Type II string effective action and UV behavior of maximal supergravities

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Plan

1. Review: Non-renormalization theorems for higher derivative couplings in 10 dimensions and implications for the UV behavior of N=8 supergravity.

2. The type II string effective action according to **L loop eleven dimensional supergravity.**

3. Conclusions

UV behavior of N=8 Supergravity from string theory

Consider the four-graviton amplitude in string theory at genus h.
By explicit calculation one finds (*)

$$genus\ 1: \quad A_4^{(1)} = (1 + O(\alpha' s))\, R^4$$

$$genus\ 2: \quad A_4^{(2)} = s^2\ (1 + O(\alpha' s))\, R^4$$

By using the pure spinor formalism, one further finds (+)

$$genus\ 3: \quad A_4^{(3)} = s^3\ (1 + O(\alpha' s))\, R^4$$

$$genus\ 4: \quad A_4^{(4)} = s^4\ (1 + O(\alpha' s))\, R^4$$

$$genus\ 5: \quad A_4^{(5)} = s^5\ (1 + O(\alpha' s))\, R^4$$

$$genus\ 6: \quad A_4^{(6)} = s^6\ (1 + O(\alpha' s))\, R^4$$

These results alone imply that there should not be any ultraviolet divergence in N=8 4d supergravity until at least ***9 loops****.*

They suggest the pattern:

$$genus\ h: \quad A_4^{(h)} = s^h\ (1 + O(\alpha' s))\, R^4$$

(*) [Green, Schwarz, Brink], [Green, Vanhove, 2000],[D'Hoker, Gutperle, Phong]
(+) [Berkovits, 2006]

Supergravity amplitudes from string theory

Genus h four-graviton amplitude in string theory:

$$A_4^h = \alpha'^{\beta_h - 1} e^{2(h-1)\phi} \quad s^{\beta_h} (1 + O(\alpha' s)) R^4$$

$$= \kappa_{(10)}^{2(h-1)} \alpha'^{3-4h+\beta_h} s^{\beta_h} (1 + O(\alpha' s)) R^4, \qquad \kappa_{(10)}^2 = \alpha'^4 e^{2\phi}$$

h-loop supergravity amplitude is obtained by taking the limit alpha' to zero.
The inverse string length is interpreted as a UV cutoff.

$$A_4^h = \kappa_{(10)}^{2(h-1)} \Lambda^{8h-6-2\beta_h} \quad s^{\beta_h} (1 + O(\alpha' s)) R^4$$

The presence of $S^\beta R^4$ means that the leading divergences Λ^{8h+2} of individual h-loop Feynman diagrams cancel and the UV divergence is reduced by a factor $\Lambda^{-8-2\beta}$

Maximal supergravity in lower dimensions:

Start with string loop amplitude on a 10-d torus with the radii = sqrt(alpha').

In the limit alpha' -> 0 all massive KK states, winding numbers and excited string states decouple.

In terms of the d-dimensional gravitational constant kappa_d^2 ,

$$A^h_{4,d} = \kappa^{2(h-1)}_{(d)} \Lambda^{(d-2)h-6-2\beta_h} \qquad s^{\beta_h} (1 + O(\alpha' s)) R^4 ,$$

$$\kappa_d^2 = \alpha'^{(d-2)/2} e^{2\phi} , \qquad radii = \sqrt{\alpha'}$$

Therefore UV divergences are absent in dimensions satisfying the bound

$$(d-2)h - 6 - 2\beta < 0 \quad \Rightarrow \quad d < 2 + \frac{2\beta_h + 6}{h}$$

KNOWN CASES:

- h = 1: beta_1=0 , therefore UV finite for d < 8
- h = 2: beta_2=2 therefore UV finite for d < 7
- h = 3: beta_3=3 [Bern, Carrasco, Dixon, Johansson, Kosower and Roiban 2007]. Therefore UV finite for d < 6
- h > 3: expected beta_h > or = 3. This gives **d < 2 + 12/h. Thus d = 4 finite up to h < 6.**

For h > 3 we use the non-renormalization theorems obtained in the pure spinor formalism [Berkovits, 2006].

One finds

- **$\text{beta}_h = h$ for h =2, 3, 4, 5 ,6**
- **beta_h > or = 6 for h > 5.**

Hence there would be no UVdivergences if

$$d < 2 + 18/h \quad , \quad h \geq 6$$

$$d < 4 + 6/h \quad , \quad h = 2,\ldots,5$$

d = 4:

$$4 < 2 + 18/h \quad \Rightarrow \quad h < 9$$

d = 4: UV finite at least up to h = 8

Now using IIA/M theory duality [Green, J.R., Vanhove, 2007] **:** $\text{beta}_h = h$ for all $h > 1$

Hence there would be no UV divergences if

$$d < 2 + \frac{2\beta_h + 6}{h} = 2 + \frac{2h + 6}{h} \Rightarrow \boxed{d < 4 + \frac{6}{h} \quad , \quad h > 1}$$

d = 4: UV FINITE FOR ALL h

Remarkably, this is the same condition as maximally supersymmetric Yang-Mils in d dimensions

[Bern, Dixon, Dunbar, Perelstein, Rozowsky, 1998], [Howe,Stelle, 2002]

Checked explicitly up to h = 3 [Bern, Carrasco, Dixon, Johansson, Kosower and Roiban 2007].

M-graviton amplitudes

The Berkovits non-renormalization theorems can easily be generalized to the M-graviton amplitude. This leads to the conditions for finiteness of an R^M term,

$$
\begin{array}{ll}
d < 4 + 6/h \quad , & 1 < h < 4 + M/2 \\
d < 2 + (14 + M)/2 \quad , & h \geq 4 + M/2
\end{array}
$$

In d = 4, this implies finiteness if h < 7 + M/2.

Duality arguments now show

$$
A_M^h = s^{h+4-M} R^M (1 + O(\alpha' s))
$$

Following previous analysis, this shows that there is no UV divergence in the M-graviton supergravity amplitude for

$$
d < 4 + 6/h \quad , \quad h > 1
$$

According to this, the d=4 N=8 supergravity theory would be completely finite in the UV.

Summary: loop order at which we expect the first divergence to occur in d dimensional Maximal supergravities

D	β = h, up to h=3 (explicit calculation)	β = h, up to h=5 (Berkovits theorems)	β = h, for all h (string dualities)
4	6	9	finite
5	4	6	6
6	3	3	3
7	2	2	2
8	1	1	1
9	1	1	1
10	1	1	1

Superspace arguments:
- Howe, Stelle: D = 4 is finite up to 6 loops
- Kallosh: D = 4 is finite up to 7 loops (constructs 8 loop counterterm).

NON-RENORMALIZATION THEOREMS IN STRING THEORY

Type II effective action: Consider the terms

$$R^4 \; , D^2 R^4 , D^4 R^4 \; , D^6 R^4 \; , \ldots , D^{2k} R^4$$

Terms involving **other supergravity fields**, or terms with the **same number of derivatives (like e.g. *R^m* or $|G|^{4n}R^4$ terms),** are expected to be connected to these terms by N=2 supersymmetry.

The statement that

$$A_4^{(h)} = s^h \left(1 + O(\alpha' s)\right) R^4$$
$$A_4^{(h+1)} = s^{h+1} \left(1 + O(\alpha' s)\right) R^4$$
$$\ldots$$

is equivalent to saying that

The term s^h R^4 = D^{2h} R^4 does not receive any quantum correction beyond genus h

(since $A^{(h+1)}$ starts contributing to s^{h+1} R^4)

What would be the expected contributions from M-theory?

String theory analogy: integrating out string excitations gives rise to an effective action of the general form:

$$S = \sum_{n=3}^{\infty} c_n \, T^{-n} \int d^{10}x \sqrt{-g}\, D^{2(n-3)} R^4 \quad , \qquad T = \frac{1}{2\pi\alpha'}$$

Similarly, in d=11 on **S**[1] one would expect that M2 brane excitations should give contributions of the form

$$S = \sum_{n,k\geq 0}^{\infty} c_{nk} \, T_2^{-n} R_{11}^{-m} \int d^{10}x \, R_{11} \sqrt{-g}\, D^{2k} R^4 \, , \qquad [T_2] = (l_P)^{-3}$$

$$8 + 2k + m = 3n + 11$$

Convert to type IIA variables: $ds^2 = R_{11}^{-1} ds_{IIA}^2 + R_{11}^2 (dx^{11} - C_\mu dx^\mu)^2 \, , \quad R_{11}^3 = g_A^2 \, l_P^3$

$$\Rightarrow S = \sum_{k=0}^{\infty} \sum_{n=0}^{k} \; c_{nk} \, g_s^{2(k-n-1)} \int d^{10}x \, (\sqrt{-g}\, D^{2k} R^4)_{IIA}$$

Genus: $\Rightarrow \quad h = k - n \leq k \quad !!$

For M5-branes, $[T_5] = L^{-6}$, leading to corrections **h = k – 2n.**

TYPE IIA EFFECTIVE ACTION FROM ELEVEN DIMENSIONS

$$A_4^L = const\, s^{\beta_L}(1 + O(s))R^4$$

$$S = \kappa_{11}^{2(L-1)} \Lambda^n R_{11}^{-m} \int d^{11}x \sqrt{-g}\, D^{2\beta_L + 2r} R^4 \quad , \qquad r = 0, 1, 2, \ldots$$

$$0 \le n \le 9L - 6 - 2\beta_L$$

Convert to type IIA variables:

$$S = \Lambda^n g_A^{2(k-3L+\frac{n}{3}+2)} \int d^{10}x \, (\sqrt{-g}\, D^{2k} R^4)_A$$

$$genus\ h : g_A^{2h-2} \rightarrow \qquad h = k + \frac{n}{3} - 3(L-1) \le k + 1 - \frac{2\beta_L}{3}, \quad if\ n > 0$$

$$h = k - 3(L-1) \le k \quad if \quad n = 0$$

Using that $S^2 R^4 = D^4 R^4$ is not renormalized beyond L=2

$$\beta_L \ge 2 \quad \longrightarrow \quad h < k - \frac{1}{3} \quad for \quad n > 0$$

$$so\ in\ general \quad h \le k$$

*Thus we find a maximum genus for every $D^{2k}R^4$. This is irrespective of the value of L. The highest genus contribution is **h = k** and comes from **L = 1**.*

The argument only uses power-counting and the relation between type IIA and M-theory parameters. **So the argument equally applies to all interactions of the same dimension as $D^{2k}R^4$, in particular, R^{4+k}, etc.**

Thus the highest-loop contribution for a term of dimension 2k+8, with k > 1, is genus k.

As an aside remark, note that this implies that the **M-theory effective action** can only contain terms with 6n+2 derivatives (e.g. $\mathbf{R^{3n+1}}$ or $\mathbf{D^{6n-6}R^4}$) [J.R. and Tseytlin, 1998]

$$S = \int d^{11}x \sqrt{-g} \left(R + c_1 R^4 + c_2 R^7 + c_3 R^{10} + \ldots \right)$$

Reason: a term $\mathbf{R^{4+k}}$ produces a dependence on the coupling $g_A^{2k/3}$. For terms where k is not a multiple of 3, one gets 1/3 powers of the coupling. Such dependence cannot possible arise if the genus expansion stops.

The type II string effective action according to

L loop eleven dimensional supergravity.

Regularize supergravity amplitudes by a cutoff $\Lambda = 1/l_P$

- Eleven-dimensional supergravity on S^1 = Type IIA

$$A(S,T,R_{11}) = A(s,t,g_s)$$

- Eleven-dimensional supergravity on T^2 = Type IIB

$$A_4(S,T,\Omega,V) = A_4(s,t,\Omega,r_B)$$
$$\Omega = \chi + i\exp(-\phi) \quad , \quad \Omega_2 = \exp(-\phi) = g_s^{-1}$$

Low momentum expansion gives analytic terms $S^k R^4$ and massless threshold terms s^k Log(s) R^4

TREE-LEVEL FOUR GRAVITON AMPLITUDE

$$A_4 = K\frac{1}{stu}\frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} = K\frac{1}{stu}\exp[2\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{2k+1}(s^{2k+1}+t^{2k+1}+u^{2k+1})]$$

$$= K\frac{1}{stu}(1+2\zeta(3)stu+\ldots)$$

$$K = \xi_1^{\mu\mu'}\xi_2^{\nu\nu'}\xi_3^{\rho\rho'}\xi_4^{\sigma\sigma'}K_{\mu\nu\rho\sigma}(k_i)K_{\mu'\nu'\rho'\sigma'}(k_i)\,, \quad K_{\mu\nu\rho\sigma}(k_i) = -\tfrac{1}{4}st\eta_{\mu\rho}\eta_{\nu\sigma}+\ldots$$

$$[K] \to "R^4"\,, \quad s+t+u=0\,, \quad s \to \frac{\alpha' s}{4}\,, \quad \alpha' = 4$$

Remarkably, it is a polynomial in two kinematical structures σ_2 and σ_3 :

$$A(\sigma_2,\sigma_3) = \sum_{p,q=0}^{\infty} T_{(p,q)}\sigma_2^p\sigma_3^q$$

$$\sigma_2 \equiv s^2+t^2+u^2\,, \quad \sigma_3 \equiv s^3+t^3+u^3 = 3\,stu$$

$$T_{(p,q)} = \sum c_k^{(p,q)}\,\zeta(r_1)\ldots\zeta(r_k)$$

The **tree-level four-graviton effective action** is

$$S = \int d^{10}x \sqrt{-g}\, e^{-2\phi} (R + \alpha'^3 2\zeta(3) R^4 + \alpha'^5 \zeta(5) D^4 R^4 + \alpha'^6 \tfrac{2}{3}\zeta(3)^2 D^6 R^4$$

$$+ \alpha'^7 \tfrac{1}{2}\zeta(7) D^8 R^4 + \alpha'^8 \tfrac{2}{3}\zeta(3)\zeta(5) D^{10} R^4$$

$$+ \alpha'^9 [\tfrac{1}{4}\zeta(9) D^{12} R^4 + \tfrac{2}{27}(2\zeta(3)^3 + \zeta(9)) \widetilde{D}^{12} R^4] + \cdots)$$

• In **type IIB**, each $D^{2k}R^4$ term in the exact effective action will contain, in addition, perturbative (higher genus) and non-perturbative corrections. For the exact coupling, the zeta functions must be replaced by a modular function of the SL(2,Z) duality group.

$$f_k(g_s) D^{2k} R^4 = (\frac{c_0}{g_s^2} + c_1 + c_2 g_s^2 + \ldots + (\text{nonpert}))\ D^{2k} R^4$$

•In **type IIA**, there are perturbative corrections only.

To the extent we checked, all coefficients in the higher genus amplitudes are always products of Riemann zeta functions.
(this includes e.g. up to genus 9 coefficients)

L = 1 eleven-dimensional supergravity amplitude on T^2

$$S^{L=1} = \int dx^9 r_B \sqrt{-g_{IIB}}\,(r_B^{-2}\Lambda^3 R^4 + Z_{3/2}(\Omega)R^4 + \sum_{k=2}^{\infty} c_k r_B^{-2k} Z_{k-1/2}(\Omega)\, D^{2k} R^4$$

$$(i.e.\quad \sum_{k=2}^{\infty} r_B^{-2k}(genus\,1 + genus\; k + \text{nonpert})D^{2k}R^4$$

where

$$(\Delta_\Omega - r(r-1))Z_r = 0\,, \qquad \Delta_\Omega = \Omega_2^2(\partial_{\Omega_1}{}^2 + \partial_{\Omega_2}{}^2)$$

$$Z_r(\Omega,\overline{\Omega}) = \sum_{(m,n)\neq(0,0)} \frac{\Omega_2^r}{|m+n\Omega|^{2r}}$$

$$= 2\zeta(2r)\Omega_2^r + \gamma_k \Omega_2^{1-r} + \sum_{n,w=1}^{\infty} c_{nw}^{(k)} \cos(2\pi wn\Omega_1)K_k(2\pi wn\Omega_2)$$

$$\Omega = \chi + i\exp(-\phi) \quad , \quad \Omega_2 = \exp(-\phi) = g_s^{-1}$$

[J.R., Tseylin, 1997], [Green, Kwon, Vanhove, 2000]

$\mathbf{L = 2}$ eleven-dimensional supergravity amplitude on T^2

$$A_4^{L=2}(s,t) = r_{IIB}\Big(g_s^{-\frac{1}{2}} Z_{5/2}(\Omega) D^4 R^4 + g_s E_{(3/2,3/2)}(\Omega) D^6 R^4 + \frac{g_s^2}{r_{IIB}^2} E(\Omega)\, D^8 R^4$$
$$+ \frac{g_s^3}{r_{IIB}^4} F(\Omega)\, D^{10} R^4 + \frac{g_s^4}{r_{IIB}^6} G^{(i)}(\Omega)\, D^{12} R^4 + \frac{g_s^4}{r_{IIB}^6} G^{(ii)}(\Omega)\, \tilde{D}^{12} R^4 \Big) + \ldots$$

$$E_{(\frac{3}{2},\frac{3}{2})}\ D^6 R^4 : \quad (\Delta_\Omega - 12) E_{(\frac{3}{2},\frac{3}{2})} = -6\, Z_{\frac{3}{2}}^{\ 2}$$

[Green, Vanhove, 2005]

$$E\ D^8 R^4 : \ E = E_0 + E_1 + E_2 + E_3$$
$$(\Delta_\Omega - 6) E_1 = Z_{\frac{3}{2}}\ Z_{\frac{1}{2}}$$
$$(\Delta_\Omega - 20) E_2 = Z_{\frac{3}{2}}\ Z_{\frac{1}{2}}$$
$$(\Delta_\Omega - 42) E_3 = Z_{\frac{3}{2}}\ Z_{\frac{1}{2}}$$

$$D^{10} R^4 : \quad F = F_0 + F_1 + F_2 + F_3 + F_4$$
$$(\Delta_\Omega - \lambda_j) F_j = Z_{\frac{3}{2}}\ Z_{\frac{3}{2}} + w_j Z_{\frac{1}{2}}\ Z_{\frac{1}{2}}$$
$$\lambda_j = 2, 12, 30, 56, 90$$

$$D^{12} R^4 : \quad G^{x,y} = G_0 + G_1 + G_2 + G_3 + G_4 + G_5 + G_6$$
$$(\Delta_\Omega - \lambda_j) G_j^x = f_j^x Z_{\frac{3}{2}}\ Z_{\frac{5}{2}} + c_j^x Z_{\frac{1}{2}}\ Z_{\frac{1}{2}}$$
$$\lambda_j = 2j(2j+1) = 0, 6, 20, 42, 72, 110, 156$$

E, F, $G^{(i)}$, $G^{(ii)}$ **are new** [Green, J.R.,Vanhove, june 2008]. The source terms that appeared could have been predicted using supersymmetry.

L LOOPS IN ELEVEN DIMENSIONS

$$S_L = S_L^{div} + S_L^{finite}$$

$$S_L^{div} = \sum_{m=[2k/3]_+}^{3L-3} \Lambda^{9L-6-3m}\, V^{k-\frac{3m}{2}} \int d^9x\, V\sqrt{-G}\; f_{(m,k)}^{(L)}(\Omega,\overline{\Omega}) D^{2k} R^4$$

$$= \sum_{m=[2k/3]_+}^{3L-3} \Lambda^{9L-6-3m} g_B^{k-\frac{m}{2}} r_B^{2m-2k-1} \int d^9x \sqrt{-g}\; f_{(m,k)}^{(L)}(\Omega,\overline{\Omega}) D^{2k} R^4$$

$$S_L^{finite} = V^{3+k-\frac{9L}{2}} \int d^9x\, V\sqrt{-G}\; f_{(k)}^{(L)}(\Omega,\overline{\Omega}) D^{2k} R^4$$

$$= g_B^{k+1-\frac{3L}{2}} r_B^{6L-2k-5} \int d^9x \sqrt{-g}\; f_{(k)}^{(L)}(\Omega,\overline{\Omega}) D^{2k} R^4$$

For L = 1 and L = 2, these expressions reproduce the several terms previously obtained by explicit calculation.

The terms that decompactify in ten dimensions are linear with r_B .

For the finite part, this is the term **k = 3L – 3.** For the divergent part, it is **m = k + 1**

L = 3 eleven-dimensional supergravity amplitude on T^2

This was not computed explicitly. Combining the general formula with some genus 1 data one gets

$$A_4^{L=3}(s,t) = r_B^5 \zeta(5) D^6 R^4 + (r_B g_B^{3/2} E_{7/2} + r_B^3 g_B E_X + r_B^5 g_B^{1/2} E_Y) D^8 R^4 + \ldots$$

Similarly, one can write down the expected L loop modular functions for general L.

Consider the IIB effective action in **nine** dimensions.

The modular group is SL(2,Z) x R^+ , where SL(2,Z) acts on the complexified coupling constant Ω and R^+ acts on the size of the circle r_B .

Its action is naturally defined on the dimensionless volume nu of the compactification manifold measured in the ten-dimensional Planck length unit

$$\nu \equiv g_B^{1/2} / r_B^2$$

$$S_{10d} = \int dx^{10} \sqrt{-g_E^{(10)}} \; (R - \tfrac{1}{2} \frac{\partial_\mu \Omega \partial^\mu \overline{\Omega}}{\Omega_2^2})$$

$$(e^{(10)})^r_\mu = \begin{pmatrix} \nu^{-\frac{1}{28}} e^r_\mu & \nu^{-\frac{1}{2}} e^9_\mu \\ 0 & \nu^{-\frac{1}{2}} \end{pmatrix}$$

$$S_{9d} = \int dx^9 \sqrt{-g_E^{(9)}} \; (R - \tfrac{2}{7} \nu^{-2} \partial_\mu \nu \partial^\mu \nu - \tfrac{1}{2} \frac{\partial_\mu \Omega \partial^\mu \overline{\Omega}}{\Omega_2^2})$$

$$\Rightarrow \Delta^{(9)} = \Delta_\Omega + \tfrac{7}{4} \nu^2 \partial_\nu^2 + \tfrac{9}{4} \nu \partial_\nu$$

TAKE GENERAL L LOOP FORMULA APPLIED TO R^4, D^4R^4 AND D^6R^4

$$R^4: \quad \int dx^9 \sqrt{-g_{IIB}}\; r_B\,(Z_{3/2}(\Omega) + r_B^{-2}\, 4\zeta(2))\; R^4 \equiv \int dx^9 \sqrt{-g_{IIB}}\; M_0(\Omega, r_B)\; R^4$$

(L = 1) **(L = 1) Λ³**

$$\left(\Delta^{(9)} - \tfrac{6}{7}\right) M_0 = 0$$

$$\Delta^{(9)} = \Delta_\Omega + \tfrac{7}{4}\nu^2\partial_\nu^2 + \tfrac{9}{4}\nu\partial_\nu\,, \quad \nu \equiv g_B^{1/2}/r_B^2$$

$$D^4R^4: \quad \int dx^9 \sqrt{-g_{IIB}}\; r_B\,(Z_{5/2}(\Omega) + r_B^2\, \tfrac{8}{15}\zeta(2)\zeta(3) + r_B^{-4}\, \tfrac{4}{15} Z_{3/2}(\Omega))\; D^4R^4$$

(L = 2) **(L = 2) Λ³** **(L = 1)**

$$\equiv \int dx^9 \sqrt{-g_{IIB}}\; M_4(\Omega, r_B) D^4R^4$$

$$\left(\Delta^{(9)} - \tfrac{30}{7}\right) M_4 = 0$$

$$D^6R^4: \quad \int dx^9 \sqrt{-g_{IIB}}\, r_B \left(E_{(\frac{3}{2},\frac{3}{2})}(\Omega) + r_B^{-2} Z_{3/2}(\Omega) + r_B^{-6}\frac{12}{63} Z_{5/2}(\Omega) + r_B^{4}\frac{24}{63}\zeta(5) + r_B^{-4}\frac{48}{5}\zeta(2)\right) D^6R^4$$

$$\equiv \int dx^9 \sqrt{-g_{IIB}}\, M_6(\Omega, r_B) D^6R^4$$

$$\left(\Delta^{(9)} - \tfrac{90}{7}\right) M_6 = -6 M_0{}^2$$

(L = 2) **(L = 2)** **(L = 1)** **(L = 3)Λ^3** **(L = 2)**

These results can be compared to the 8d modular function of R^4 [Kiritsis, Pioline, 1997] and the 8d modular functions of D^4R^4 and D^6R^4 [Basu, 2007]

Taking the 9d limit on these modular functions, we reproduce the above results.

The differential equations that determine the different modular functions also dictate that they contain a finite number of perturbative contributions.
Namely that the corresponding higher derivative coupling is not renormalized beyond a given genus

$$D^8R^4: \quad \int dx^9 \sqrt{-g_{IIB}}\, r_B \left(\tfrac{2}{315} r_B^{-8} Z_{7/2} + r_B^{-2} E(\Omega) + r_B^{-4} Z_{1/2} + Z_{7/2} + r_B^{2} E_X + r_B^{4} E_Y + r_B^{6} \frac{\zeta(7)}{525}\right) D^8R^4$$

(L = 1) (L = 2) (L = 2) (L = 3) (L = 3) (L = 3) (L = 4)

$$\equiv \int dx^9 \sqrt{-g_{IIB}}\, M_8(\Omega, r_B) D^8R^4$$

(?) $M_8(\Omega, r_B) = M_8^{(\lambda_1)}(\Omega, r_B) + \ldots + M_8^{(\lambda_n)}(\Omega, r_B)$

$\left(\Delta^{(9)} - \lambda_i\right) M_8^{(\lambda_i)} = ?$

Problem: presence of terms $O(\exp(-1/r_B))$ in genus one amplitude [GRV 2008].

This implies that the complete M_8 contains terms $O(\exp(-1/r_B))$.

These are difficult to calculate. But they could be generated automatically once the correct differential equation is known.

CONCLUSIONS

1. UV BEHAVIOR OF N = 8 SUPERGRAVITY

We have argued that for all $D^{2k}R^4$ couplings the perturbative expansion stops at genus k
(Agreement up to 18 derivatives with the non-renormalization theorems derived by Berkovits).

This seems to imply finiteness of N= 8 supergravity.

Conversely, finiteness of N = 8 supergravity would imply that all $D^{2k}R^4$ are not renormalized in string theory beyond some given genus.

2. TYPE II EFFECTIVE ACTION

With a few string theory inputs one can determine the full string-theory quantum S matrix just from loops in 11D supergravity.

New modular functions for higher derivative couplings up to $D^{12} R^4$. They obey Poisson equations with different eigenvalues.
It would be important to see if also modular functions arising from L = 3 supergravity are determined by differential equations having as source terms modular functions of lower derivative terms (this would give independent evidence on non-renormalization theorems).

General structure of modular functions arising from L loop supergravity (combining power counting and string dualities).
The *complete* modular function multiplying a given coupling R^4, D^4R^4 or D^6R^4, in nine dimensions, (which is a sum of several pieces originating from different loop orders in 11d supergravity) is determined by a simple differential equation.

Structure of differential equations expected from supersymmetry

$$\delta\Phi = (\delta^{(0)} + \alpha'^3 \delta^{(3)} + \alpha'^5 \delta^{(5)} + \ldots)\Phi$$
$$S = S^{(0)} + \alpha'^3 S^{(3)} + \alpha'^5 S^{(5)} + \ldots$$

$$\rightarrow 0 = \delta^{(6)} S^{(0)} + \delta^{(3)} S^{(3)} + \delta^{(0)} S^{(6)}$$

This leads to the Poisson equation defining $E_{(3/2,3/2)}$ with source term $Z_{3/2}Z_{3/2}$

Similarly, one can predict the source terms for E,F,Gi,Gii