \partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\cos 2\varphi$$

φ, θ are $SO(4)$-invariants:

$$\mu^2 \cos 2\varphi = \partial_+ X^m \partial_- X^m$$

$$\mu^3 \sin^2\varphi\, \partial_\pm\theta = \mp\frac{1}{2}\epsilon_{mnkl} X^m \partial_+ X^n \partial_- X^k \partial_\pm^2 X^l$$

"String on $R_t \times S^n$" interpretation

conformal gauge plus $t = \mu\tau$ to fix conformal diffeomorphisms:

$\partial_\pm X^m \partial_\pm X^m = \mu^2$ are Virasoro constraints

Similar construction for AdS_n case,

i.e. string on $AdS_n \times S^1_\psi$ with $\psi = \mu\tau$

e.g. reduced theory for $AdS_3 \times S^1$

$$\widetilde{L} = \partial_+\phi\partial_-\phi + \coth^2\varphi\, \partial_+\chi\partial_-\chi - \frac{\mu^2}{2}\cosh 2\phi$$

Comments:

- Virasoro constraints are solved by a special choice of variables related nonlocally to the original coordinates
- Although the reduction is not explicitly Lorentz invariant the resulting Lagrangian turns out to be 2d Lorentz invariant
- The reduced theory is formulated in terms of manifestly $SO(n)$ invariant variables: "blind" to original global symmetry
- reduced theory is equivalent to the original theory as integrable system: the respective Lax pairs are gauge-equivalent
- PR may be thought of as a formulation in terms of physical d.o.f. – coset space analog of flat-space l.c. gauge (where 2d Lorentz is unbroken)
- In general reduced theory can not be quantum-equivalent to the original one (e.g., conformal symmetry was assumed in the reduction procedure)

PR for bosonic F/G-coset model

To find reduced theory for $AdS_5 \times S^5$ GS model need to understand PR of F/G coset sigma models as G/H gauged WZW models modified by relevant integrable potential and then generalize to GS supercoset

F/G-coset sigma model:

symmetric space condition ($\mathfrak{f}$, $\mathfrak{g}$ are Lie algebras of F and G)

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g} \,, \qquad [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} \,, \qquad [\mathfrak{g}, \mathfrak{p}] \subset \mathfrak{p} \,, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{g}$$

with $\langle \mathfrak{g}, \mathfrak{p} \rangle = 0$ (choose $\langle a, b \rangle = \mathrm{Tr}(ab)$)

Lagrangian:

$$L = -\mathrm{Tr}(P_+ P_-) \,, \qquad P_\pm = (f^{-1}\partial_\pm f)_\mathfrak{p} \,,$$

$$J = f^{-1} df = \mathcal{A} + P \,, \qquad \mathcal{A} = J_\mathfrak{g} \in \mathfrak{g} \,, \quad P = J_\mathfrak{p} \in \mathfrak{p} \,.$$

Symmetries: G gauge transformations $f \to fg$;

global F-symmetry: $f \to f_0 f$, $f_0 = \mathrm{const} \in F$

classical conformal invariance

Equations of motion in terms of currents

let $J = \mathcal{A} + P$ be fundamental variables, not f

$$D_+ P_- = 0\,, \quad D_- P_+ = 0\,, \qquad D = d + [\mathcal{A},\ ] \qquad \text{– EOM}$$

$$D_- P_+ - D_+ P_- + [P_+, P_-] + \mathcal{F}_{+-} = 0 \qquad \text{– Maurer-Cartan}$$

$$\mathrm{Tr}(P_+ P_+) = -\mu^2\,, \quad \mathrm{Tr}(P_- P_-) = -\mu^2 \qquad \text{– Virasoro}$$

Main idea: – first solve EOM and Virasoro and then MC
using special choice of G gauge condition and conformal diffs
then find reduced action giving eqs. resulting from MC
gauge fixing that solves the first Virasoro constraint

$$P_+ = \mu\, T = \text{const}\,, \qquad T \in \mathfrak{p} = \mathfrak{f} \ominus \mathfrak{g}, \qquad \mathrm{Tr}(TT) = -1$$

choice of special element $T \to$ decomposition of the algebra of F

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g}\,, \qquad \mathfrak{p} = T \oplus \mathfrak{n}\,, \qquad \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}\,, \qquad [T, \mathfrak{h}] = 0\,,$$

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}\,, \qquad [\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}\,, \qquad [T, \mathfrak{m}] \subset \mathfrak{n}\,, \qquad [T, \mathfrak{n}] \subset \mathfrak{m}\,.$$

$\mathfrak{h}$ is a centraliser of T in $\mathfrak{g}$

EOM $D_-P_+ = 0$ is solved by

$$(\mathcal{A}_-)_{\mathfrak{m}} = 0\,, \qquad \mathcal{A}_- = (\mathcal{A}_-)_{\mathfrak{h}} \equiv A_-$$

second Virasoro constraint is solved by

$$P_- = \mu\, g^{-1} T g\,, \qquad g \in G$$

EOM $D_+P_- = 0$ is solved by

$$\mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1} A_+ g$$

To summarise:
solved EOM's and Virasoro constraints introducing
new dynamical field variables

G-valued field g , $\qquad$ $\mathfrak{h}$-valued fields $A_+,\ A_-,$ $\qquad$ $[T, A_\pm] = 0$

what remains is the Maurer-Cartan equation on $g, A_\pm$

Relation to G/H gauged WZW model

G/H gWZW action with potential:

$$
\begin{aligned}
L = & - \frac{1}{2}\mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) + \text{WZ term} \\
& - \mathrm{Tr}\Big(A_+\,\partial_- g g^{-1} - A_-\, g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+ A_-\Big) \\
& - \mu^2 \mathrm{Tr}(T g^{-1} T g)
\end{aligned}
$$

Pohlmeyer-reduced theory for F/G coset sigma model
(as first proposed by Bakas, Park, Shin 95)
and thus also for strings on $R_t \times F/G$ or $F/G \times S^1_\psi$
integrable potential: relation at the level of Lax pairs
special case of non-abelian Toda theory:
"symmetric space Sine-Gordon model"
(Hollowood, Miramontes et al 96)

What to do with A_+, A_-: integrate out or gauge-fix

Reduced equation of motion in the "on-shell" gauge $A_\pm = 0$:

Non-abelian Toda equqtions:

$$\partial_-(g^{-1}\partial_+ g) - \mu^2[T, g^{-1}Tg] = 0 ,$$
$$(g^{-1}\partial_+ g)_{\mathfrak{h}} = 0 , \qquad (\partial_- g g^{-1})_{\mathfrak{h}} = 0 .$$

$$F/G = SO(n+1)/SO(n) = S^n : \;\; G/H = SO(n)/SO(n-1)$$

$$g = \begin{pmatrix} k_1 & k_2 & \ldots & k_n \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix} , \qquad \sum_{1=1}^{n} k_l k_l = 1$$

get (in general non-Lagrangian) EOM for k_m

$$\partial_-\Big(\frac{\partial_+ k_\ell}{\sqrt{1-\sum_{m=2}^{n} k_m k_m}}\Big) = -\mu^2 k_\ell , \qquad \ell = 2, \ldots, n .$$

Linearising around the vacuum $g = 1$ (i.e. $k_1 = 1, \; k_\ell = 0$)

$$\partial_+\partial_- k_\ell + \mu^2 k_\ell + O(k_\ell^2) = 0$$

massive spectrum: non-trivial S-matrix with H global symmetry

$F/G = SO(n+1)/SO(n) = S^n$:
parametrization of g in Euler angles
$g = e^{T_{n-2}\theta_{n-2}}...e^{T_1\theta_1}e^{2T\varphi}e^{T_1\theta_1}...e^{T_{n-2}\theta_{n-2}}$
and integrating out $H = SO(n-1)$ gauge field $A_\pm$
leads to reduced theory that generalizes SG and CSG

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + G_{pq}(\varphi,\theta)\partial_+\theta^p\partial_-\theta^q + \frac{\mu^2}{2}\cos 2\varphi$$

no B_{mn} coupling
gWZW for $G/H = SO(n)/SO(n-1)$

$$ds^2_{n=2} = d\varphi^2 \ , \qquad\qquad ds^2_{n=3} = d\varphi^2 + \cot^2\varphi \, d\theta^2$$

ironically, return of old metrics of "de Sitter" or "S^n"
gWZW models (Bars, Nemeschansky,...)
$G/H = SO(4)/SO(3)$ (Fradkin, Linetsky 91)

$$ds^2_{n=4} = d\varphi^2 + \cot^2\varphi \, (d\theta_1 + \cot\theta_1 \tan\theta_2 \, d\theta_2)^2 + \tan^2\varphi \, \frac{d\theta_2^2}{\sin^2\theta_1}$$

Bosonic strings on $AdS_n \times S^n$

straightforward generalization:

Lagrangian and the Virasoro constraints

$L = \mathrm{Tr}(P_+^A P_-^A) - \mathrm{Tr}(P_+^S P_-^S)\,,$

$\mathrm{Tr}(P_\pm^S P_\pm^S) - \mathrm{Tr}(P_\pm^A P_\pm^A) = 0$

fix conformal symmetry by

$\mathrm{Tr}(P_\pm^S P_\pm^S) = \mathrm{Tr}(P_\pm^A P_\pm^A) = -\mu^2$

then PR applies independently in each sector:

get direct sum of reduced systems for S^n and AdS_n

linked by Virasoro, i.e. common μ

e.g. for $F/G = AdS_2 \times S^2$:

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$$

$AdS_5 \times S^5$ superstring sigma-model

$AdS_5 \times S^5 = \frac{SU(2,2)}{Sp(2,2)} \times \frac{SU(4)}{Sp(4)}$

supercoset GS sigma model

$\frac{\widehat{F}}{G} = \frac{PSU(2,2|4)}{Sp(2,2)\times Sp(4)}$

basic superalgebra $\widehat{\mathfrak{f}} = psu(2,2|4)$

bosonic part $\mathfrak{f} = su(2,2) \oplus su(4) \cong so(2,4) \oplus so(6)$

admits Z_4-grading: (Berkovits, Bershadsky, et al 89)

$$\widehat{\mathfrak{f}} = \mathfrak{f}_0 \oplus \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3 \,, \qquad [\mathfrak{f}_i, \mathfrak{f}_j] \subset \mathfrak{f}_{i+j \,\mathrm{mod}\, 4}$$

$\mathfrak{f}_0 = \mathfrak{g} = sp(2,2) \oplus sp(4)$

current ($J = f^{-1}\partial_a f, \;\; f \in \widehat{F}$) decomposes as

$$J_a = f^{-1}\partial_a f = \mathcal{A}_a + Q_{1a} + P_a + Q_{2a}$$

$$\mathcal{A} \in \mathfrak{f}_0, \quad Q_1 \in \mathfrak{f}_1, \quad P \in \mathfrak{f}_2, \quad Q_2 \in \mathfrak{f}_3 \,.$$

GS Lagrangian:

$$L_{\rm GS} = \frac{1}{2}\,{\rm STr}(\sqrt{-g}g^{ab}P_aP_b + \varepsilon^{ab}Q_{1a}Q_{2b})\,,$$

very simple structure – but not standard coset model:
fermionic currents in WZ term only
conformal gauge: $\sqrt{-g}g^{ab} = \eta^{ab}$

$$L_{\rm GS} = {\rm STr}[P_+P_- + \frac{1}{2}\,(Q_{1+}Q_{2-} - Q_{1-}Q_{2+})]$$

$${\rm STr}(P_+P_+) = 0\,, \qquad {\rm STr}(P_-P_-) = 0$$

PR procedure: solve first EOM and Virasoro
κ-gauge condition: $Q_{1-} = 0\,, \qquad Q_{2+} = 0$
solves the last (fermionic) pair of EOM

as in the bosonic F/G case can fix the "reduction gauge"

$$P_+ = \mu\, T\,, \qquad T = \frac{i}{2}{\rm diag}(1,1,-1,-1|1,1,-1,-1)$$

$$P_- = \mu\, g^{-1} T g\,, \quad \mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1} A_+ g\,, \quad \mathcal{A}_- = A_-$$

T defines $\mathfrak{h}$ by $[\mathfrak{h}, T] = 0$:

$\mathfrak{h} = su(2) \oplus su(2) \oplus su(2) \oplus su(2)$

new parametrisation: $G = Sp(2,2) \times Sp(4)$-valued field g

and $\mathfrak{h}$-valued field $A_\pm$

AdS_5 and S^5 sectors now coupled by fermions

remains residual κ-symmetry to be fixed

use T to generalise decomposition of bosonic part

remaining fermionic components

$$\Psi_{{}_R} = \frac{1}{\sqrt{\mu}} \Psi_1^{\|}\,, \qquad\qquad \Psi_{{}_L} = \frac{1}{\sqrt{\mu}} \Psi_2^{\|}\,,$$

transform under $H \times H$ as $\Psi_{{}_R} \to \bar{h}^{-1} \Psi_{{}_R} \bar{h}\,,\ \Psi_{{}_L} \to h^{-1} \Psi_{{}_L} h\,.$

Lagrangian of PR theory for $AdS_5 \times S^5$ superstring

(Grigoriev, AT 07; related work: Mikhailov, Schafer-Nameki 07)

fermionic generalization of "gWZW+ potential" theory for

$$\frac{G}{H} = \frac{Sp(2,2)}{SU(2)\times SU(2)} \times \frac{Sp(4)}{SU(2)\times SU(2)}$$

$$\begin{aligned} L &= L_{\rm gWZW}(g, A_+, A_-) + \mu^2\, {\rm STr}(g^{-1}TgT) \\ &+ {\rm STr}\left(\Psi_{_L}[T, D_+\Psi_{_L}] + \Psi_{_R}[T, D_-\Psi_{_R}]\right) \\ &+ \mu\, {\rm STr}\left(g^{-1}\Psi_{_L} g \Psi_{_R}\right) \end{aligned}$$

direct sum of PR theories for AdS_5 and S^5

"glued together" by components of fermions

$$\begin{aligned} L &= \widetilde{L}_{S^5}(g, A_+, A_-) + \widetilde{L}_{AdS_5}(g, A_+, A_-) \\ &+ \psi_{_L} D_+ \psi_{_L} + \psi_{_R} D_+ \psi_{_R} + \mu\ (\text{interaction terms}) \end{aligned}$$

all gauge symmetries fixed; standard kin. terms (cf. GS action)

The corresponding Lax pair encoding the equations of motion

$$\mathcal{L}_- = \partial_- + A_- + \ell^{-1}\sqrt{\mu}g^{-1}\Psi_{_L}g + \ell^{-2}\mu g^{-1}Tg\,,$$
$$\mathcal{L}_+ = \partial_+ + g^{-1}\partial_+g + g^{-1}A_+g + \ell\sqrt{\mu}\Psi_{_R} + \ell^2\mu T\,.$$

use that $[T,[T,\Psi_{_{L,R}}]] = -\Psi_{_{L,R}}$

Comments:

- gWZW model coupled to the fermions interacting minimally and through the "Yukawa term"
- 8 real bosonic and 16 real fermionic independent variables
- 2d Lorentz invariant with Ψ_R, Ψ_L as 2d Majorana spinors
- 2d supersymmetry? yes, at the linearised level, and yes in $AdS_2 \times S^2$ case: $n = 2$ super sine-Gordon
- μ-dependent interaction terms are equal to original GS Lagrangian; gWZW produces MC eq.: path integral derivation via change from fields to currents?
- quadratic in fermions (like susy version of gWZW); integrating out $A_\pm$ gives quartic fermionic terms (reflecting curvature)
- linearisation of EOM in the gauge $A_\pm = 0$ around $g = \mathbf{1}$ describes 8+8 massive bosonic and fermionic d.o.f. with mass μ: same as in BMN limit
- symmetry of resulting relativistic S-matrix: $H = [SU(2)]^4$ – same as bosonic part of magnon S-matrix symmetry $[PSU(2|2)]^2$

Example: superstring on $AdS_2 \times S^2$

Explicit parametrisation:

$$T = \frac{1}{2}\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

$$g = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & \cos\varphi & i\sin\varphi \\ 0 & 0 & i\sin\varphi & \cos\varphi \end{pmatrix}$$

$$\Psi_R = \begin{pmatrix} 0 & 0 & 0 & i\gamma \\ 0 & 0 & -\beta & 0 \\ 0 & i\beta & 0 & 0 \\ \gamma & 0 & 0 & 0 \end{pmatrix}, \quad \Psi_L = \begin{pmatrix} 0 & 0 & 0 & \rho \\ 0 & 0 & -i\nu & 0 \\ 0 & \nu & 0 & 0 \\ i\rho & 0 & 0 & 0 \end{pmatrix}$$

PR Lagrangian: same as $n = 2$ supersymmetric sine-Gordon!

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$$
$$+ \beta\partial_-\beta + \gamma\partial_-\gamma + \nu\partial_+\nu + \rho\partial_+\rho$$
$$- 2\mu\left[\cosh\phi\ \cos\varphi\,(\beta\nu + \gamma\rho) + \sinh\phi\ \sin\varphi\,(\beta\rho - \gamma\nu)\right] .$$

indeed, equivalent to

$$\widetilde{L} = \partial_+\Phi\partial_-\Phi^* - |W'(\Phi)|^2$$
$$+\psi^*_{L}\partial_+\psi_{L} + \psi^*_{R}\partial_-\psi_{R} + \left[W''(\Phi)\psi_{L}\psi_{R} + W^{*\prime\prime}(\Phi^*)\psi^*_{L}\psi^*_{R}\right] .$$

bosonic part is of $AdS_2 \times S^2$ bosonic reduced model if

$$W(\Phi) = \mu\cos\Phi\ , \qquad |W'(\Phi)|^2 = \frac{\mu^2}{2}(\cosh 2\phi - \cos 2\varphi)\ .$$

$$\psi_{L} = \nu + i\rho\,, \qquad \psi_{R} = -\beta + i\gamma\ ,$$

Example: superstring on $AdS_3 \times S^3$

Green-Schwarz superstring on $AdS_3 \times S^3$
supported by RR 3-form flux: coset model

$$\frac{PSU(1,1|2) \times PSU(1,1|2)}{SU(2) \times SU(1,1)}$$

superalgebra $psu(1,1|2)$ admits a Z_4-grading
GS Lagrangian: in terms of Z_4-components of $J_\pm = \widehat{f}^{-1}\partial_\pm \widehat{f}$

$$J_\pm = \mathcal{A}_\pm + P_\pm + Q_{1\pm} + Q_{2\pm} \,,$$

$$\mathcal{A} \in \widehat{\mathfrak{f}}_0, \quad Q_1 \in \widehat{\mathfrak{f}}_1, \quad P \in \widehat{\mathfrak{f}}_2, \quad Q_2 \in \widehat{\mathfrak{f}}_3$$

$$L_{\rm GS} = {\rm STr}\Big[P_+P_- + \frac{1}{2}(Q_{1+}Q_{2-} - Q_{1-}Q_{2+})\Big] \,,$$

conformal gauge constraints: ${\rm STr}(P_+P_+) = 0$ and ${\rm STr}(P_-P_-) = 0$
Solving for gauge fields $A_\pm$

$$L_{tot} = L_1 + L_2 + L_3 = L_B + \text{fermionic terms}$$

bosonic terms: direct sum of the CSG action and its “hyperbolic” counterpart – reduced bosonic string in $AdS_3 \times S^3$:

$$L_B = \partial_+\varphi\partial_-\varphi + \cot^2\varphi\,\partial_+\theta\partial_-\theta$$
$$+\,\partial_+\phi\partial_-\phi + \coth^2\phi\,\partial_+\chi\partial_-\chi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$$

$$L_1 = \partial_+\varphi\partial_-\varphi + \frac{1}{2}(1+\cos 2\varphi)\,\partial_+\theta\partial_-\theta$$
$$+\partial_+\phi\partial_-\phi - \frac{1}{2}(1+\cosh 2\phi)\,\partial_+\chi\partial_-\chi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$$

$$L_2 = \alpha\partial_-\alpha + \beta\partial_-\beta + \gamma\partial_-\gamma + \delta\partial_-\delta + \lambda\partial_+\lambda + \nu\partial_+\nu + \rho\partial_+\rho$$
$$-2\mu\Big(\sinh\phi\sin\varphi(\lambda\beta - \nu\alpha + \rho\delta - \sigma\gamma) + \cosh\phi\cos\varphi\Big[\cos(\chi$$
$$+\lambda\delta - \nu\gamma) + \sin(\chi+\theta)(\rho\alpha + \sigma\beta - \lambda\gamma - \nu\delta)\Big]\Big)$$

$$L_3 = \frac{[\partial_+\chi\ (1+\cosh 2\phi)\ - 2(\alpha\beta - \gamma\delta)][\partial_-\chi\ (1+\cosh 2\phi)\ +}{2(\cosh 2\phi - 1)}$$

$$+\frac{[\partial_+\theta\ (1+\cos 2\varphi)\ + 2(\alpha\beta - \gamma\delta)][\partial_-\theta\ (1+\cos 2\varphi)\ - 2(\lambda\nu -}{2(1-\cos 2\varphi)}$$

identify the fermions $\alpha, \beta, \gamma, \delta$ and $\lambda, \nu, \rho, \sigma$ with 2d MW spinors – 2d supersymmetry ?

Open questions

- Quantum equivalence of reduced theory and GS theory?

  Check of UV finiteness? Yes in $AdS_2 \times S^2$. In $AdS_3 \times S^3$?

- Path integral argument of equivalence?

  Potential term is original action

  $\mathrm{Tr}(P_+P_-) = \mu^2 \mathrm{Tr}(Tg^{-1}Tg)$

  while gWZW should come from change of variables.

  Rough idea: string in $R_t \times F/G$ coset

  $L = -(\partial t)^2 + \mathrm{Tr}(f^{-1}df + B)^2 \ , \quad f \in F, \ \ B \in \mathfrak{g}$

  string path integral in conformal+ $t = \mu\tau$ gauge:

  $$\int DfDB\, \delta(T_{++} - \mu^2)\, \delta(T_{--} - \mu^2) \quad e^{iI(f,B)}$$

  then replace $f^{-1}df$ by C

  $$\int DCDBDv\, \delta(T_{++}{-}\mu^2)\delta(T_{--}{-}\mu^2) \exp[i \int (C{+}B)^2{+}v(dC$$

set $(C+B)_+ = \mu T$, $(C+B)_- = \mu g^{-1} T g$; change from C, B, v to $g \in G, A \in \mathfrak{h}$: $[\mathfrak{h}, T] = 0$

Transformation may work only in genuine quantum-conformal ($AdS_n \times S^n$) case.

- Indication of equivalence: semiclassical expansion

  near analog of (S, J) rigid string in $AdS_5 \times S^5$ leads to the same characteristic frequencies – same 1-loop partition function (Roiban, AT 08, to appear)

- Tree-level S-matrix for elementary excitations?

  Manifest $SU(2) \times SU(2) \times SU(2) \times SU(2)$ symmetry?

  Hidden bigger symmetry? Relation to magnon S-matrix in BA?

- better understanding the relationship between the original and the reduced system: symmetries, vacua, values of conserved charges, etc.; which observables can be related?