Type II string effective action and UV behavior of maximal supergravities

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Based on work in coll. with M.B. Green and P. Vanhove,
Plan

1. Review: Non-renormalization theorems for higher derivative couplings in 10 dimensions and implications for the UV behavior of N=8 supergravity.

2. The type II string effective action according to L loop eleven dimensional supergravity.

3. Conclusions
Consider the four-graviton amplitude in string theory at genus $h$. By explicit calculation one finds (*)

$$
genus 1: \quad A_4^{(1)} = (1 + O(\alpha' s)) R^4$$

$$
genus 2: \quad A_4^{(2)} = s^2 (1 + O(\alpha' s)) R^4$$

By using the pure spinor formalism, one further finds (+)

$$
genus 3: \quad A_4^{(3)} = s^3 (1 + O(\alpha' s)) R^4$$

$$
genus 4: \quad A_4^{(4)} = s^4 (1 + O(\alpha' s)) R^4$$

$$
genus 5: \quad A_4^{(5)} = s^5 (1 + O(\alpha' s)) R^4$$

$$
genus 6: \quad A_4^{(6)} = s^6 (1 + O(\alpha' s)) R^4$$

These results alone imply that there should not be any ultraviolet divergence in N=8 4d supergravity until at least $9$ loops.

They suggest the pattern:

$$
genus h: \quad A_4^{(h)} = s^h (1 + O(\alpha' s)) R^4$$

(*) [Green, Schwarz, Brink], [Green, Vanhove, 2000],[D'Hoker, Gutperle, Phong]

(+) [Berkovits, 2006]
Supergravity amplitudes from string theory

Genus h four-graviton amplitude in string theory:

\[ A_4^h = \alpha'^{n-1} e^{2(h-1)\phi} s^{\beta_h} (1 + O(\alpha' s)) R^4 \]

\[ = \kappa_{(10)}^{2(h-1)} \alpha'^{3-4h+\beta} s^{\beta_h} (1 + O(\alpha' s)) R^4, \quad \kappa_{(10)}^2 = \alpha'^4 e^{2\phi} \]

h-loop supergravity amplitude is obtained by taking the limit \( \alpha' \) to zero. The inverse string length is interpreted as a UV cutoff.

\[ A_4^h = \kappa_{(10)}^{2(h-1)} \Lambda^{8h-6-2\beta_h} s^{\beta_h} (1 + O(\alpha' s)) R^4 \]

The presence of \( S^\beta R^4 \) means that the leading divergences \( \Lambda^{8h+2} \) of individual h-loop Feynman diagrams cancel and the UV divergence is reduced by a factor \( \Lambda^{-8-2\beta} \)
Maximal supergravity in lower dimensions:

Start with string loop amplitude on a 10-d torus with the radii = sqrt(alpha’).

In the limit alpha’ -> 0 all massive KK states, winding numbers and excited string states decouple.

In terms of the d-dimensional gravitational constant \( \kappa_d^2 \),

\[
A_{4,d}^h = \kappa_{(d)}^{2(h-1)} \Lambda^{(d-2)h-6-2\beta_h} s^{\beta_h} (1 + O(\alpha' s)) R^4,
\]

\[
\kappa_d^2 = \alpha'^{(d-2)/2} e^{2\phi}, \quad \text{radii} = \sqrt{\alpha'}
\]

Therefore UV divergences are absent in dimensions satisfying the bound

\[
(d - 2)h - 6 - 2\beta < 0 \quad \Rightarrow \quad d < 2 + \frac{2\beta_h + 6}{h}
\]

**KNOWN CASES:**

- \( h = 1 \): beta_1=0 , therefore UV finite for \( d < 8 \)
- \( h = 2 \): beta_2=2 therefore UV finite for \( d < 7 \)
- \( h = 3 \): beta_3=3 [Bern, Carrasco, Dixon, Johansson, Kosower and Roiban 2007]. Therefore UV finite for \( d < 6 \)
- \( h > 3 \): expected beta_h > or = 3. This gives \( d < 2 + 12/h \). Thus \( d = 4 \) finite up to \( h < 6 \).
For $h > 3$ we use the non-renormalization theorems obtained in the pure spinor formalism [Berkovits, 2006].

One finds

- $\beta_h = h \text{ for } h = 2, 3, 4, 5, 6$
- $\beta_h \geq 6 \text{ for } h > 5.$

Hence there would be no UV divergences if

$$d < 2 + \frac{18}{h}, \quad h \geq 6$$
$$d < 4 + \frac{6}{h}, \quad h = 2, \ldots, 5$$

$d = 4:$

$$4 < 2 + \frac{18}{h} \Rightarrow h < 9$$

$d = 4$: UV finite at least up to $h = 8$
Now using IIA/M theory duality [Green, J.R., Vanhove, 2007]: \( \beta_h = h \) for all \( h > 1 \)

Hence there would be no UV divergences if

\[
d < 2 + \frac{2\beta_h + 6}{h} = 2 + \frac{2h + 6}{h} \quad \Rightarrow \quad d < 4 + \frac{6}{h}, \quad h > 1
\]

\[
d = 4: \text{UV FINITE FOR ALL } h
\]

Remarkably, this is the same condition as maximally supersymmetric Yang-Mils in \( d \) dimensions

[Bern, Dixon, Dunbar, Perelstein, Rozowsky, 1998], [Howe, Stelle, 2002]

Checked explicitly up to \( h = 3 \) [Bern, Carrasco, Dixon, Johansson, Kosower and Roiban 2007].
M-graviton amplitudes

The Berkovits non-renormalization theorems can easily be generalized to the M-graviton amplitude. This leads to the conditions for finiteness of an $R^M$ term,

\[
d < 4 + 6 / h \quad , \quad 1 < h < 4 + M / 2 \\
d < 2 + (14 + M) / 2 \quad , \quad h \geq 4 + M / 2
\]

In $d = 4$, this implies finiteness if $h < 7 + M/2$.

Duality arguments now show

\[
A_M^h = s^{h+4-M} R^M (1 + O(\alpha' s))
\]

Following previous analysis, this shows that there is no UV divergence in the M-graviton supergravity amplitude for

\[
d < 4 + 6 / h \quad , \quad h > 1
\]

According to this, the $d=4 \ N=8$ supergravity theory would be completely finite in the UV.
Summary: loop order at which we expect the first divergence to occur in d dimensional Maximal supergravities

<table>
<thead>
<tr>
<th>D</th>
<th>$\beta = h$, up to $h=3$ (explicit calculation)</th>
<th>$\beta = h$, up to $h=5$ (Berkovits theorems)</th>
<th>$\beta = h$, for all $h$ (string dualities)</th>
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Superspace arguments:
- Howe, Stelle: $D = 4$ is finite up to 6 loops
- Kallosh: $D = 4$ is finite up to 7 loops (constructs 8 loop counterterm).
Type II effective action: Consider the terms

\[ R^4, D^2R^4, D^4R^4, D^6R^4, \ldots, D^{2k}R^4 \]

Terms involving other supergravity fields, or terms with the same number of derivatives (like e.g. \( R^m \) or \( |G|^{4n}R^4 \) terms), are expected to be connected to these terms by N=2 supersymmetry.

The statement that

\[
A_4^{(h)} = s^h \left( 1 + O(\alpha' s) \right) R^4
\]

\[
A_4^{(h+1)} = s^{h+1} \left( 1 + O(\alpha' s) \right) R^4
\]

...  

is equivalent to saying that

The term \( s^h R^4 = D^{2h} R^4 \) does not receive any quantum correction beyond genus \( h \)

(since \( A^{(h+1)} \) starts contributing to \( s^{h+1} R^4 \))
What would be the expected contributions from M-theory?

String theory analogy: integrating out string excitations gives rise to an effective action of the general form:

\[
S = \sum_{n=3}^{\infty} c_n T^{-n} \int d^{10}x \sqrt{-g} D^{2(n-3)} R^4, \quad T = \frac{1}{2\pi \alpha'}
\]

Similarly, in d=11 on S^1 one would expect that M2 brane excitations should give contributions of the form

\[
S = \sum_{n,k \geq 0} c_{nk} T_2^{-n} R_{11}^{-m} \int d^{10}x R_{11} \sqrt{-g} D^{2k} R^4, \quad [T_2] = (l_P)^{-3}
\]

\[8 + 2k + m = 3n + 11\]

Convert to type IIA variables:

\[
ds^2 = R_{11}^{-1} ds_{IIA}^2 + R_{11}^2 (dx^{11} - C_\mu dx^\mu)^2, \quad R_{11}^3 = g_A^2 l_P^3
\]

\[
\Rightarrow S = \sum_{k=0}^{\infty} \sum_{n=0}^{k} c_{nk} g_s^{2(k-n-1)} \int d^{10}x \left( \sqrt{-g} D^{2k} R^4 \right)_{IIA}
\]

Genus: \[\Rightarrow h = k - n \leq k \quad !!\]

For M5-branes, \([T_5] = L^{-6}\), leading to corrections \(h = k - 2n\).
TYPE IIA EFFECTIVE ACTION FROM ELEVEN DIMENSIONS

\[ A_4^L = \text{const} \, s^{\beta_L} \, (1 + O(s)) R^4 \]

\[ S = \kappa_{11}^{2(L-1)} \Lambda^n R_{11}^{-m} \int d^{11} x \sqrt{-g} \, D^{2\beta_k + 2r} R^4 , \quad r = 0, 1, 2, \ldots \]

\[ 0 \leq n \leq 9L - 6 - 2\beta_L \]

Convert to type IIA variables:

\[ S = \Lambda^n g_A^{2(k - 3L + \frac{n}{3} + 2)} \int d^{10} x (\sqrt{-g} \, D^{2k} R^4)_A \]

Genus \( h : g_A^{2h-2} \rightarrow \)

\[ h = k + \frac{n}{3} - 3(L - 1) \leq k + 1 - \frac{2 \beta_L}{3} , \quad \text{if} \ n > 0 \]

\[ h = k - 3(L - 1) \leq k \quad \text{if} \quad n = 0 \]

Using that \( S^2 R^4 = D^4 R^4 \) is not renormalized beyond \( L=2 \)

\[ \beta_L \geq 2 \quad \rightarrow \quad h < k - \frac{1}{3} \quad \text{for} \quad n > 0 \]

so in general \( h \leq k \)

\( \text{Thus we find a maximum genus for every} \ D^{2k} R^4. \text{This is irrespective of the value of} \ L. \text{The highest genus contribution is} \ h = k \text{ and comes from} \ L = 1. \)
The argument only uses power-counting and the relation between type IIA and M-theory parameters. So the argument equally applies to all interactions of the same dimension as D$^{2k}R^4$, in particular, R$^{4+k}$, etc.

Thus the highest-loop contribution for a term of dimension 2$k$+8, with $k > 1$, is genus $k$.

As an aside remark, note that this implies that the M-theory effective action can only contain terms with 6$n$+2 derivatives (e.g. $R^{3n+1}$ or $D^{6n-6}R^4$) [J.R. and Tseytlin, 1998]

$$S = \int d^{11}x \sqrt{-g} \left( R + c_1 R^4 + c_2 R^7 + c_3 R^{10} + \ldots \right)$$

**Reason:** a term R$^{4+k}$ produces a dependence on the coupling g$_A^{2k/3}$. For terms where $k$ is not a multiple of 3, one gets 1/3 powers of the coupling. Such dependence cannot possible arise if the genus expansion stops.
The type II string effective action according to

**L loop eleven dimensional supergravity.**

Regularize supergravity amplitudes by a cutoff \( \Lambda = 1/l_p \)

- Eleven-dimensional supergravity on \( S^1 = \) Type IIA

\[
A(S, T, R_{11}) = A(s, t, g_s)
\]

- Eleven-dimensional supergravity on \( T^2 = \) Type IIB

\[
A_4(S, T, \Omega, V) = A_4(s, t, \Omega_2, r_B)
\]

\[
\Omega = \chi + i \exp(-\phi), \quad \Omega_2 = \exp(-\phi) = g_s^{-1}
\]

Low momentum expansion gives analytic terms \( S^k R^4 \) and massless threshold terms \( s^k \log(s) R^4 \)
TREE-LEVEL FOUR GRAVITON AMPLITUDE

\[ A_4 = K \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} = K \frac{1}{stu} \exp \left[ 2 \sum_{k=1}^{\infty} \frac{\xi(2k+1)}{2k+1} (s^{2k+1} + t^{2k+1} + u^{2k+1}) \right] \]

\[ = K \frac{1}{stu} (1 + 2\xi(3)stu + ...) \]

\[ K = \xi_1^{\mu\nu} \xi_2^{\rho\sigma} \xi_3^{\nu\sigma} \xi_4^{\rho\nu} K_{\mu\nu\rho\sigma} (k_i) K_{\mu\nu\rho\sigma} (k_i), \quad K_{\mu\nu\rho\sigma} (k_i) = -\frac{1}{4} st \eta_{\mu\rho} \eta_{\nu\sigma} + ... \]

\[ [K] \rightarrow "R^4", \quad s + t + u = 0, \quad s \rightarrow \frac{\alpha' s}{4}, \quad \alpha' = 4 \]

Remarkably, it is a polynomial in two kinematical structures \( \sigma_2 \) and \( \sigma_3 \):

\[ A(\sigma_2, \sigma_3) = \sum_{p,q=0}^{\infty} T_{(p,q)} \sigma_2^p \sigma_3^q \]

\[ \sigma_2 \equiv s^2 + t^2 + u^2, \quad \sigma_3 \equiv s^3 + t^3 + u^3 = 3 stu \]

\[ T_{(p,q)} = \sum_k c_k^{(p,q)} \xi(r_1) ... \xi(r_k) \]
The tree-level four-graviton effective action is

\[ S = \int d^{10}x \sqrt{-g} \ e^{-2\phi} (R + \alpha'^3 2\zeta(3)R^4 + \alpha'^5 \zeta(5)D^4R^4 + \alpha'^6 \frac{2}{3}\zeta(3)^2D^6R^4 \\
+ \alpha'^7 \frac{1}{2}\zeta(7)D^8R^4 + \alpha'^8 \frac{2}{3}\zeta(3)\zeta(5)D^{10}R^4 \\
+ \alpha'^9 \left[ \frac{1}{4}\zeta(9)D^{12}R^4 + \frac{2}{27}(2\zeta(3)^3 + \zeta(9))\tilde{D}^{12}R^4 \right] + \cdots) \]

- In type IIB, each \( D^{2k}R^4 \) term in the exact effective action will contain, in addition, perturbative (higher genus) and non-perturbative corrections. For the exact coupling, the zeta functions must be replaced by a modular function of the SL(2,Z) duality group.

\[ f_k(g_s)D^{2k}R^4 = \left( \frac{c_0}{g_s^2} + c_1 + c_2g_s^2 + \ldots + \text{(nonpert)} \right) \ D^{2k}R^4 \]

- In type IIA, there are perturbative corrections only.

To the extent we checked, all coefficients in the higher genus amplitudes are always products of Riemann zeta functions. (this includes e.g. up to genus 9 coefficients)
L = 1 eleven-dimensional supergravity amplitude on $T^2$

\[ S^{L=1} = \int dx^9 r_B \sqrt{-g_{IIB}} \left( r_B^{-2} \Lambda^3 R^4 + Z_{3/2}(\Omega) R^4 + \sum_{k=2}^{\infty} c_k r_B^{-2k} Z_{k-1/2}(\Omega) D^{2k} R^4 \right) \]

(i.e. \[ \sum_{k=2}^{\infty} r_B^{-2k} (\text{genus } 1 + \text{genus } k + \text{nonpert}) D^{2k} R^4 \])

where

\[ (\Delta_\Omega - r(r-1)) Z_r = 0 , \quad \Delta_\Omega = \Omega_2^2 (\partial_{\Omega_1}^2 + \partial_{\Omega_2}^2) \]

\[ Z_r(\Omega, \overline{\Omega}) = \sum_{(m,n) \neq (0,0)} \frac{\Omega_2^r}{m + n\Omega}^{2r} \]

\[ = 2\zeta(2r)\Omega_2^r + \gamma_k \Omega_2^{1-r} + \sum_{n,w=1}^{\infty} c^{(k)}_{nw} \cos(2\pi nw\Omega_1) K_k(2\pi nw\Omega_2) \]

\[ \Omega = \chi + i \exp(-\phi) , \quad \Omega_2 = \exp(-\phi) = g^{-1}_s \]

[J.R., Tseylin, 1997], [Green, Kwon, Vanhove, 2000]
L = 2 eleven-dimensional supergravity amplitude on $T^2$

\[ A_{4}^{L=2}(s, t) = r_{II} \left( g_{s}^{-\frac{1}{2}} Z_{5/2}(\Omega) D^{4} R^{4} + g_{s} E_{(3/2, 3/2)}(\Omega) D^{6} R^{4} + \frac{g_{s}^{2}}{r_{II}^{2}} E(\Omega) D^{8} R^{4} \right. \]
\[ \left. + \frac{g_{s}^{3}}{r_{II}^{4}} F(\Omega) D^{10} R^{4} + \frac{g_{s}^{4}}{r_{II}^{6}} G^{(i)}(\Omega) D^{12} R^{4} + \frac{g_{s}^{4}}{r_{II}^{6}} G^{(ii)}(\Omega) \tilde{D}^{12} R^{4} \right) + \ldots \]

\[
\begin{align*}
E_{(\frac{1}{2}, \frac{1}{2})} D^{6} R^{4} : & \quad (\Delta_{\Omega} - 12) E_{(\frac{3}{2}, \frac{3}{2})} = -6 Z_{\frac{3}{2}}^{2} \\
E D^{8} R^{4} : & \quad E = E_{0} + E_{1} + E_{2} + E_{3} \\
(\Delta_{\Omega} - 6) E_{1} = & \quad Z_{\frac{3}{2}} Z_{\frac{1}{2}} \\
(\Delta_{\Omega} - 20) E_{2} = & \quad Z_{\frac{3}{2}} Z_{\frac{1}{2}} \\
(\Delta_{\Omega} - 42) E_{3} = & \quad Z_{\frac{3}{2}} Z_{\frac{1}{2}} \\
\end{align*}
\]

\[
\begin{align*}
D^{10} R^{4} : & \quad F = F_{0} + F_{1} + F_{2} + F_{3} + F_{4} \\
(\Delta_{\Omega} - \lambda_{j}) F_{j} = & \quad Z_{\frac{1}{2}} Z_{\frac{1}{2}} + w_{j} Z_{\frac{1}{2}} Z_{\frac{1}{2}} \\
\lambda_{j} = & \quad 2, 12, 30, 56, 90 \\
\end{align*}
\]

\[
\begin{align*}
D^{12} R^{4} : & \quad G^{x,y} = G_{0} + G_{1} + G_{2} + G_{3} + G_{4} + G_{5} + G_{6} \\
(\Delta_{\Omega} - \lambda_{j}) G_{x}^{j} = & \quad f_{j}^{x} Z_{\frac{1}{2}} Z_{\frac{1}{2}} + c_{j}^{x} Z_{\frac{1}{2}} Z_{\frac{1}{2}} \\
\lambda_{j} = & \quad 2 j (2 j + 1) = 0, 6, 20, 42, 72, 110, 156 \\
\end{align*}
\]

**E, F, G^{(i)}, G^{(ii)} are new** [Green, J.R., Vanhove, June 2008]. The source terms that appeared could have been predicted using supersymmetry.
L LOOPS IN ELEVEN DIMENSIONS

\[ S_L = S_L^{\text{div}} + S_L^{\text{finite}} \]

\[ S_L^{\text{div}} = \sum_{m=[2k/3]}^{3L-3} \Lambda^{9L-6-3m} V^{k-m} \int d^9 x V \sqrt{-G} f^{(L)}_{(m,k)} (\Omega, \bar{\Omega}) D^{2k} R^4 \]

\[ = \sum_{m=[2k/3]}^{3L-3} \Lambda^{9L-6-3m} g_B^{k-m} r_B^{2m-2k-1} \int d^9 x \sqrt{-g} f^{(L)}_{(m,k)} (\Omega, \bar{\Omega}) D^{2k} R^4 \]

\[ S_L^{\text{finite}} = V^{3+k-\frac{3L}{2}} \int d^9 x V \sqrt{-G} f^{(L)}_{(k)} (\Omega, \bar{\Omega}) D^{2k} R^4 \]

\[ = g_B^{k+1-\frac{3L}{2}} r_B^{6L-2k-5} \int d^9 x \sqrt{-g} f^{(L)}_{(k)} (\Omega, \bar{\Omega}) D^{2k} R^4 \]

For \( L = 1 \) and \( L = 2 \), these expressions reproduce the several terms previously obtained by explicit calculation.

The terms that decompactify in ten dimensions are linear with \( r_B \).

For the finite part, this is the term \( k = 3L - 3 \). For the divergent part, it is \( m = k + 1 \).
This was not computed explicitly. Combining the general formula with some genus 1 data one gets

\[
A_4^{L=3}(s, t) = r_B^5 \zeta(5) D^6 R^4 + (r_B g_B^{3/2} E_{7/2} + r_B^3 g_B E_X + r_B^5 g_B^{1/2} E_Y) D^8 R^4 + \ldots
\]

Similarly, one can write down the expected L loop modular functions for general L.
Consider the IIB effective action in nine dimensions.

The modular group is $\text{SL}(2,\mathbb{Z}) \times \mathbb{R}^+$, where $\text{SL}(2,\mathbb{Z})$ acts on the complexified coupling constant $\Omega$ and $\mathbb{R}^+$ acts on the size of the circle $r_B$.

Its action is naturally defined on the dimensionless volume $\nu$ of the compactification manifold measured in the ten-dimensional Planck length unit

$$\nu \equiv g_B^{1/2} / r_B^2$$

$$S_{10d} = \int dx^{10} \sqrt{-g^{(10)}_E} \left( R - \frac{1}{2} \frac{\partial_\mu \Omega \partial^{\mu} \overline{\Omega}}{\Omega^2} \right)$$

$$\left( e^{(10)} \right)^r_\mu = \begin{pmatrix} \nu^{-\frac{1}{28}} e^r_\mu & \nu^{-\frac{1}{2}} e^9_\mu \\ 0 & \nu^{-\frac{1}{2}} \end{pmatrix}$$

$$S_{9d} = \int dx^9 \sqrt{-g^{(9)}_E} \left( R - \frac{2}{7} \nu^{-2} \partial_\mu \nu \partial^{\mu} \nu - \frac{1}{2} \frac{\partial_\mu \Omega \partial^{\mu} \overline{\Omega}}{\Omega^2} \right)$$

$$\Rightarrow \Delta^{(9)} = \Delta_\Omega + \frac{7}{4} \nu^2 \partial_\nu^2 + \frac{9}{4} \nu \partial_\nu$$
TAKE GENERAL L LOOP FORMULA APPLIED TO $R^4$, $D^4R^4$ AND $D^6R^4$

$$R^4 : \int dx^9 \sqrt{-g_{IIB}} \ r_B (Z_{3/2} (\Omega) + r_B^{-2} 4 \zeta (2)) \ R^4 \equiv \int dx^9 \sqrt{-g_{IIB}} \ M_0 (\Omega, r_B) \ R^4$$

$$\left( \Lambda^{(9)} - \frac{6}{7} \right) M_0 = 0$$

$$\Delta^{(9)} = \Delta_{\Omega} + \frac{7}{4} \nu^2 \partial^2_\nu + \frac{9}{4} \nu \partial_\nu \ , \ \nu \equiv g_B^{1/2} / r_B^2$$

$$D^4 R^4 : \int dx^9 \sqrt{-g_{IIB}} \ r_B (Z_{5/2} (\Omega) + r_B^{-2} \frac{8}{15} \zeta (2) \zeta (3) + r_B^{-4} \frac{4}{15} Z_{3/2} (\Omega)) \ D^4 R^4$$

$$\equiv \int dx^9 \sqrt{-g_{IIB}} \ M_4 (\Omega, r_B) D^4 R^4$$

$$\left( \Delta^{(9)} - \frac{30}{7} \right) M_4 = 0$$
These results can be compared to the 8d modular function of $R^4$ [Kiritsis, Pioline, 1997] and the 8d modular functions of $D^4R^4$ and $D^6R^4$ [Basu, 2007].

Taking the 9d limit on these modular functions, we reproduce the above results.

The differential equations that determine the different modular functions also dictate that they contain a finite number of perturbative contributions. Namely that the corresponding higher derivative coupling is not renormalized beyond a given genus.
\[ D^8 R^4 : \int dx^9 \sqrt{-g_{IB}} \ r_B \left( \frac{2}{315} r_B^{-8} Z_{7/2} + r_B^{-2} E(\Omega) + r_B^{-4} Z_{1/2} + Z_{7/2} + r_B^2 E_X + r_B^4 E_Y + r_B^6 \frac{\zeta(7)}{525} \right) D^8 R^4 \]

\[ \equiv \int dx^9 \sqrt{-g_{IB}} \ M_8(\Omega, r_B) D^8 R^4 \]

\[ (\Delta^9 - \lambda_i) M_8^{(\lambda_i)} = ? \]

Problem: presence of terms \( O(\exp(-1/r_B)) \) in genus one amplitude \[\text{[GRV 2008]}\].

This implies that the complete \( M_8 \) contains terms \( O(\exp(-1/r_B)) \).

These are difficult to calculate. But they could be generated automatically once the correct differential equation is known.
CONCLUSIONS

1. UV BEHAVIOR OF N = 8 SUPERGRAVITY
   We have argued that for all $D^{2k}R^4$ couplings the perturbative expansion stops at genus $k$ (Agreement up to 18 derivatives with the non-renormalization theorems derived by Berkovits).

   This seems to imply finiteness of $N=8$ supergravity.

   Conversely, finiteness of $N = 8$ supergravity would imply that all $D^{2k}R^4$ are not renormalized in string theory beyond some given genus.

2. TYPE II EFFECTIVE ACTION
   With a few string theory inputs one can determine the full string-theory quantum $S$ matrix just from loops in 11D supergravity.

   New modular functions for higher derivative couplings up to $D^{12}R^4$. They obey Poisson equations with different eigenvalues.
   It would be important to see if also modular functions arising from $L = 3$ supergravity are determined by differential equations having as source terms modular functions of lower derivative terms (this would give independent evidence on non-renormalization theorems).

   General structure of modular functions arising from $L$ loop supergravity (combining power counting and string dualities).
   The complete modular function multiplying a given coupling $R^4$, $D^4R^4$ or $D^6R^4$, in nine dimensions, (which is a sum of several pieces originating from different loop orders in 11d supergravity) is determined by a simple differential equation.
Structure of differential equations expected from supersymmetry

\[ \delta \Phi = (\delta^{(0)} + \alpha_1^{13} \delta^{(3)} + \alpha_5^{15} \delta^{(5)} + \ldots) \Phi \]
\[ S = S^{(0)} + \alpha_1^{13} S^{(3)} + \alpha_5^{15} S^{(5)} + \ldots \]

\[ \rightarrow 0 = \delta^{(6)} S^{(0)} + \delta^{(3)} S^{(3)} + \delta^{(0)} S^{(6)} \]

This leads to the Poisson equation defining \( E_{(3/2,3/2)} \) with source term \( Z_{3/2} Z_{3/2} \)

Similarly, one can predict the source terms for \( E,F,G_i,G_{ii} \)