Finiteness (or not) of N=8 Supergravity

Wonders of Gauge Theory and Supergravity Conference

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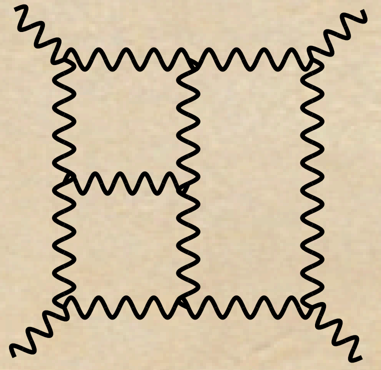
Ultraviolet Divergences in Gravity

- Simple power counting in gravity and supergravity theories leads to a naïve degree of divergence

$$\Delta = (D-2)L + 2$$

in D spacetime dimensions. So, for D=4, L=3, one expects $\Delta = 8$. In dimensional regularization, only logarithmic divergences are seen ($\frac{1}{\varepsilon}$ poles, $\varepsilon = D-4$), so 8 powers of momentum would have to come out onto the external lines of such a diagram.



- Local supersymmetry implies that the pure curvature part of such a D=4, 3-loop divergent structure must be built from the square of the Bel-Robinson tensor Deser, Kay & K.S.S

$$\int \sqrt{-g}T_{\mu\nu\rho\sigma}T^{\mu\nu\rho\sigma}\,, \qquad T_{\mu\nu\rho\sigma} = R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}R_{\rho\alpha\sigma\beta} + {}^{*}R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}\,{}^{*}R_{\rho\alpha\sigma\beta}$$

Grisaru, Van de Ven & Zanon

- This is directly related to the α'^3 corrections in the superstring effective action, except that in the string context such contributions occur with finite coefficients. The question remains whether such string theory contributions develop poles in $(\alpha')^{-1}$ as one takes the zero-slope limit $\alpha' \to 0$ and how this bears on the ultraviolet properties of the corresponding field theory.

- The consequences of supersymmetry for the ultraviolet structure are not restricted, however, simply to the requirement that counterterms be supersymmetric invariants.

- There exist more powerful "non-renormalization theorems," the most famous of which excludes infinite renormalization within D=4, N=1 supersymmetry of chiral invariants, given in N=1 superspace by integrals over half the superspace:

$$\int d^2\theta W(\phi(x,\theta,\bar{\theta})) \ , \quad \bar{D}\phi = 0$$

- Key tools in proving non-renormalization theorems are superspace formulations and the background field method.

- For example, the Wess-Zumino model in N=1, D=4 supersymmetry is formulated in terms of a chiral superfield $\phi(x,\theta,\bar{\theta})$: $\bar{D}\phi = 0$; $\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^{\alpha}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}$

- In the background field method, one splits the superfield into "background" and "quantum" parts,

$$\phi = \varphi + Q$$

background quantum

- The chiral constraint on $Q(x,\theta,\bar{\theta})$ can be solved by introducing a "prepotential": $Q = \bar{D}^2 X \qquad (\bar{D}^3 \equiv 0)$

- Although the Wess-Zumino action includes chiral superspace integrals $I = \int d^4x d^4\theta \bar{\phi}\phi + Re \int d^4x d^2\theta \phi^3$ when written in terms of the total field ϕ, the parts involving the quantum field Q appearing inside loop diagrams can be re-written as $\int d^4x d^4\theta = \int d^4x d^2\theta d^2\bar{\theta}$ full superspace integrals using the "integration=differentiation" property of Berezin integrals.

- Upon expanding into background and quantum parts, one finds that the chiral interaction terms can be re-written as full superspace integrals, e.g.

$$\int d^4x d^2\theta Q^2\varphi = \int d^4x d^4\theta X\bar{D}^2X\varphi$$

- Thus all counterterms written using the background field φ must be writable as full-superspace integrals.

- The strength of such supersymmetric non-renormalization theorems depends on the extent of linearly realizable, or "off-shell" supersymmetry. This is the extent of supersymmetry for which the algebra can close without use of the equations of motion.
- Knowing the extent of this off-shell supersymmetry is tricky, and may involve formulations (*e.g.* harmonic superspace) with infinite numbers of auxiliary fields.
- For maximal N=4 Super Yang-Mills and maximal N=8 supergravity, the linearly realizable supersymmetry has been known since the 80's to be at least half the full supersymmetry of the theory. This was used to show the finiteness of D=4, N=4 SYM theory.

Howe, K.S.S. & Townsend

- The key point about non-renormalization theorems is that allowed counterterms have to be written as full $\int d^{4M}\theta$ superspace integrals for the linearly realized M-extended supersymmety, where the integrands must be written using a clearly defined set of basic objects (analogous to the WZ background field φ), and where the integrated counterterms have to satisfy all applicable gauge symmetries and also must be locally constructed (*i.e.* written without using such operators as $\Box^{-1}$). Haag
- The full extent of a theory's supersymmetry, even though it may be non-linear, also restricts the infinities since the *leading* counterterms have to be invariant under the original unrenormalized supersymmetry transformations.

- Assuming that 1/2 supersymmetry is linearly realizable and requiring gauge and supersymmetry invariances, together with other relevant automorphism symmetries, one derives predictions for the first divergent loop orders in maximal (N=4 ↔ 16 supercharge) SYM and (N=8 ↔ 32 sc.) SUGRA: Howe, K.S.S & Townsend

Max. SYM first divergences, assuming half SUSY off-shell (8 supercharges)

Dimension D	10	8	7	6	5	4
Loop order L	1	1	2	3	4	∞
Gen. form	$\partial^2 F^4$	F^4	$\partial^2 F^4$	$\partial^2 F^4$	F^4	finite

Max. SUGRA first divergences, assuming half SUSY off-shell (16 supercharges)

Dimension D	11	10 *	8	7	6 *	5	4
Loop order L	2	2	1	2	3	2	3
Gen. form	$\partial^{12} R^4$	$\partial^{10} R^4$	R^4	$\partial^6 R^4$	$\partial^6 R^4$	R^4	R^4

- The D=10 and D=6 max supergravity * cases are peculiar: one might have thought there could be $\partial^2 R^4$ counterterms one loop earlier. But these are cases where *on-shell* supersymmetry and automorphism symmetries rule this out. Drummond, Heslop, Howe & Kerstan

Unitarity-based calculations

Bern, Dixon, Dunbar, Kosower, Perelstein, Rozowsky et al.

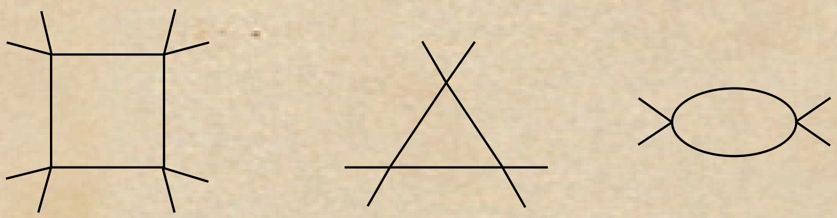
- Within the last decade, there have been significant advances in the computation of loop corrections in quantum field theory.
- These developments include the organization of amplitudes into a new kind of perturbation theory starting with maximal helicity violating amplitudes (MHV), then next-to-MHV (NMHV), *etc.*
- They also incorporate a specific use of dimensional regularization together with a clever use of unitarity cutting rules.

- Normally, one thinks of unitarity relations such as the optical theorem as giving information only about the imaginary parts of amplitudes. However, if one keeps all orders in an expansion in $\varepsilon = 4 - D$, then loop integrals like $\int d^{(4-\varepsilon)}p$ require integrands to have an additional momentum dependence $f(s) \to f(s)s^{-\varepsilon/2}$, where s is a momentum invariant. Then, since $s^{-\varepsilon/2} = 1 - (\varepsilon/2)\ln(s) + \ldots$ and $\ln(s) = \ln(|s|) + i\pi\Theta(s)$, one can learn about the real parts of an amplitude by retaining imaginary terms at order ε.

- This gives rise to a procedure for the *cut construction* of higher-loop diagrams.

- Another key element in the unitarity-based analysis of amplitudes is the Passarino-Veltman procedure for the reduction of Feynman-diagram propagators, replacing numerator factors like $2k \cdot p$ where $p^2 = 0$ by $(k+p)^2 - k^2$ and then canceling corresponding denominators.

- This procedure can yield a variety of final irreducible configurations in the reduced diagram, including boxes, triangles and bubbles.



- Important simplifications occur if one can show there are ultimately no bubbles or triangles in the reduced amplitude.

- For maximal supergravity amplitudes, another specific relation allowing amplitudes to be evaluated is the Kawai-Lewellen-Tye relation between open- and closed-string amplitudes. This gives rise to tree-level relations between max. SUGRA and max. SYM field-theory amplitudes, *e.g.*

$$M_4^{\mathrm{tree}}(1,2,3,4) = -is_{12}A_4^{\mathrm{tree}}(1,2,3,4)\,A_4^{\mathrm{tree}}(1,2,4,3)$$

- Combining this with unitarity-based calculations, in which all amplitudes are ultimately reduced to integrals of products of tree amplitudes, one has a way to obtain higher-loop supergravity amplitudes from SYM amplitudes.

- In this way, a different set of anticipated first loop orders for ultraviolet divergences has arisen from the unitarity-based approach:

Max. SYM first divergences, unitarity-based predictions

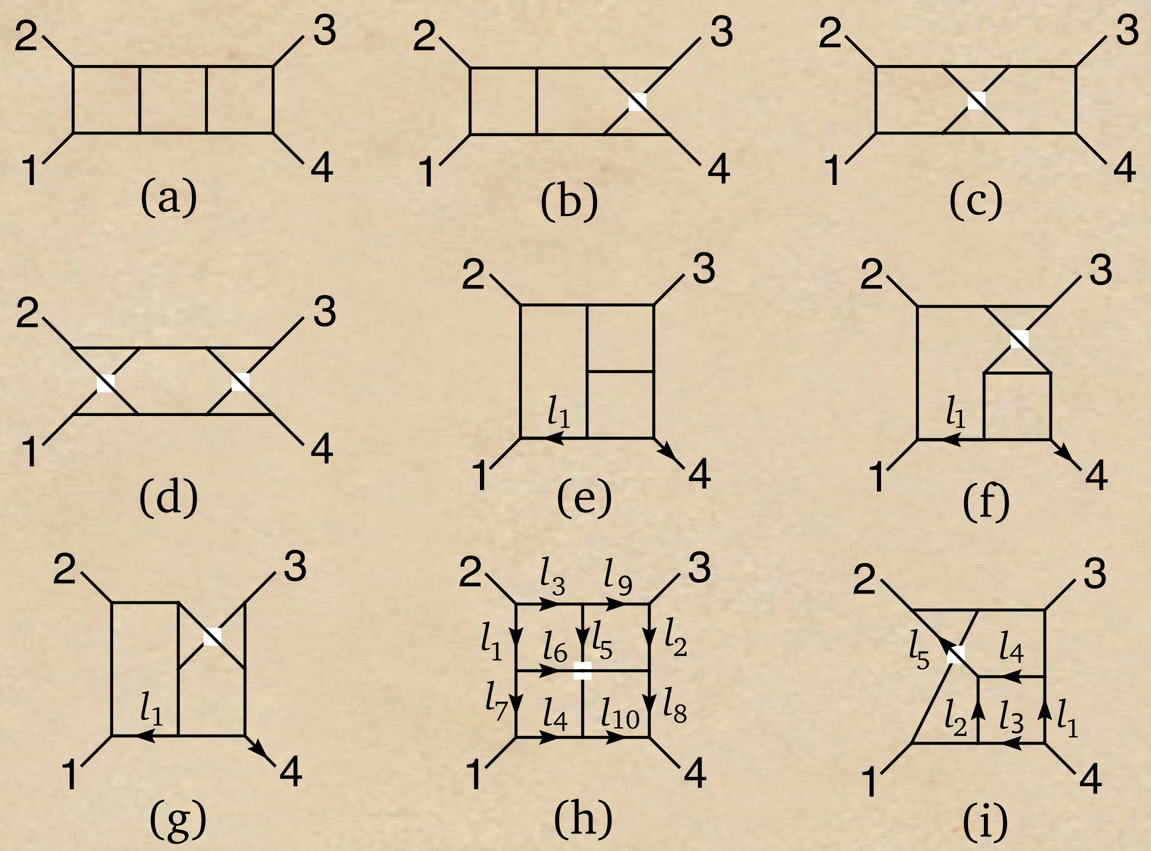
Dimension D	10	8	7	6	5	4
Loop order L	1	1	2	3	6	∞
Gen. form	$\partial^2 F^4$	F^4	$\partial^2 F^4$	$\partial^2 F^4$	$\partial^2 F^4$	finite

Max. SUGRA first divergences, unitarity-based predictions

Dimension D	11	10	8	7	6	5	4
Loop order L	2	2	1	2	3	4	5
Gen. form	$\partial^{12} R^4$	$\partial^{10} R^4$	R^4	$\partial^6 R^4$	$\partial^6 R^4$	$\partial^6 R^4$	$\partial^4 R^4$

- These anticipations are based on iterated 2-particle cuts, however. Full calculations can reveal different behavior.

- The main recent development is the completion of the 3-loop calculation: Bern, Carrasco, Dixon, Johansson, Kosower & Roiban.



- Diagrams (a-g) can be evaluated using iterated two-particle cuts, but diagrams (h) & (i) cannot. The result is finite at L=3 in D=4, but the surprize is that the finite parts have an unexpected six powers of momentum that come out onto the external lines, giving a $\partial^6 R^4$ leading effective action correction.

- What can one say about the possibility of yet higher-order cancelations? The hints of this depend on the no-triangle hypothesis for the end result of the graph reduction procedure. This structure been argued to follow for four-point diagrams directly from N=8 supersymmetry at one loop for the non-local effective action. Kallosh
- It has also been argued that maximal supergravity has a new type of reduction procedure, leaving a result with neither triangles, bubbles nor rational pieces after the reduction procedure. Bjerrum-Bohr & Vanhove

Counterterms counterattack

- The 3-loop N=8 supergravity calculation is a remarkable *tour de force*, but does it indicate that there are "miracles" that cannot be understood from non-renormalization theorems?
- All known SYM divergences in the various dimensions D can be understood using non-renormalization theorems.
- Moreover, these SYM results extend to counterterms that have not yet been calculated using the unitarity-based methods. Examples are the full D=7, L=2 results for max. SYM. Here, there are both single- and double-trace structures for the Yang-Mills gauge group. Marcus & Sagnotti

- Recently it has been realized that N=4 SYM can be quantized with 9=8+1 off-shell supersymmetries, at the price of manifest Lorentz invariance. Baulieu, Berkovits, Bossard & Martin

- The usual problem with finding an off-shell formalism for SYM is the imbalance between the number of non-gauge bosonic and fermionic degrees of freedom. In D=10, there are 9 bosonic and 16 fermionic propagating fields, giving a deficit of 7 bosonic. This doesn't fit into any finite combination of SO(9,1) representations. However, it will fit into SO(1,1)x$Spin_7$ representations. One first makes a decomposition into SO(1,1)xSO(8) reps, separating the D=10 Majorana-Weyl spinor into two SO(8) chiral spinors. Then, under the SO(8) ➔ $Spin_7$ decomposition, one chirality remains an **8** while the other splits into **7+1**. **8+1** SUSYs can then be taken off-shell.

- This construction can also be viewed from a Kaluza-Klein perspective after reduction to D=2, where the SO(1,1)xSO(8) decomposition is natural. The ***8+1*** formalism then natually corresponds to (8,1) D=2 SUSY.
- This might be considered similar to SO(1,1)xSO(8) light-cone reductions. The latter, however, do not respect all D=10 gauge symmetries, while the SO(1,1)x$Spin_7$ formalism does.

Bossard, Howe & K.S.S (WIP)

- A similar formulation for maximal supergravity exists with 17=16+1 off-shell supersymmetries in D=2. This corresponds to off-shell (16,1) supersymmetry in D=2.
- Lifting the 17-SUSY D=2 maximal SG formulation to higher dimensions remains complicated, however.

- To see how these 1/2 SUSY + 1 formalisms work, consider the (8,1) SYM multiplet (linearized):

Baulieu, Berkovits, Bossard & Martin

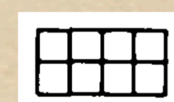
$$D_{\alpha+}\phi_A = (\Sigma_\alpha)_{AA'}\psi_{A'+}$$
$$D_-\phi_A = \psi_{A-}$$
$$D_-\psi_{A'+} = G_{A'}$$
$$D_{\alpha+}\psi_{A-} = (\Sigma_\alpha)_{AA'}G_{A'}$$

$$\alpha,\ A,\ A' = 1,\ldots,8$$

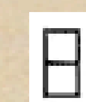
- This multiplet contains 8 scalar and 16 fermion components plus and 8 auxiliaries $G_{A'}$. In order to get SYM, one of these 8 auxiliaries has to be identified with F_{++--}, so $SO(8) \rightarrow \mathrm{Spin}_7$.

- A minimalist perspective on the ***8+1*** max. SYM and the ***16 +1*** max. SG formalisms focuses on their usefulness in attacking the eligibility of counterterms involving integration over *half* the corresponding full on-shell superspaces, i.e. *8* integrations for SYM and 16 for SG. These two "half SUSY" counterterms have similar structures in D=4: Howe, K.S.S. & Townsend

$$\Delta I_{SYM} = \int (d^4\theta d^4\bar{\theta})_{105}\,\mathrm{tr}(\phi^4)_{105}$$

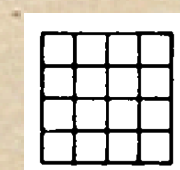


105 ϕ_{ij}

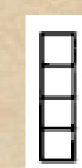


6 of SU(4)

$$\Delta I_{SG} = \int (d^8\theta d^8\bar{\theta})_{232848}(W^4)_{232848}$$



232848 W_{ijkl}



70 of SU(8)

- The implications of gauge invariance are not fully resolved, but it seems likely that the "half SUSY +1" formalisms will be just enough to rule out the F^4 SYM and R^4 SG counterterms.

- The "half SUSY +1" formalisms appear to be the largest possible finite-component formalisms for max. SYM and max. SG. But there exist also harmonic superspace formalisms with infinite numbers of ordinary component fields. The largest known example of this is the N=3 (i.e.12-supercharge) off-shell formulation of N=4 SYM.

Galperin, Ivanov, Kalitzin, Ogievetsky & Sokatchev

- The N=3 harmonic superspace SYM action has a Chern-Simons type integrand:

$$I_{SYM} = \int d^4x\, du\, \left(D_2 D_3 \bar{D}^1 \bar{D}^2\right)^2 Q^{(3)}$$

$$dQ^{(3)} = \mathrm{tr}(F \wedge F) \qquad u \in (U(1) \times U(1) \times U(1)) \backslash U(3)$$

- Lifting such harmonic superspace formulations to higher dimensions is tricky, but if it can be lifted, it would also rule out the $\mathrm{tr}(F^4)$ L=4, D=5 half-SUSY counterterm.

- One kind of invariant which would be allowed by 12 supercharge Feynman rules is known to occur in D=7, L=2 max. SYM and is of the generic form $\partial^2 \mathrm{tr}(F^4) + \mathrm{tr}(F^5)$. The full 16 supercharge on-shell invariant form of this counterterm is given by the full superspace integral of the Konishi operator $\mathrm{tr}(\bar{\phi}^{ij}\phi_{ij})$. (Marcus & Sagnotti)
  - For on-shell fields, the full superspace integral of this operator vanishes for abelian groups, but is non-vanishing for non-abelian groups.
- However, there are also known "double trace" D=7, L=2 counterterms of the general form $\partial^2(\mathrm{tr}(F^2))^2$. These require a more delicate treatment to preserve gauge invariance. (Marcus & Sagnotti)

Algebraic Renormalization

Dixon
Piguet & Sorella
Hennaux
Stora
Baulieu & Bossard

- Another approach to analyzing the divergences in supersymmetric gauge theories starts from the Callan-Symanzik equation for the renormalization of the Lagrangian as a operator insertion, e.g. governing mixing with the half-SUSY operator $S^{(4)} = \mathrm{tr}(F^4)$. Letting the classical action be $S^{(2)}$, the C-Z equation in dimension D is $\mu\frac{\partial}{\partial\mu}[S^{(2)}\cdot\Gamma] = (4-D)[S^{(2)}\cdot\Gamma] + \gamma_{(4)}g^{2n_{(4)}}[S^{(4)}\cdot\Gamma] + \cdots$, where $n_{(4)} = 4,\ 2,\ 1$ for $D = 5,\ 6,\ 8$.
- From this one learns that $(n_{(4)} - 1)\beta_{(4)} = \gamma_{(4)}$ so the beta function for the $S^{(4)} = \mathrm{tr}(F^4)$ operator is determined by the anomalous dimension $\gamma_{(4)}$.

- Combining the supersymmetry generator with a commuting spinor parameter to make a scalar operator $Q = \bar{\varepsilon}Q$, the expression of SUSY invariance for a D-form density in D-dimensions is $Q\mathcal{L}_D + d\mathcal{L}_{D-1} = 0$. Combining this with the SUSY algebra $Q^2 = -i(\bar{\varepsilon}\gamma^\mu\varepsilon)\partial_\mu$ and using the Poincaré Lemma, one finds $i_{i(\bar{\varepsilon}\gamma\varepsilon)}\mathcal{L}_D + S_{(Q)|\Sigma}\mathcal{L}_{D-1} + d\mathcal{L}_{D-2} = 0$.
- Hence one can consider the cocycles of the extended nilpotent differential $d + S_{(Q)|\Sigma} + i_{i(\bar{\varepsilon}\gamma\varepsilon)}$ acting on formal sums $\mathcal{L}_D + \mathcal{L}_{D-1} + \mathcal{L}_{D-2} + \cdots$.
- The supersymmetry Ward identities then imply that the whole cocycle is renormalized in a coherent way. In order for an operator like $S^{(4)}$ to mix with the classical action $S^{(2)}$, their cocycles need to have the same structure.

- To see better how the algebraic renormalization analysis works, consider once again the Wess-Zumino model. Consider the cocycle for a chiral superspace invariant $\int d^2\theta W(\phi,\bar{\phi})$ where $W(\phi,\bar{\phi})$ is either a chiral integrand $W(\phi)$ or something like $\phi\bar{D}^2\bar{\phi}$ for a kinetic term.

- In D=4, the top form in the W cocycle is the 4-form

$$\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}dx^\mu dx^\nu dx^\rho dx^\rho \; Q^\beta Q_\beta W(\phi,\bar{\phi})$$

- Applying $d+S_{(Q)|\Sigma}+i_{i(\bar{\varepsilon}\gamma\varepsilon)}$ to this, one gets first the 3-form $\frac{i}{3}\varepsilon_{\mu\nu\rho\sigma}dx^\mu dx^\nu dx^\rho \bar{\varepsilon}_{\dot{\alpha}}\sigma^{\sigma\dot{\alpha}\beta}Q_\beta W(\phi,\bar{\phi})$
and then the 2-form $\bar{\varepsilon}_{\dot{\alpha}}\sigma^{\dot{\alpha}}_{\mu\nu\dot{\beta}}\bar{\varepsilon}^{\dot{\beta}}dx^\mu dx^\nu W(\phi,\bar{\phi})$ after which the next cochain element becomes cohomologically trivial, $d\mathcal{L}_1=0$.

- Thus, if $W = W(\phi)$ for a superpotential correction, the cocycle consists of form degrees 4,3 and 2.
- On the other hand, if $W(\phi, \bar{\phi}) = \bar{Q}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}} F(\phi, \bar{\phi})$, *e.g.* $W(\phi, \bar{\phi}) = \phi \bar{D}^2 \bar{\phi}$ for a WZ kinetic term, then the cocycle actually ends a form degree earlier: adding the exact element obtained by $d + S_{(Q)|\Sigma} + i_{i(\bar{\varepsilon}\gamma\varepsilon)}$ acting on $i\bar{\varepsilon}_{\dot{\alpha}} \sigma_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}} Q^{\dot{\beta}} dx^{\mu} dx^{\nu} F + \bar{\varepsilon}_{\dot{\alpha}} \sigma_{\mu}^{\dot{\alpha}\beta} \varepsilon_{\beta} dx^{\mu} F$, the 2-form cochain element itself becomes cohomologically trivial, leaving then a cocycle with just form degrees 4 and 3.

- Now, the cocycle of the classical SYM Lagrangian density (viewed as a top form $\mathcal{L}_D$) admits only 5 form degrees, with the last one being proportional to the BPS composite operator $\mathrm{tr}(\phi^i\phi^j - \frac{1}{10-D}\delta^{ij}\phi_k\phi^k)$ whose half-superspace integral gives the on-shell action.

- On the other hand, the cocycle of the operator $S^{(4)}$ is longer, admitting non-trivial components of all form degrees.

Bossard, Howe & K.S.S. (WIP)

- Thus, the half-SUSY operator $S^{(4)} = \mathrm{tr}(F^4)$ cannot mix under renormalization with the classical action $S^{(2)}$.

- It is expected that the Konishi operator cocycle will pass these cohomological tests, however, and thus be admitted as a counterterm

- Thus, from the analysis of counterterms and their supersymmetry properties from a variety of points of view, the renormalization of max. SYM theory in dimensions 4 and higher agrees fully with all unitarity-based and earlier Feynman-diagram calculations.

- Similar agreement with known and anticipated unitarity calculational results are expected in supergravity.

Dimension D	10	8	7	6	5	4
Loop order L	1	1	2	3	6	∞
Gen. form	$\partial^2 F^4$	F^4	$\partial^2 F^4$	$\partial^2 F^4$	$\partial^2 F^4$	finite

Dimension D	11	10	8	7	6	5	4
Loop order L	2	2	1	2	3	4	5
Gen. form	$\partial^{12} R^4$	$\partial^{10} R^4$	R^4	$\partial^6 R^4$	$\partial^6 R^4$	$\partial^6 R^4$	$\partial^4 R^4$

- Despite the involved nature of some of the arguments, note that a simple overall picture remains possible: the highest operators that are *protected* against mixing with the classical action under renormalization are the half-SUSY SYM operator $\mathrm{tr}F^4$ and its supergravity counterpart R^4.

- So, what will be the final story for maximal supergravity: protection of up to the half-SUSY operators and then no more, *or* a series of truly miraculous D=4 cancelations to all orders? The question remains unresolved, but according to an old physics tradition, bets have been taken, for bottles of wine.

The key test will be in
$D = 5,\ L = 4$

Which will be the payoff?

or