Reciprocity in QCD and $\mathcal{N}=4$ SYM

Giuseppe Marchesini

Milan-Bicocca University and INFN

Content

- Anomalous dimensions in DIS and e^+e^- $\quad (\gamma_{\mathrm{DIS}}(N)$ and $\gamma_{e^+e^-}(N))$
- Gribov-Lipatov reciprocity between $\gamma_{\mathrm{DIS}}(N)$ and $\gamma_{e^+e^-}(N)$
- First checks in QCD for DIS and e^+e^-
- $\mathcal{N}=4$ SYM case
- Running of α_s
- Conclusions

Work done in collaboration with Yuri Dokshitzer (since 2006)

Question: are $\gamma_{\rm DIS}(N) = \gamma_{-}(N)$ and $\gamma_{e^+e^-}(N) = \gamma_{+}(N)$ related?

Drell-Levy-Yan, Gribov-Lipatov 1970's:
DIS and e^+e^- inclusive production are crossing related:

$$\gamma_{\pm}(N) = \int_0^1 \frac{dx}{x} x^N \, \tilde{\gamma}(x) \, , \qquad \tilde{\gamma}(x) = -x \, \tilde{\gamma}(1/x)$$

Problem: not true beyond one-loop

Question: how much $\gamma_{\pm}(N)$ is inheriting soft components?

Universality of soft radiation: 1958 Low

$$\tilde{\gamma}_{\rm p}^{\rm Low}(x) = C_{\rm p} \, \alpha_s^{\rm ph} \frac{x}{(1-x)_+} + B_{\rm p}(\alpha_s)\delta(1-x)$$

Possibility of predicting large parts of higher order terms

Relations between $\gamma_{\rm DIS}(N) = \gamma_-(N)$ and $\gamma_{e^+e^-}(N) = \gamma_+(N)$

Reciprocity Relation (2006 Y.Dokshitzer, G.Salam &GM)

$$\gamma_\sigma(N) = \mathcal{P}\left(N + \sigma\gamma_\sigma(N)\right)\,, \qquad \sigma = \pm\,, \qquad (\text{no running}\,\alpha_s)$$

- Conjecture based on mass singularities of multi-parton phase space

- $\mathcal{P}(N)$ independent of channel DIS or e^+e^- $(\sigma = \pm 1)$

True in $\mathcal{N} = 4$ SYM for $\gamma_-(N)$ (B.Basso and G.Korchemsky hep-th/0612247)

RR equivalent to $(J^2 = N(N+1)$ and $\tilde{J}^2 = J^2 e^{\gamma_E})$

$$\gamma_-(N) = A(\alpha_s)\ln \bar{J}^2 + \sum_{p\geq 0} J^{-2p}\, f^{(p)}(\alpha_s, \bar{J}^2)$$

DIS and e^+e^- relations

$$\mathrm{DIS}: \quad \mathrm{e\,P} \to \mathrm{e}' + \cdots$$

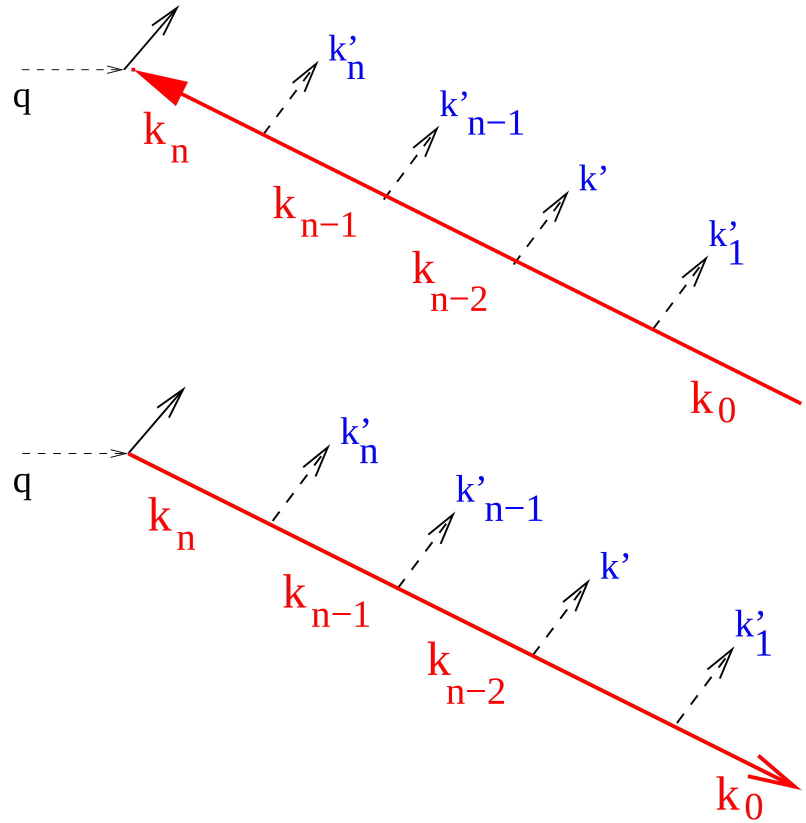
$$-q^2 = -(e - e')^2 = Q^2 \gg \Lambda^2_{QCD}$$

$$x_B = \frac{Q^2}{2Pq}, \qquad \frac{d\ln\sigma_{DIS}}{d\ln Q^2 dx_B} = \Sigma_-(x_B, Q^2)$$

$$e^+e^-\mathrm{inclusive}: \quad e^+e^- \to P + \cdots$$

$$q^2 = (e^+ - e^-)^2 = Q^2 \gg \Lambda^2_{QCD}$$

$$x_F = \frac{2Pq}{Q^2}, \qquad \frac{d\ln\sigma_{e^+e^-}}{dx_F} = \Sigma_+(x_F, Q^2)$$

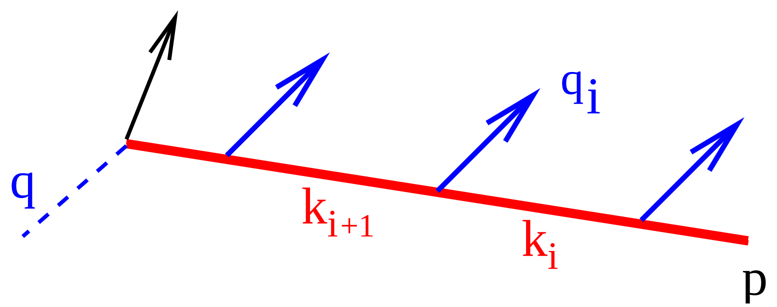


$$\Sigma_\sigma(N, Q^2) = C_\sigma(N, \alpha_s(Q^2)) \cdot D_\sigma(N, Q^2, \mu^2), \qquad x \Leftrightarrow N \quad \text{(Mellin moment)}$$

$$\ln D_\sigma(N, Q^2, \mu^2) = \int_{\mu^2}^{\#Q^2} \frac{dk^2}{k^2}\, \gamma_\sigma(N, \alpha_s(k^2))$$

\# kinem. boundary can be absorbed into $C_\sigma(N, \alpha_s)$, μ mass-singularity cutoff

Collinear ordering for γ_{DIS} or $\gamma_{e^+e^-}$



$$\frac{|k_{i+1}^2|}{k_{i+1}^+} = \frac{|k_i^2|}{k_i^+} + \frac{{q_i}^2}{{q_i}^+} + \frac{q_i^+ \, k_i^+}{k_{i+1}^+} \left(\frac{\vec{q}_{i,t}}{q_i^+} - \frac{\vec{k}_{i,t}}{k_i^+} \right)^2$$

Collinear kinematics: $\frac{|k_{i+1}^2|}{k_{i+1}^+} > \frac{|k_i^2|}{k_i^+}$

$$|k_i^2| \; < \; |k_{i+1}^2| \, \frac{k_i^+}{k_{i+1}^+} \qquad \begin{cases} \mathrm{DIS}: & \dfrac{k_i^+}{k_{i+1}^+} = z_i^{-1} \;\Rightarrow\; |k_i^2| \; < \; |k_{i+1}^2| \, z_i^{-1} \\ e^+e^-: & \dfrac{k_i^+}{k_{i+1}^+} = z_i \;\Rightarrow\; k_i^2 \; < \; k_{i+1}^2 \, z_i \end{cases}$$

Same "splitting kernel" for DIS $(k_i \rightarrow k_{i+1} \, q_i)$ and e^+e^- $(k_{i+1} \rightarrow k_i \, q_i)$

Reciprocity relation:

	Process	variables	collinear ordering
S-case, $\sigma=-1$:	$P\,q \to X$	$x_B = \dfrac{\lvert q\rvert^2}{2Pq}$	$\lvert k_i^2\rvert < \lvert k_{i+1}^2\rvert\; z_i^{-1}$
T-case, $\sigma=+1$:	$q \to X\;P$	$x_F = \dfrac{2Pq}{q^2}$	$k_i^2 < k_{i+1}^2\; z_i$

$\mathrm{S} \leftrightarrow \mathrm{T}$ kinematics: $x_B \leftrightarrow x_F^{-1}$ and $\lvert k^2\rvert\, z^{-1} \leftrightarrow k^2\, z$

Accounting for z rescaling of k^2 $\Rightarrow$ Reciprocity relation equation

$$\partial_\tau\, D_\sigma(N,Q^2) = \int_0^1 \frac{dz}{z}\, z^N P(z,\alpha_s)\, D_\sigma(N,Q^2\, z^\sigma) = \gamma_\sigma(N)\cdot D_\sigma(N,Q^2)$$

where $P(z,\alpha_s)$ is unique giving (running α_s, see later)

$$\gamma_\sigma(N) = \mathcal{P}(N+\sigma\gamma_\sigma(N)) = \int_0^1 \frac{dz}{z} P(z,\alpha_s)\, z^{N+\sigma\gamma_\sigma(N)}$$

Questions

- Is RR true at higher orders: $\tilde{P}(x) = -x\tilde{P}(1/x)$
- Low scaling at large x: $\tilde{P}(x) = \tilde{\gamma}^{\mathrm{Low}}(x) + \cdots$
- Small x in DIS and e^+e^-
- $\mathcal{N} = 4$ SYM case, twist $L = 2$
- $\mathcal{N} = 4$ SYM case, twist $L = 3$
- Running α_s

Is RR true at high orders?

Non-Singlet: ${\gamma_\sigma}^{\rm NS} = \sum \alpha_s^\ell \, {\gamma_\sigma}^{(\ell)}(N)$ $\quad \mathcal{P}^{\rm NS} = \sum \alpha_s^\ell \, \mathcal{P}^{(\ell)}(N)$

$${\gamma_\sigma}^{(1)}(N) = \mathcal{P}^{(1)}(N)$$

$1970's$ GL

$${\gamma_\sigma}^{(2)}(N) = \mathcal{P}^{(2)}(N) + \sigma \mathcal{P}^{(1)} \partial_N \mathcal{P}^{(1)}$$

$${\gamma_\sigma}^{(3)}(N) = \mathcal{P}^{(3)}(N) + \sigma \partial_N \left(\mathcal{P}^{(1)} \mathcal{P}^{(2)} \right) + \tfrac{1}{2} \mathcal{P}^{(1)} \left(\mathcal{P}^{(1)} \partial_N + 2 \partial_N \mathcal{P}^{(1)} \right) \partial_N \mathcal{P}^{(1)}$$

$\mathcal{P}(N, \alpha_s)$ captures the essential parts of $\gamma_\sigma(N, as)$

$$\gamma_+^{(2)}(N) - \gamma_-^{(2)}(N) = 2 \mathcal{P}^{(1)}(N) \partial_N \mathcal{P}^{(1)}(N)$$

1980 CFM

$$\gamma_+^{(3)}(N) - \gamma_-^{(3)}(N) = 2 \partial_N \left(\mathcal{P}^{(1)}(N) \, \mathcal{P}^{(2)}(N) \right)$$

2006 MMV

Non-Singlet three loops: $\gamma_{+}^{(3)}(N) - \gamma_{+}^{(3)}(N) = 2\partial_N \Big(\mathcal{P}^{(1)}(N)\,\mathcal{P}^{(2)}(N)\Big)$

$$
\begin{aligned}
\delta P^{(2)+}(x) \;\equiv\; & P^{(2)+}_{\sigma=1}(x) - P^{(2)+}_{\sigma=-1}(x) \;=\\
& + 16\,C_F^3\Big(p_{\rm qq}(x)\Big[311/24\,{\rm H}_0 + 4/3\,{\rm H}_0\zeta_2 - 169/9\,{\rm H}_{0,0} + 8{\rm H}_{0,0}\zeta_2 - 22{\rm H}_{0,0,0}\\
& - 268/9\,{\rm H}_{1,0} + 8{\rm H}_{1,0}\zeta_2 - 44/3\,{\rm H}_{1,0,0} - 268/9\,{\rm H}_2 + 8{\rm H}_2\zeta_2 - 44/3\,{\rm H}_{2,0} - 44/3\,{\rm H}_3\Big]\\
& + (1+x)\Big[-4{\rm H}_{0,0}\zeta_2 + 25/2\,{\rm H}_{0,0,0} + {\rm H}_{2,0} + 2{\rm H}_3\Big] - (1-x)\Big[325/18\,{\rm H}_0 + 50/3\,{\rm H}_{1,0}\\
& + 50/3\,{\rm H}_2\Big] + (3-5x){\rm H}_0\zeta_2 - (173/18 - 691/18\,x){\rm H}_{0,0}\Big)\\
& + 16\,C_F^2(C_A - 2C_F)\Big(p_{\rm qq}(x)\Big[151/24\,{\rm H}_0 + {\rm H}_0\zeta_3 + 13/6\,{\rm H}_0\zeta_2 - 169/18\,{\rm H}_{0,0} + 8{\rm H}_{0,0}\zeta_2\\
& - 13/2\,{\rm H}_{0,0,0} - 8{\rm H}_{0,0,0,0} - 134/9\,{\rm H}_{1,0} + 4{\rm H}_{1,0}\zeta_2 - 22/3\,{\rm H}_{1,0,0} - 6{\rm H}_{1,0,0,0} - 134/9\,{\rm H}_2\\
& + 4{\rm H}_2\zeta_2 - 22/3\,{\rm H}_{2,0} - 2{\rm H}_{2,0,0} - 22/3\,{\rm H}_3 - 2{\rm H}_{3,0} - 6{\rm H}_4\Big] + p_{\rm qq}(-x)\Big[-8{\rm H}_{-3,0}\\
& + 8{\rm H}_{-2}\zeta_2 + 8{\rm H}_{-2,-1,0} + 3{\rm H}_{-2,0} - 14{\rm H}_{-2,0,0} - 4{\rm H}_{-2,2} + 8{\rm H}_{-1,-2,0} + 16{\rm H}_{-1,-1,0,0}\\
& + 8{\rm H}_{-1,0}\zeta_2 + 6{\rm H}_{-1,0,0} - 18{\rm H}_{-1,0,0,0} - 4{\rm H}_{-1,2,0} - 8{\rm H}_{-1,3} - 7{\rm H}_0\zeta_3 + 3/2\,{\rm H}_0\zeta_2\\
& - 8{\rm H}_{0,0}\zeta_2 - 9/2\,{\rm H}_{0,0,0} + 8{\rm H}_{0,0,0,0} + 2{\rm H}_{3,0} + 6{\rm H}_4\Big] - (1+x)\Big[4{\rm H}_{-2,0} + 8{\rm H}_{-1,0,0}\Big]\\
& + (1-x)\Big[4{\rm H}_{-3,0} + 4{\rm H}_{-2,0,0} - 88/9\,{\rm H}_0 + 3{\rm H}_0\zeta_3 - 28/3\,{\rm H}_{1,0} - 28/3\,{\rm H}_2\Big] - 4x{\rm H}_0\zeta_2\\
& - (50/9 - 184/9\,x){\rm H}_{0,0} - 4x{\rm H}_{0,0}\zeta_2 + (11/2 + 35/2\,x){\rm H}_{0,0,0} + 8x{\rm H}_{0,0,0,0}\Big)\\
& + 16\,C_F^2 n_f\Big(p_{\rm qq}(x)\Big[-11/12\,{\rm H}_0 - 2/3{\rm H}_0\zeta_2 + 11/9\,{\rm H}_{0,0} + 2{\rm H}_{0,0,0} + 20/9\,{\rm H}_{1,0}\\
& + 4/3\,{\rm H}_{1,0,0} + 20/9\,{\rm H}_2 + 4/3\,{\rm H}_{2,0} + 4/3\,{\rm H}_3\Big] - (1+x)\,{\rm H}_{0,0,0} + (1-x)\Big[13/9\,{\rm H}_0\\
& + 4/3\,{\rm H}_{1,0} + 4/3\,{\rm H}_2\Big] + (8/9 - 28/9\,x){\rm H}_{0,0}\Big)\,,
\end{aligned}
$$

A. Mitov,S.Moch,A.Vogt, Phys.Lett.B638(2006)61

To what extent ${\gamma_\sigma}^a(N)$ determined by Low soft radiation?

High order terms driven from $\tilde{P}(x)$ and Low classical radiation

$$\text{Use}: \ \gamma^a_\sigma(N) = \mathcal{P}^a(N+\sigma\,\gamma^a_\sigma(N))\,, \qquad a \ = \ qq\,, gg \quad (\beta_0 = 0)$$

$$\mathcal{P}^a(x) = \tilde{\gamma}^a_{\rm Low}(x) = \frac{A^a\,x}{(1-x)_+} + B^a\delta(1-x)\,, \quad A^a = \frac{C_a\alpha_s^{\rm ph}}{\pi}$$

$$\text{Result}: \ \tilde{\gamma}^a_\sigma(x) = \tilde{\gamma}^a_{\rm Low}(x) \ + \ C^a_\sigma \ln(1-x) + D^a_\sigma + \ldots$$

$$C^a_\sigma \ = \ -\sigma(A^a)^2\,, \qquad D^a_\sigma \ = \ -\sigma A^a\,B^a \quad (+\,\beta_0\,\text{term})$$

2-loop, $\sigma=\pm1$ G.Curci,W.Furmanski,R.Petronzio,NPB175(1980)27

3-loop, $\sigma=-1$ S.Moch,J.Vermaseren,A.Vogt,NPB691(2004)129

3-loop, $\sigma=\pm1$ A. Mitov,S.Moch,A.Vogt, hep-ph/06040563

RR and coherence at small-x (i.e. $N \to 0$)

Leading order: DL expansion: $\gamma_\sigma^{\rm DL}(N) = \sum_p \frac{\bar\alpha_s^p}{N^{2p-1}} + \cdots$

In DIS: no DL at high order

$$\gamma_-^{\rm DL}(N) = \frac{\bar\alpha_s}{N}\,, \qquad \bar\alpha_s = \frac{N_c\,\alpha_s}{\pi} \qquad k_t - \text{ordering}$$

Using RR obtain $\gamma_+^{\rm DL}(N)$ from $\gamma_-^{\rm DL}(N)$:
-compute $\mathcal{P}^{\rm DL}(N)$ from $\gamma_- = \mathcal{P}(N - \gamma_-)$
-compute $\gamma_+^{\rm DL}(N)$ from $\gamma_+ = \mathcal{P}(N + \gamma_+)$

$$\gamma_+^{\rm DL}(N) = \frac{1}{4}\left(\sqrt{N^2 + 8\bar\alpha_s} - N\right) \qquad \theta - \text{ordering}$$

Using RR: DIS k_t-ordering $\Longleftrightarrow$ e^+e^- θ-ordering

Higher order analysis similar up to N^{5}LO (no running coupling)

$$\gamma(N) = \sum_p c_p^{(1)} \frac{\bar{\alpha}_s^p}{N^{2p-1}} + \sum_p c_p^{(2)} \frac{\bar{\alpha}_s^p}{N^{2p-2}} \cdots = \mathcal{O}\left(\sqrt{\alpha_s}\right) + \mathcal{O}\left(\alpha_s\right) \cdots \quad (N \sim \sqrt{\alpha_s})$$

$$\text{NLO}: \qquad \gamma_-^{\text{NLO}} = \frac{\bar{\alpha}_s}{N} - a \cdot \bar{\alpha}_s + 0 \cdot \frac{\bar{\alpha}_s^2}{N^2} \qquad a = \frac{11}{12} + \frac{n_f}{6N^3}$$

$$\text{Use RR}: \qquad \gamma_+^{\text{NLO}} = \frac{1}{4}\left(\sqrt{N^2+8\bar{\alpha}_s} - N\right) - \frac{a}{2}\bar{\alpha}_s\left(1 - \frac{N}{\sqrt{N^2+8\bar{\alpha}_s}}\right)$$

$0 \cdot \frac{\bar{\alpha}_s^2}{N^2}$ (BFKL) $\Rightarrow$ angular ordering result unmodified

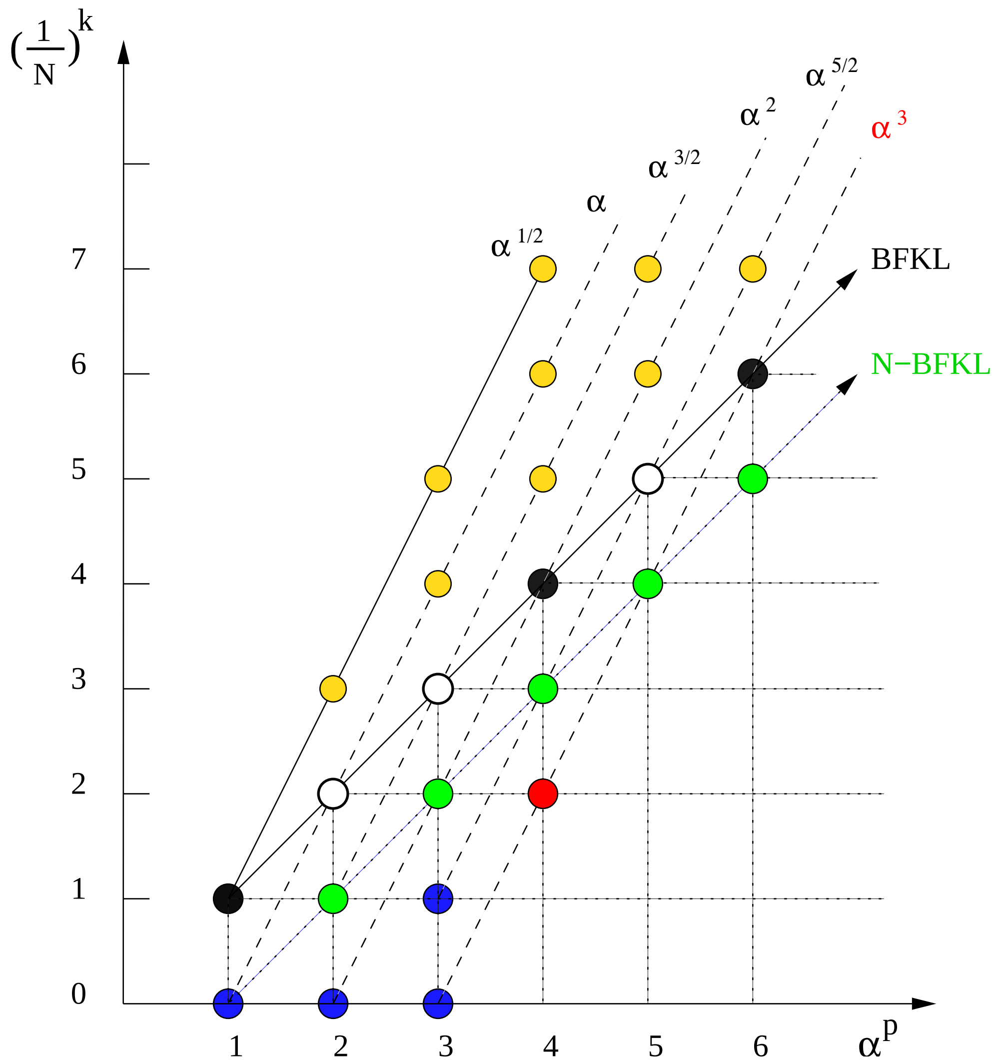
1984 Malaza-Webber and Gaffney-Mueller and Dokshitzer-Khoze-Mueller-Troian book

$$\text{NNLO}: \qquad \gamma_-^{\text{NNLO}} = \frac{\bar{\alpha}_s}{N} - a \cdot \bar{\alpha}_s + 0 \cdot \frac{\bar{\alpha}_s^2}{N^2} + \frac{\bar{\alpha}_s^2}{N} + 0 \cdot \frac{\bar{\alpha}_s^3}{N^3}$$

$0 \cdot \frac{\bar{\alpha}_s^3}{N^3}$ (BFKL) $\equiv$ redefinition of coupling in NLO

$\Rightarrow$ angular ordering result unmodified (1984 Malaza-Webber and Gaffney-Mueller)

Higher orders: $\gamma_\pm(N) = \sum_{pk} c_\pm^{pk} \frac{\alpha_s^p}{N^k}$



Give $c_-^{p,k}$ for $2k-1 \leq p$

obtain $c_+^{p,k}$ for all k

DIS case

α_s^2/N^2 and α_s^3/N^3 missing

e^+e^- case

exact ang.ord. to NNLO

Malaza-Webber and Gaffney-Mueller

Dokshitzer-Khoze-Mueller-Troian book

Three loop anomalous dimension in $\mathcal{N}=4$ SYM

Related to QCD: $C_A = C_F = T_f/2 = N_c$

No running coupling: $\beta(\alpha_s) = 0$

Spacelike anomalous dimension eigenvectors: $\{\gamma(N), \gamma(N-2), \gamma(N-4)\}$

Use $\mathcal{N}=4$ SYM: not very different from QCD Lipatov, Kotikov 2000

Consider $\gamma^{(\ell)}_{\rm QCD}(N)$ the ℓ-loop in QCD

$$\gamma^{(\ell)}_{\rm QCD}(N) = \sum_{r=0}^{\tau} \gamma^{(r,\,\ell)}_{\rm QCD}(N)\,, \quad \tau = 2\ell - 1 \quad \text{transcendentality}$$

$$\gamma^{(\tau,\,\ell)}_{\rm QCD}(N) = \gamma^{(\ell)}_{\mathcal{N}=4\,\rm SYM}(N) \quad \text{with } C_A{=}C_F$$

Use MVV to obtain $\gamma^{(3)}_{\mathcal{N}=4\,\mathrm{SYM}}(N)$

A.Kotikov, L.Lipatov, A.Onishchenko, V.Velizhanin, Phys.Lett.B595:521,2004

Compute $\mathcal{P}(N)$ from RR: $\gamma_-(N) = \mathcal{P}\left(N - \gamma_-(N)\right)$

Questions:

- Validity of RR: $P(x) = -xP(x^{-1})$
- To what extent SYM $\mathcal{N} = 4$ is semiclassical (Low scaling)

Validity of RR for $P(x) = x^{-1}P(x^{-1})$?

Study of $\mathcal{N} = 4$ SYM at three loop (Dokshitzer&GM Phys.Lett. B (2007))

$$P_1(x) = x\left(\frac{1}{1-x}\right)_+$$

$$P_2(x) = x\left(\frac{\ln^2 x}{2!(1-x)}\right)_+ - \frac{x}{2(1+x)}\Phi_{-3}(x) \quad + \text{const.}$$

$$P_3(x) = x\left(\frac{\ln^4 x}{4!(1-x)}\right)_+ + \frac{2x}{3(1+x)}\Phi_{-5}(x) \quad + \text{const.} + \text{factorized pieces}$$

$$\Phi_{s+1}(x) = \int_x^1 \frac{dz}{s!z}\ln^s\left(\frac{z(1+x)^2}{x(1+z)^2}\right) = (-)^{s+1}\Phi_{s+1}\left(\frac{1}{x}\right)$$

Then, at three loops: $P(x) = -xP\left(\frac{1}{x}\right)$

To what extent SYM $\mathcal{N}=4$ is semiclassical (Low)

$$P^{\rm Low}(x,\alpha_s) = \frac{x\,\bar\alpha_s^{\rm ph}}{(1-x)_+} + B(\bar\alpha_s^{\rm ph})\,\delta(1-x) \qquad \bar\alpha_s^{\rm ph} = \frac{\alpha_s^{\rm ph}\,N_c}{\pi}$$

$$P^{\rm ph}(x,\alpha_s) = P^{\rm Low}(x) + P^{\rm (pos)}(x) + P^{\rm (neg)}(x)$$ quantum fluct.

with physical coupling: $\bar\alpha_s^{\rm ph} = \bar\alpha_s(1-\frac{1}{2}\zeta_2 + \frac{11}{8}\bar\alpha_s^2 + \cdots)$

two-loop
$$\begin{cases} P_2^{\rm (pos)}(x) \simeq P^{\rm Low}(x)\cdot(1-x)^2 \sim (1-x) \\ P_2^{\rm (neg)}(x) \simeq P^{\rm Low}(-x)\cdot(1-x)^3 \sim (1-x)^3 \end{cases}$$

three-loop
$$\begin{cases} P_3^{\rm (pos)}(x) \simeq P^{\rm Low}(x)\cdot(1-x)^4 \sim (1-x)^3 \\ P_3^{\rm (neg)}(x) \simeq P^{\rm Low}(-x)\cdot(1-x)\ln(1-x) \sim (1-x)\ln(1-x) \end{cases}$$

Extend to twist3 anomalous dimensions computed at four loops

M.Beccaria, hep-th/0704.3585
A.Kotikov, L.Lipatov, A.Rej, M.Staudacher and V.Velizhanin, hep-th/0704.3586

$$
\begin{aligned}
\gamma_3^{(1)} &= 4\,S_1\,, \\
\gamma_3^{(2)} &= -2\,(S_3 + 2\,S_1 S_2) \\
\gamma_3^{(3)} &= 5\,S_5 + 6\,S_2\,S_3 - 8\,S_{3,1,1} + 4\,S_{4,1} - 4\,S_{2,3} + S_1(4\,S_2^2 + 2\,S_4 + 8\,S_{3,1}), \\
\gamma_3^{(4)} &= \tfrac{1}{2}\,S_7 + 7\,S_{1,6} + 15\,S_{2,5} - 5\,S_{3,4} - 29\,S_{4,3} - 21\,S_{5,2} - 5\,S_{6,1} \\
&\quad -40\,S_{1,1,5} - 32\,S_{1,2,4} + 24\,S_{1,3,3} + 32\,S_{1,4,2} - 32\,S_{2,1,4} + 20\,S_{2,2,3} \\
&\quad +40\,S_{2,3,2} + 4\,S_{2,4,1} + 24\,S_{3,1,3} + 44\,S_{3,2,2} + 24\,S_{3,3,1} + 36\,S_{4,1,2} \\
&\quad +36\,S_{4,2,1} + 24\,S_{5,1,1} + 80\,S_{1,1,1,4} - 16\,S_{1,1,3,2} + 32\,S_{1,1,4,1} \\
&\quad -24\,S_{1,2,2,2} + 16\,S_{1,2,3,1} - 24\,S_{1,3,1,2} - 24\,S_{1,3,2,1} - 24\,S_{1,4,1,1} \\
&\quad -24\,S_{2,1,2,2} + 16\,S_{2,1,3,1} - 24\,S_{2,2,1,2} - 24\,S_{2,2,2,1} - 24\,S_{2,3,1,1} \\
&\quad -24\,S_{3,1,1,2} - 24\,S_{3,1,2,1} - 24\,S_{3,2,1,1} - 24\,S_{4,1,1,1} - 64\,S_{1,1,1,3,1} - 8\,\beta\,S_1\,S_3
\end{aligned}
$$

Arguments of harmonic sums: $S_{\vec{m}} = S_{\vec{m}}(N/2)$

Questions: validity of RR; dominance of Low scaling

Validity of RR (M.Beccaria, Yu.Dokshitzer and GM hep-th/0705.2639)

$$\mathcal{P}^{(1)} = 4\,S_1$$

$$\mathcal{P}^{(2)} = -\,2\,S_3$$

$$\mathcal{P}^{(3)} = 3\,S_5 - 2\,\Phi_{1,1,3} - 2\zeta_2\,S_3$$

$$\mathcal{P}^{(4)} = 4\,S_1 \cdot \left(2\,\widehat{\Phi}_{1,1,1,3} - \widehat{\Phi}_{1,5} - \widehat{\Phi}_{3,3} - \widehat{S}_3\right) + 16\,\Phi_{1,1,1,1,3} - 4\big(\Phi_{3,1,3} + \Phi_{1,3,3} + \Phi_{1,1,5}\big) - \tfrac{5}{2}\,S_7 + 2\,\zeta_2 \cdot \big(3\,S_5 - 2\,\Phi_{1,1,3}\big)$$

$$\tilde{\Phi}_{b,\vec{m}}(x) = -\frac{x}{x-1}\int_x^1 \frac{dz\,(z+1)}{z^2}\,\frac{1}{(b-1)!}\,\ln^{b-1}\frac{z}{x}\cdot\tilde{\Phi}_{\vec{m}}(z) \qquad \tilde{\Phi}_a(x) = \tilde{S}_a(x)$$

At four loops:

- since $S_{\vec{m}}(n)$ with $n = \frac{N}{2}$: $P(x) = -x^2\,P(1/x)$
- dominance of Low scaling: $\mathcal{P}(N) = 4\,S_1(n)\cdot(\alpha_s^{\rm ph} + \mathcal{A}(n)) + \mathcal{B}(n)$ with $\mathcal{A} \sim \alpha_s^3$ and $\mathcal{B} \sim \alpha_s^2$

Running α_s

Use dispersive method

$$\partial_{\ln Q^2} D_N(Q^2) = \gamma_\sigma(N) \cdot D_N(Q^2) = \int_0^1 \frac{dz}{z} z^N P(z, \alpha_s) D_N(Q^2 z^\sigma)$$

one finds $\alpha_s = \alpha_s(|1 - z^\sigma| Q^2)$

$$\mathcal{P}_\sigma(N) = \int_0^1 \frac{dz}{z} z^N P(z, \alpha_s(|1 - z^\sigma| Q^2))$$

with $\mathcal{P}_\sigma(N)$ depending on channel through α_s

For comparison with known results necessary to know the scheme

Summary: long way into the anomalous dimensions

- Reciprocity relation works: splitting kernel $\mathcal{P}(N)$ universality
- Relation between DIS and e^+e^- restored
- Large pieces of $\gamma_\pm(N)$ inherited from $\mathcal{P}(N)$
- Quantum fluctuations reduced to essentials: Low scaling
- High energy physics (DIS) and jet physics (e^+e^-) connection
- $\mathcal{N}=4$ SYM a laboratory for studying QCD (L=2,L=3)

Work in progress: improve RR to reduce quantum fluctuations, the matrix problem and identifying running of coupling