

Scattering amplitudes in N=4 super-Yang-Mills at weak and strong coupling

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Outline

- Scattering and Wilson loops: a butchered summary and preview
quantum corrections at strong coupling
- Scattering at weak coupling
structure of amplitudes
perturbation theory and unitarity
symmetries
cues from strong coupling; large number of legs
- 6-points at 2-loops
- BDS vs. Amplitude (vs. DHKS/BHT) and the remainder function
- Outlook

Scattering amplitudes at strong coupling from AdS/CFT

Alday, Maldacena

- use T -duality to relate vertex operators to extended strings
 - use warping of AdS to argue that large momentum approx. is exact
- ◇ mathematically – same calculation as for vev of Wilson loops
- ↳ minimal surfaces with prescribed boundary conditions
 - ↳ WL = closed polygon w/ light-like edges
 - ↳ regularization is required
- ◇ $\frac{1}{\sqrt{\lambda}}$ expansion = perturbation around the minimal surface
- ◇ Relevant action (in absence of regularization)

Kallosh, Tseytlin

$$S_E = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \left[\frac{1}{z^2} (\partial^a x^m \partial_a x_m + \partial^a z^s \partial_a z^s) + 4\epsilon^{ab} \bar{\theta} (\partial_a x^m \Gamma_m + \partial_a z^s \Gamma_s) \partial_b \theta \right]$$

- Known result(s) in $\mathcal{N} = 4$: min. surface ending on 4-sided polygon
 - higher polygons resist despite attempts

Alday, Maldacena

- Conjectured dual of gauge theory dim. reg.:

$$AdS_5 \times S^5 \mapsto AdS_{5-2\epsilon} \times S^{5+2\epsilon}; \text{ extra scale}$$

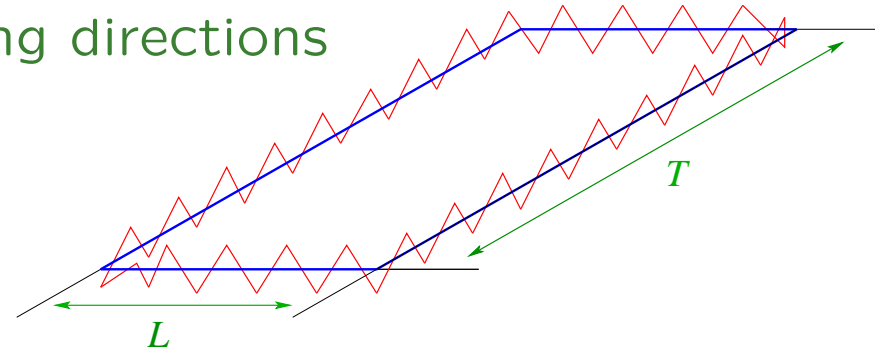
- Some features of $\frac{1}{\sqrt{\lambda}}$ corrections

Kruczenski, RR, Tirziu, Tseytlin

- Not clear how to construct the complete worldsheet action
- At tree-level, regularization is due to nontrivial conformal factor in induced metric; 1-loop Z is insensitive to it \mapsto no regularization!
- Need worldsheet solution to all orders in ϵ
- gauge theory: $\ln Z$ should contain $\ln \lambda \leftrightarrow$ ws perturbation theory?
- other regulators depart from gauge theory information
- Stick to tree level...

◇ In general – difficult problem simplifies for large number of legs

- many gluons moving in alternating directions
- Further approximation: $T \gg L$;
find leading term in T/L
- Space-like rectangle



◇ Leading order: $q\bar{q}$ potential multiplied by distance

$$\ln\langle W \rangle = \sqrt{\lambda} \frac{4\pi^2}{\Gamma\left(\frac{1}{4}\right)^4} \frac{T}{L} \quad \lambda \gg 1$$

- dependence on T/L suggests that it arises from nontrivial function
- other L and T dependence: $L + T$, LT together with regulator

Weak coupling implications: **New conjecture**

$$\ln \frac{A_n^{\text{MHV}}}{A_n^{\text{tree, MHV}}} = \ln \langle W_n \rangle + \mathcal{O}(\epsilon)$$

order by order in perturbation theory

Drummond, Henn, Korchemsky, Sokatchev

Brandhuber, Heslop, Travaglini

◇ Evidence:

- 4-points at 1-loop

Drummond, Korchemsky, Sokatchev

- n-points at 1-loop

Brandhuber, Heslop, Travaglini

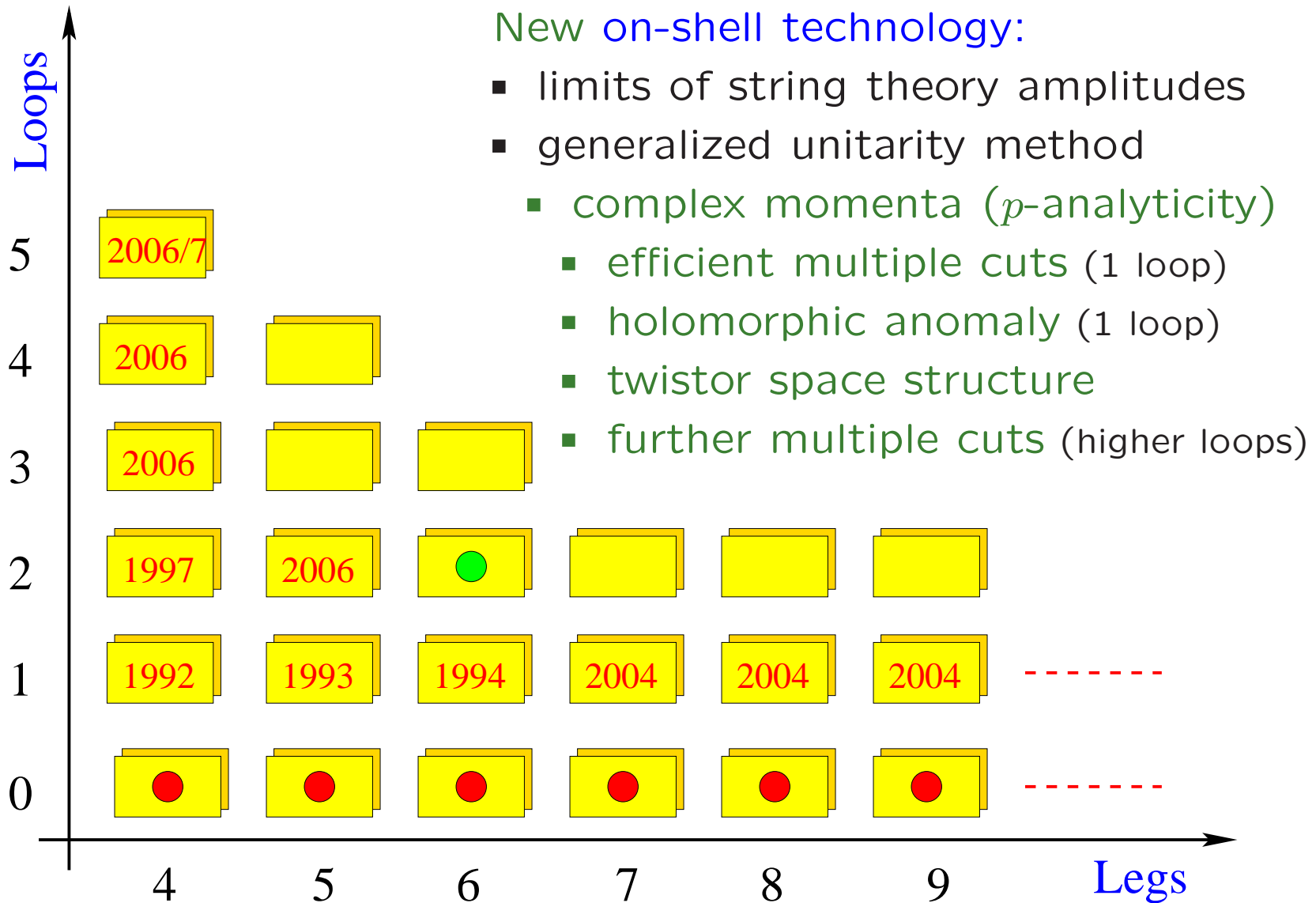
holds in any gauge theory!

- 4- and 5-points at 2-loop

Drummond, Henn, Korchemsky, Sokatchev

- How far does it go? Why does it work at all?

Current perturbative analytic results in $\mathcal{N} = 4$ SYM



Properties of scattering amplitudes

- General grounds: leading IR pole at L loops is $\frac{1}{\epsilon^{2L}}$

overlap of

- soft : $\int \frac{d\omega}{\omega^{1+\epsilon}} \propto \frac{1}{\epsilon}$
- collinear : $\int \frac{dk_{\top}}{k_{\top}^{1+\epsilon}} \propto \frac{1}{\epsilon}$

- IR pole structure (including subleading poles) is predicted by the soft/collinear factorization and exponentiation theorem

- Extensive QCD literature: Akhoury (1979), Mueller (1979), Collins (1980), Sen (1981), Sterman (1987), Botts, Sterman (1989), Catani, Trentadue (1989), Korchemsky (1989), Magnea, Sterman (1990), Korchemsky, Marchesini (1992), Catani (1998), Sterman, Tejada-Yeomans (2002)

- Simplifications at large N : IR in terms of $\beta(\lambda)$, cusp anomaly $\gamma_K(\lambda)$ or the large spin limit of the twist-2 anomalous dimension, “collinear” anomalous dimension $\mathcal{G}_0(\lambda)$

- $\mathcal{N} = 4$ SYM – further simplifications $\beta = 0$

Soft/Collinear factorization

Magnea, Sterman
Sterman, Tejeda-Yeomans

◇ Rescaled amplitude factorizes in three parts:

$$\mathcal{M}_n = S(k, \frac{Q}{\mu}, \alpha_s(\mu), \epsilon) \times \left[\prod_{i=1}^n J_i(\frac{Q}{\mu}, \alpha_s(\mu), \epsilon) \right] \times h_n(k, \frac{Q}{\mu}, \alpha_s(\mu), \epsilon)$$

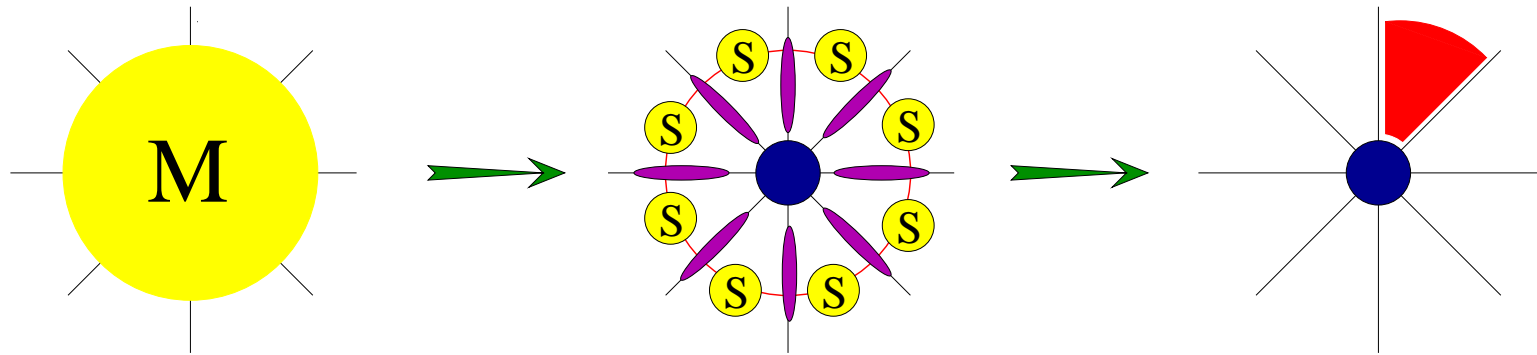
- $S(k, \mu, \alpha_s(\mu), \epsilon)$ soft function; captures the soft gluon radiation; defined up to overall function
- $J_i(k, \mu, \alpha_s(\mu), \epsilon)$ independent of color flow; all collinear dynamics
- $h_n(\mu, \alpha_s(\mu), \epsilon)$ is finite as $\epsilon \rightarrow 0$

◇ Independence of Q : factorization vs. evolution

● Consequences of the large N limit:

Bern, Dixon, Smirnov

- 1) trivial color structure: S can be absorbed in J
- 2) planarity: gluon exchange is confined to neighboring legs



$$\mathcal{M}_n = \times \left[\prod_{i=1}^n \mathcal{M}^{[gg \rightarrow 1]} \left(\frac{s_{i,i+1}}{\mu}, \lambda, \epsilon \right) \right]^{1/2} \times h_n(k, \lambda, \epsilon)$$

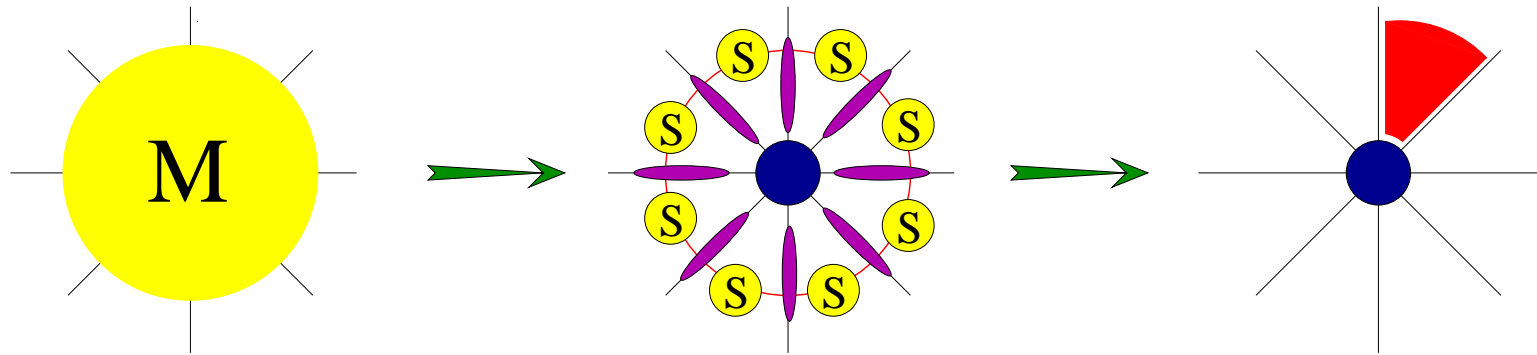
- Sudakov form factor: decay of a scalar into 2 gluons
- Factorization \mapsto differential (RG) equation for $\mathcal{M}^{[gg \rightarrow 1]}$

Mueller (1979); Collins (1980); Sen(1981); Korchemsky, Radyushkin (1987);
Korchemsky (1989); Magnea, Sterman (1990)

$$\frac{d}{d \ln Q^2} \mathcal{M}^{[gg \rightarrow 1]} \left(\frac{Q^2}{\mu^2}, \lambda, \epsilon \right) = \frac{1}{2} \left[K(\epsilon, \lambda) + G\left(\frac{Q^2}{\mu^2}, \lambda, \epsilon\right) \right] \mathcal{M}^{[gg \rightarrow 1]} \left(\frac{Q^2}{\mu^2}, \lambda, \epsilon \right)$$

$$\left(\frac{d}{d \ln \mu} + \beta(\lambda) \frac{d}{dg} \right) (K + G) = 0 \quad \left(\frac{d}{d \ln \mu} + \beta(\lambda) \frac{d}{dg} \right) K(\epsilon, \lambda) = -\gamma_K(\lambda)$$

- Exact solution



$$\mathcal{M}_n = \times \left[\prod_{i=1}^n \mathcal{M}^{[gg \rightarrow 1]} \left(\frac{s_{i,i+1}}{\mu}, \lambda, \epsilon \right) \right]^{1/2} \times h_n(k, \lambda, \epsilon)$$

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- Exact solution for $\mathcal{N} = 4$ SYM

Bern, Dixon, Smirnov

$$\mathcal{M}_n = \exp \left[-\frac{1}{8} \sum_l a^l \left(\frac{\gamma_K^{(l)}}{(l\epsilon)^2} + \frac{2\mathcal{G}_0^{(l)}}{l\epsilon} \right) \sum_i \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{l\epsilon} \right] \times h_n$$

$$f(\lambda) = \sum_l a^l \gamma_K^{(l)} \quad \text{universal scaling function}$$

Technology of choice: generalized unitarity-based method

Bern, Dixon, Dunbar, Kosower

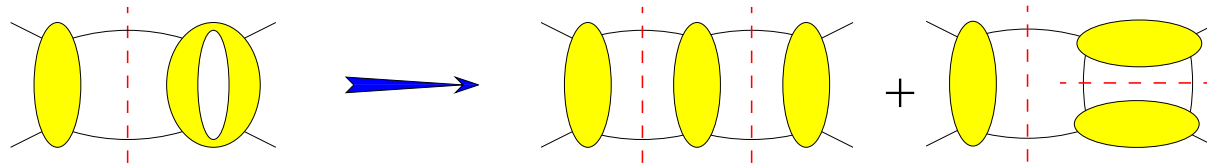
- a generalized cut isolates the part of an amplitude which contains some specified set of propagators and imposes the on-shell condition for their momenta
- Reconstruct amplitudes from (generalized) d -dimensional cuts

Technology of choice: generalized unitarity-based method

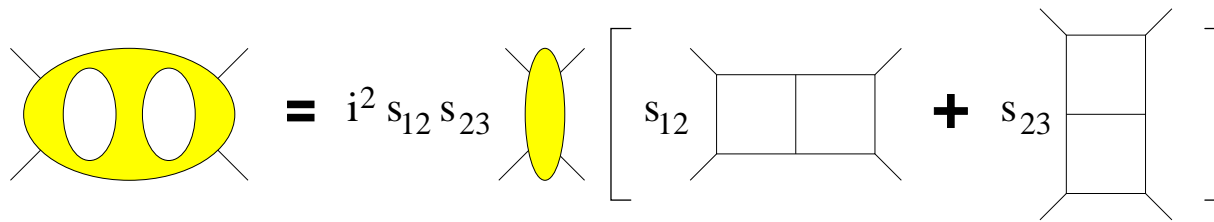
Bern, Dixon, Dunbar, Kosower

◇ 2-loop 4-point amplitudes:

Bern, Rozowsky, Yan



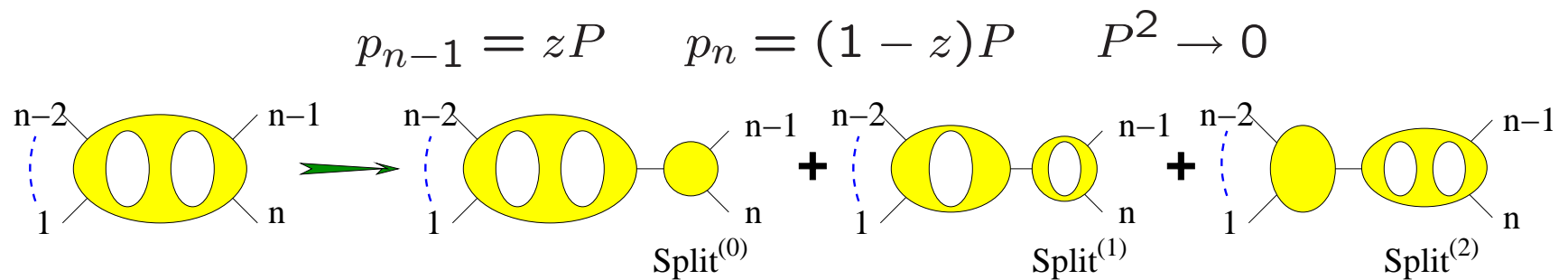
▪ double-two-particle cuts suffice to reconstruct the amplitude



◇ 2-loop splitting amplitude

Anastasiou, Bern, Dixon, Kosower

controls the behavior of amplitudes as two adjacent momenta become collinear



$$\text{Split}^{(l)} = \text{Split}^{(0)} r_S^{(l)}(z, s_{n-1, n}, \epsilon) \quad r_S^{(2)}(\epsilon) = \frac{1}{2} \left(r_S^{(1)}(\epsilon) \right)^2 + f^{(2)} r_S^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)$$

Technology of choice: generalized unitarity-based method

◇ 2-loop 4-point amplitudes:

Bern, Rozowsky, Yan

$$= i^2 s_{12} s_{23} \left[s_{12} \text{ (rectangle with vertical line)} + s_{23} \text{ (rectangle with horizontal line)} \right]$$

◇ 2-loop splitting amplitude

Anastasiou, Bern, Dixon, Kosower

$$p_{n-1} = zP \quad p_n = (1-z)P \quad P^2 \rightarrow 0$$

$$\text{Torus} \rightarrow \text{Torus} \cdot \text{Split}^{(0)} + \text{Torus} \cdot \text{Split}^{(1)} + \text{Torus} \cdot \text{Split}^{(2)}$$

Splitting amplitudes may be computed directly!

◇ Both consistent with:

Anastasiou, Bern, Dixon, Kosower

$$M_n^{(2)}(\epsilon) = \frac{1}{2} \left(M_n^{(1)}(\epsilon) \right)^2 + f^{(2)}(\epsilon) M_n^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon)$$

⇒ ABDK conjecture that it holds for any n at 2-loops

◇ Striking resemblance with the general structure of IR poles

$$\mathcal{M}_n = \exp \left[-\frac{1}{8} \sum_l a^l \left(\frac{\gamma_K^{(l)}}{(l\epsilon)^2} + \frac{2g_0^{(l)}}{l\epsilon} \right) \sum_i \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{l\epsilon} \right] \times h_n$$

$$f(\lambda) = \sum_l a^l \gamma_K^{(l)} \quad \text{cusp anomalous dimension (univ. scaling fcn.)}$$

- $\frac{1}{\epsilon^2} \sum_i \left(\frac{\mu^2}{-s_{i,i+1}} \right)^\epsilon$ – singular part of the 1-loop n-point amplitude

◇ Striking resemblance with the general structure of IR poles

$$\mathcal{M}_n = \exp \left[-\frac{1}{8} \sum_l a^l \left(\frac{\gamma_K^{(l)}}{(l\epsilon)^2} + \frac{2\mathcal{G}_0^{(l)}}{l\epsilon} \right) \sum_i \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{l\epsilon} \right] \times h_n$$

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▪ $\frac{1}{\epsilon^2} \sum_i \left(\frac{\mu^2}{-s_{i,i+1}} \right)^\epsilon$ – singular part of the 1-loop n-point amplitude

◇ $h_n \rightarrow \exp \left(\sum_l a^l h_n^{(l)} \right)$ and combine with the pole part

Bern, Dixon, Smirnov

$$\mathcal{M}_n = \exp \left[-\frac{1}{8} \sum_l a^l f^{(l)}(\epsilon) \mathcal{M}_n^{(1)}(l\epsilon) + C^{(l)} \right] \quad f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$$

Captures correctly collinear limit: $M_n^{(1)} \mapsto M_{n-1}^{(1)} + r_S^{(1)}$

Nevertheless... not a proof. There is **evidence for the conjecture:**

- 3-loop 4-points

Bern, Dixon, Smirnov

$$\text{3-loop 4-point diagram} = i^3 s_{12} s_{23} \left[s_{12}^2 \text{diagram}_1 + s_{23}^2 \text{diagram}_2 + 2s_{23}(1+k_1)^2 \text{diagram}_3 + 2s_{12}(1+k_4)^2 \text{diagram}_4 \right]$$

- Strong coupling limit of the 4-gluon amplitude

Alday, Maldacena

- 2-loop 5-points

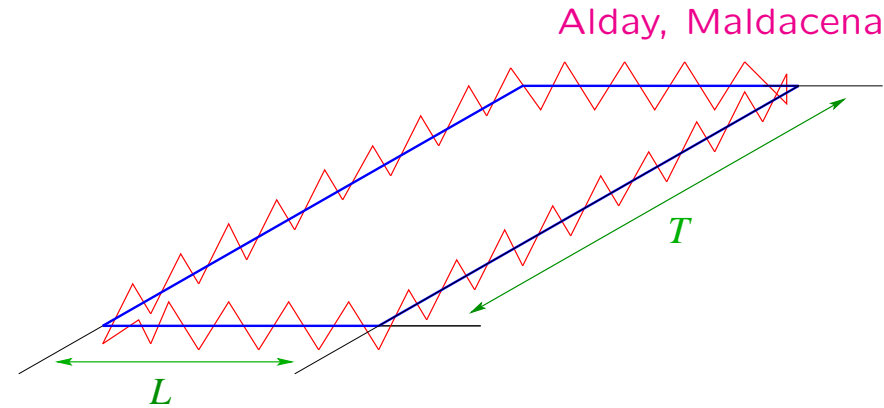
Cachazo, Spradlin, Volovich
Bern, Czakon, Dixon, Kosower, RR, Smirnov

$$\begin{aligned}
 & -\frac{1}{2} A_5^{tree} \sum_{cyc} \left\{ c_1 \text{diagram}_1 + c_2 \text{diagram}_2 + c_3 (q - k_1)^2 \text{diagram}_3 \right\} \\
 & -\frac{1}{2} A_5^{tree} \epsilon(k_1, k_2, k_3, k_4) \sum_{cyc} \left\{ c_1 \text{diagram}_4 + c_5 \text{diagram}_5 \right. \\
 & \left. - c_6 \text{diagram}_6 + c_7 \text{diagram}_7 + c_8 (q + k_1)^2 \text{diagram}_8 \right\}
 \end{aligned}$$

There are also puzzles:

- Large coupling limit of n -gluon amplitude in the large n limit and a special kinematic configuration

$$\ln \langle W_{n \rightarrow \infty} \rangle = \frac{\sqrt{\lambda}}{4} \frac{16\pi^2}{\Gamma\left(\frac{1}{4}\right)^4} \frac{T}{L} + \dots$$



$$\sum_l a^l f^{(l)}(\epsilon) \mathcal{M}_n^{(1)}(l\epsilon) = \sum_l a^l f^{(l)}(\epsilon) \langle W_n \rangle^{(1)}(l\epsilon) \xrightarrow{n \rightarrow \infty} \frac{\sqrt{\lambda}}{4} \frac{T}{L} + \dots$$

- Hexagon Wilson loop at 2-loops differs from BDS ansatz by non-trivial function of momenta

Drummond, Henn, Korchemsky, Sokatchev

- Difficulties with multi-Regge limits

Bartels, Lipatov, Sabio-Vera

Dual conformal invariance

Drummond, Henn, Sokatchev, Smirnov

4- and 5- point amplitudes: integrals invariant under dual space inversion if regularized by continuing external momenta off-shell

- $k_i = x_i - x_{i+1}$ and $\mathbf{I} : x_i^\mu \mapsto \frac{x_i^\mu}{x_i^2}$; finitness $\leftrightarrow u_{ijkl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$
- Regulator breaks dual conformal invariance \rightarrow potential anomaly
- Wilson loop renormalization: $W = Z(\epsilon)F$ Korchensky, Radyushkin

$$\mathbb{K}^\mu \ln F_n^W = \frac{1}{2} f(\lambda) \sum_{i=1}^n x_{i,i+1}^\mu \ln \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}$$

Drummond, Henn
Sokatchev, Smirnov

- ◇ Subtracted BDS ansatz $\ln F_n^{BDS} = \frac{1}{4} f(\lambda) F_n^{(1)}(0)$ captures anomaly

Probe structure of amplitude: define 2-loop “remainder fct” :

$$R_n^{(2)} \equiv \lim_{\epsilon \rightarrow 0} \left[M_n^{(2)}(\epsilon) - \left(\frac{1}{2} \left(M_n^{(1)}(\epsilon) \right)^2 + f^{(2)}(\epsilon) M_n^{(1)}(2\epsilon) + C^{(2)} \right) \right]$$

- finite
- trivial collinear limits

If believe that dual conformal symmetry holds –

⇒ $R_6^{(2)}$ is the first potentially-nonzero *remainder fct*

⇒ expressed in terms of

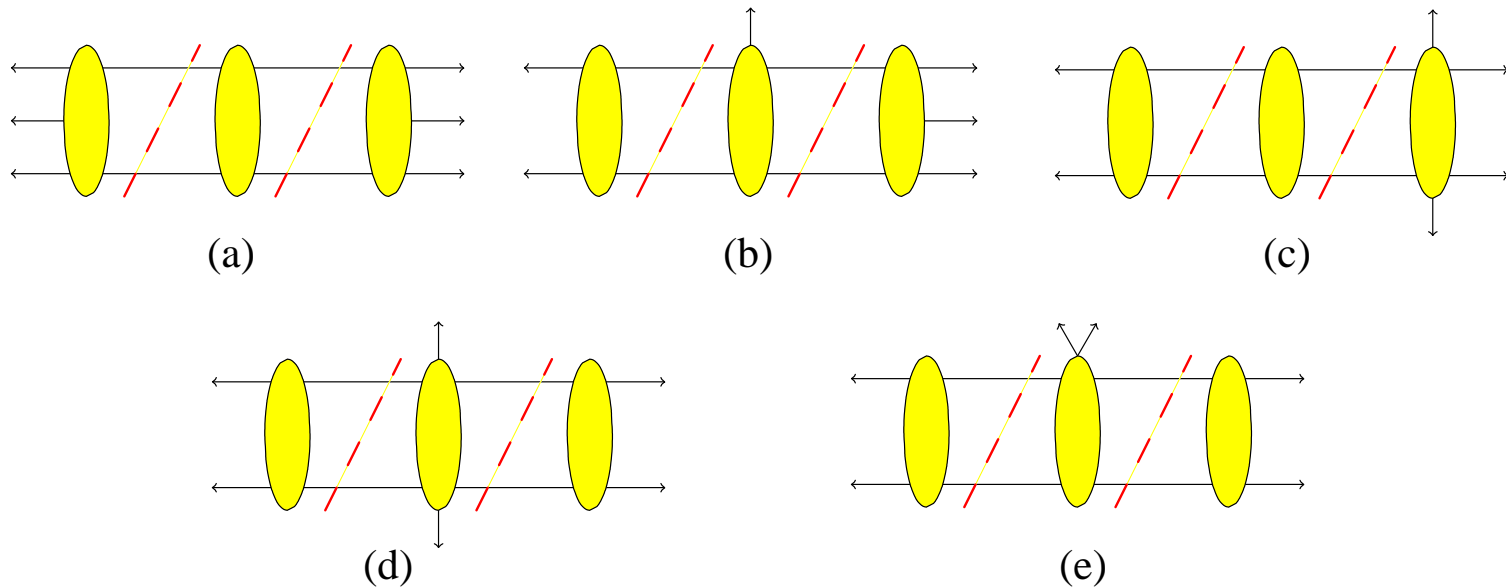
$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}, \quad u_2 = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2} = \frac{s_{23} s_{56}}{s_{234} s_{123}}, \quad u_3 = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2} = \frac{s_{34} s_{61}}{s_{345} s_{234}}$$

◇ homogeneous solution of W.I. ↔ component unrelated to IR

6-point amplitude at 2-loops Bern, Dixon, Kosower, RR, Spradlin, Vergu, Volovich

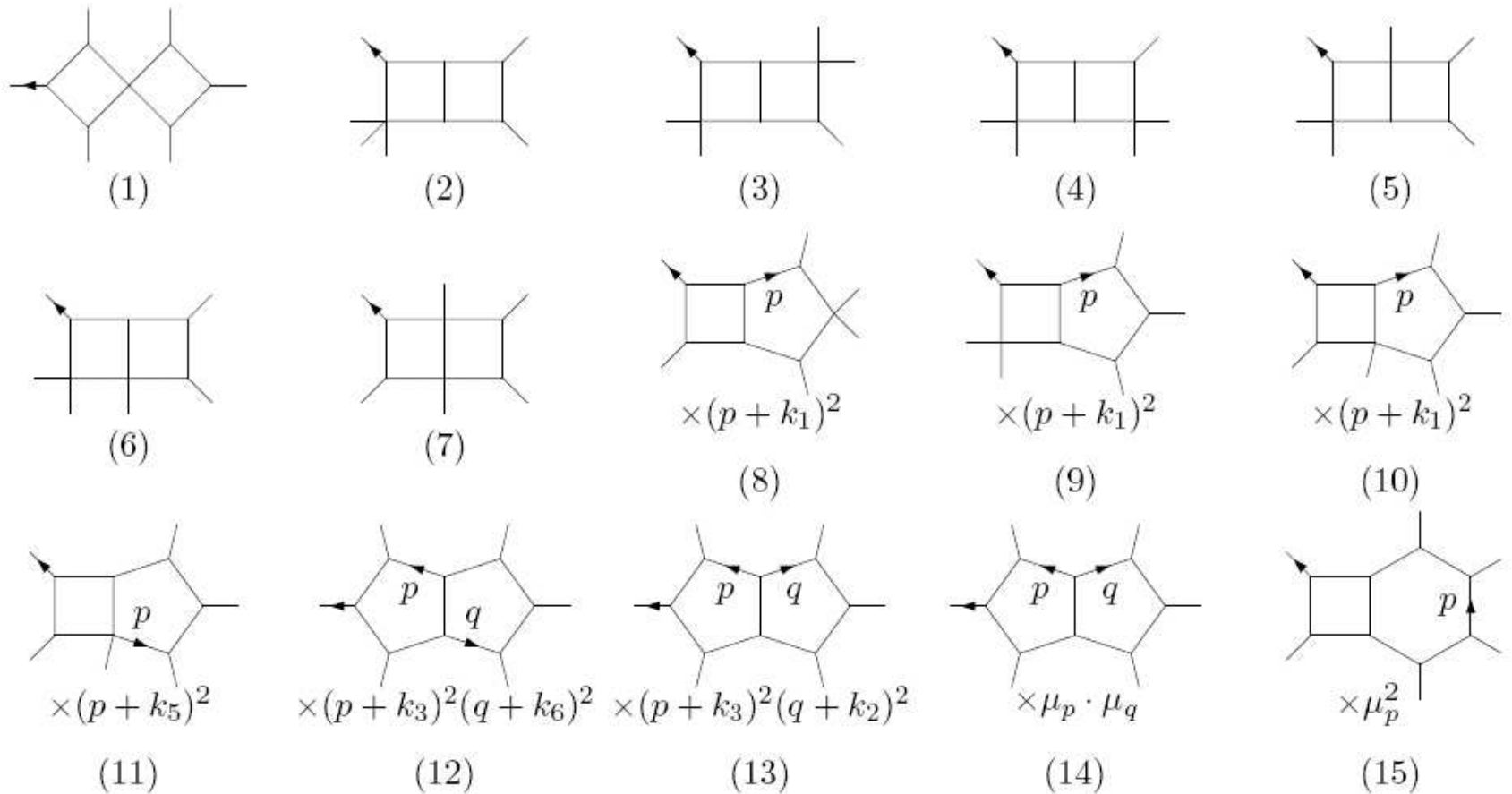
- use generalized unitarity
- no-triangle constraint \mapsto double-two-particle cuts suffice

Relevant generalized cuts:



- Main advantage: for $d = 4$ cuts each tree is MHV
- no guarantee that $d = 4$ suffice (already for 5-points odd)
- Split in $d = 4$ cuts and $d \neq 4$ cuts $M_6^{(2)}(\epsilon) = M_6^{(2),D=4}(\epsilon) + M_6^{(2),\mu}(\epsilon)$

6-point amplitude at 2-loops: use dual conformal symmetry to organize integrals



- 26 possible dual conformal integrals

The integrand:

$$M_6^{(2),D=4-2\epsilon}(\epsilon) = M_6^{(2),D=4}(\epsilon) + M_6^{(2),\mu}(\epsilon)$$

$$M_6^{(2),D=4}(\epsilon) = \frac{1}{16} \sum_{12 \text{ perms.}} \left[\frac{1}{4} c_1 I^{(1)}(\epsilon) + c_2 I^{(2)}(\epsilon) + \frac{1}{2} c_3 I^{(3)}(\epsilon) + \frac{1}{2} c_4 I^{(4)}(\epsilon) + c_5 I^{(5)}(\epsilon) \right. \\ \left. + c_6 I^{(6)}(\epsilon) + \frac{1}{4} c_7 I^{(7)}(\epsilon) + \frac{1}{2} c_8 I^{(8)}(\epsilon) + c_9 I^{(9)}(\epsilon) \right. \\ \left. + c_{10} I^{(10)}(\epsilon) + c_{11} I^{(11)}(\epsilon) + \frac{1}{2} c_{12} I^{(12)}(\epsilon) + \frac{1}{2} c_{13} I^{(13)}(\epsilon) \right]$$

$$M_6^{(2),\mu}(\epsilon) = \frac{1}{16} \sum_{12 \text{ perms.}} \left[\frac{1}{4} c_{14} I^{(14)}(\epsilon) + \frac{1}{2} c_{15} I^{(15)}(\epsilon) \right]$$

- **Strategy:** expose propagators in products of trees; id integrals
- c_{15} may be obtained from a partial $d = 4$ cut

- The coefficients:

$$c_1 = s_{16}s_{34}s_{123}s_{345} + s_{12}s_{45}s_{234}s_{345} + s_{345}^2(s_{23}s_{56} - s_{123}s_{234})$$

$$c_2 = 2s_{12}s_{23}^2$$

$$c_3 = s_{234}(s_{123}s_{234} - s_{23}s_{56})$$

$$c_4 = s_{12}s_{234}^2$$

$$c_5 = s_{34}(s_{123}s_{234} - 2s_{23}s_{56})$$

$$c_6 = -s_{12}s_{23}s_{234}$$

$$c_7 = 2s_{123}s_{234}s_{345} - 4s_{16}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345}$$

$$c_8 = 2s_{16}(s_{234}s_{345} - s_{16}s_{34})$$

$$c_9 = s_{23}s_{34}s_{234}$$

$$c_{10} = s_{23}(2s_{16}s_{34} - s_{234}s_{345})$$

$$c_{11} = s_{12}s_{23}s_{234}$$

$$c_{12} = s_{345}(s_{234}s_{345} - s_{16}s_{34})$$

$$c_{13} = -s_{345}^2s_{56}$$

$$c_{14} = -2s_{126}(s_{123}s_{234}s_{345} - s_{16}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345})$$

$$c_{15} = 2s_{16}(s_{123}s_{234}s_{345} - s_{16}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345})$$

- Some comments

- $M_6^{(2),\mu}$ nonvanishing while integrand vanishes in $D = 4$
- $M_6^{(2),D=4}$ is constructed out of pseudo-conformal integrals
- Unlike $n=4$ and $n=5$, relative weights are not $0, \pm 1$

- Features of the result

- IR divergences should be the same as in BDS
- subtract BDS \longrightarrow find conformal invariance?

$$u_1 = \frac{s_{12}s_{45}}{s_{123}s_{345}} \quad u_2 = \frac{s_{23}s_{56}}{s_{234}s_{123}} \quad u_3 = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$

- How important is $D = 4$? ($\det k_i \cdot k_j = 0$, $i, j = 1, \dots, 5$)?
- Comparison with expectation value of hexagon Wilson loop?

◇ “Direct” integration \mapsto departure from BDS ansatz

$$R_A \equiv \mathcal{M}_6^{(2)} - \mathcal{M}_6^{(2)BDS}$$

$$R_A^0 \equiv R_A(K^{(0)}) = 1.0937 \pm 0.0057 \quad K^{(0)} : (1/4, 1/4, 1/4)$$

kinematics	(u_1, u_2, u_3)	$R_A - R_A^0$
$K^{(1)}$	$(1/4, 1/4, 1/4)$	-0.018 ± 0.023
$K^{(2)}$	$(0.547253, 0.203822, 0.881270)$	-2.753 ± 0.015
$K^{(3)}$	$(28/17, 16/5, 112/85)$	-4.7445 ± 0.0075
$K^{(4)}$	$(1/9, 1/9, 1/9)$	4.12 ± 0.10
$K^{(5)}$	$(4/81, 4/81, 4/81)$	10.00 ± 0.50

- the remainder function is nontrivial
- it is dual conformal invariant

◇ Comparison with hexagon Wilson loop

$$R_A \equiv \mathcal{M}_6^{(2)} - \mathcal{M}_6^{(2)BDS}$$

$$R_A^0 \equiv R_A(K^{(0)}) = 1.0937 \pm 0.0057$$

$$R_W^0 = 13.26530$$

kinematics	(u_1, u_2, u_3)	$R_A - R_A^0$	$R_W - R_W^0$
$K^{(1)}$	$(1/4, 1/4, 1/4)$	-0.018 ± 0.023	$< 10^{-5}$
$K^{(2)}$	$(0.547253, 0.203822, 0.881270)$	-2.753 ± 0.015	-2.7553
$K^{(3)}$	$(28/17, 16/5, 112/85)$	-4.7445 ± 0.0075	-4.7446
$K^{(4)}$	$(1/9, 1/9, 1/9)$	4.12 ± 0.10	4.0914
$K^{(5)}$	$(4/81, 4/81, 4/81)$	10.00 ± 0.50	9.7255

• Agreement within errors!

• $\mathcal{M}_6 = \langle W_6 \rangle \neq \mathcal{M}_6^{BDS}$

Where else does $R_6^{(2)}$ crop up?

Does it have another (more physical) interpretation?

Where else does $R_6^{(2)}$ crop up? More physical interpretation?

Yes; a triple-collinear splitting amplitude

- triple-collinear limit:

$$k_a = z_1 P \quad k_b = z_2 P \quad k_c = z_3 P \quad P^2 \rightarrow 0$$

$$z_1 + z_2 + z_3 = 1, \quad 0 \leq z_i \leq 1$$

$$A_n^{(l)}(k_1, \dots, k_{n-2}, k_{n-1}, k_n) \mapsto \sum_{\lambda=\pm} \sum_{s=0}^l A_n^{(l-s)}(k_1, \dots, P^\lambda) \text{Split}_{-\lambda}^{(s)}(k_{n-2} k_{n-1} k_n; P)$$

s-loop triple-collinear
splitting amplitude

- MHV amplitudes \mapsto four triple-collinear splitting amplitudes

$$\text{Split}_+(k_a^+ k_b^+ k_c^+; P) = 0$$

$$\text{Split}_-(k_a^+ k_b^+ k_c^+; P); \quad \text{Split}_+(k_a^- k_b^+ k_c^+; P); \quad \text{Split}_+(k_a^+ k_b^- k_c^+; P)$$

- $$\frac{\text{Split}_-(k_a^+ k_b^+ k_c^+; P)}{\text{Split}_{\text{tree}}^-(k_a^+ k_b^+ k_c^+; P)} = r_S\left(\frac{s_{ab}}{s_{abc}}, \frac{s_{bc}}{s_{abc}}, z_1, z_3\right)$$

- first time for 6-point kinematics $k_a = z_1 P, k_b = z_2 P, k_c = z_3 P$

◇ cross ratios are arbitrary! $\rightarrow R$ survives

$$\bar{u}_1 = \frac{s_{45}}{s_{456}} \frac{1}{1 - z_3} \quad \bar{u}_2 = \frac{s_{56}}{s_{456}} \frac{1}{1 - z_1} \quad \bar{u}_3 = \frac{z_1 z_3}{(1 - z_1)(1 - z_3)}$$

- triple-collinear factorization **vs.** corrected BDS at 2-loops

$$\begin{aligned} M_6^{(2)} &\mapsto M_4^{(2)} + M_4^{(1)} r_S^{(1)} + r_S^{(2)} \\ &\mapsto M_4^{(2)BDS} + M_4^{(1)BDS} r_S^{(1)BDS} + r_S^{(2)BDS} + R_6^{(2)}(\bar{u}) \end{aligned}$$

◇ remainder function \leftrightarrow triple-collinear splitting amplitude

$$R_6^{(2)}(\bar{u}) = r_S^{(2)}\left(\frac{s_{ab}}{s_{abc}}, \frac{s_{bc}}{s_{abc}}, z_1, z_3, \epsilon\right) - r_S^{(2)BDS}\left(\frac{s_{ab}}{s_{abc}}, \frac{s_{bc}}{s_{abc}}, z_1, z_3, \epsilon\right)$$

◇ Advantage: potentially simpler integrals

- potentially resumable

Summary

- rescaled 6pt MHV amplitude = Wilson loop \neq BDS
- BDS captures the part of amplitudes determined by IR behavior
- correction visible in triple-collinear limit; equals splitting amplitude

Summary and open questions

- rescaled 6pt MHV amplitude = Wilson loop \neq BDS
- correction visible in triple-collinear limit; equals splitting amplitude
- Is it possible to find the analytic form of the remainder?
- Why are MHV amplitudes related to null Wilson loops? What about non-MHV amplitudes?
- Who ordered dual conformal invariance? What are the allowed types of contributions that break it?
- What other implications does it have? Is it relevant for non-MHV amplitudes and in what sense? (see conjecture in Sokatchev's talk)
- Is dual conformal invariance really restricted to the planar theory? What theories exhibit it? Does it have any relation to integrability of the dilatation operator?
- ...

Extra

Scattering amplitudes in CFT

- CFT:
 - interactions never turn off \rightarrow no free asymptotic states
 - IR divergences
- Regularization:
 - use dimensional regularization $d = 4 - 2\epsilon$
 - breaks conformal invariance
 - allows definition of asymptotic states
 - recovered as $\epsilon \rightarrow 0$
- Similarly to QCD:
 - on-shell gauge invariance
 - for “ $\mathcal{N} = 4$ collider” observables, need to turn them into scattering of gauge singlets
 - finiteness exposes properties of amplitudes which hold in any gauge theory