Generating functions for $N = 4$ and $N = 8$ amplitudes

Henriette Elvang (MIT)

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- arXiv:0805.0757 w/ Massimo Bianchi and Dan Freedman
- arXiv:0710.1270 w/ Dan Freedman

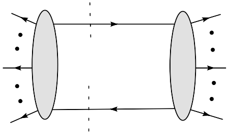
1. Motivation

Is $\mathcal{N} = 8$ supergravity perturbatively finite?

$\updownarrow$

Explicit calculations:

Loop amplitudes $\leftrightarrow$ tree amplitudes



via generalized unitarity cuts

[Bern, Dixon, Kosower, ...]

$\updownarrow$

Our work focuses on n-point on-shell tree amplitudes in $\mathcal{N} = 8$ SG and their relationship with tree amplitudes in $\mathcal{N} = 4$ SYM.

Generating function Z_n — idea

$$\begin{array}{ccc} \text{States } X_i & \leftrightarrow & \text{differential operators } D_{X_i} \\ \downarrow & & \downarrow \\ \text{Amplitude } A_n(X_1\, X_2 \ldots X_n) & = & D_{X_1}\, D_{X_2} \cdots D_{X_n}\, Z_n \end{array}$$

Original $\mathcal{N} = 4$ SYM generating function by Nair [Nair (1988,2005)] .
Further developed and extended by Georgio, Glover and Khoze [GGK (2004)] .

Our formulation in terms of derivative operators + extensions to supergravity.

- The simplest amplitudes are MHV (maximally helicity violating)
  - $\mathcal{N} = 4$ SYM: $A_n(-,-,+,\ldots,+)$ gluons.
  - $\mathcal{N} = 8$ SG: $M_n(-,-,+,\ldots,+)$ gravitons.

  MHV sector: amplitudes related to A_n and M_n, resp., via SUSY Ward identities.

- The next-to-simplest amplitudes are Next-to-MHV
  - $\mathcal{N} = 4$ SYM: $A_n(-,-,-,+,\ldots,+)$ gluons.
  - $\mathcal{N} = 8$ SG: $M_n(-,-,-,+,\ldots,+)$ gravitons.

  NMHV sector: SUSY related (but much harder to solve SUSY Ward identities).

Generating functions encode dependence on external states.

Benefits of Generating Functions

$\longrightarrow$ Easy calculation of MHV and NMHV amplitudes
— just differentiate!

$\longrightarrow$ Precise characterization of MHV and NMHV sectors,
e.g. $A_n(\lambda_+ \lambda_+ \lambda_+ \lambda_+ \phi\, \phi)$ is MHV.

$\longrightarrow$ Counts distinct processes in each sector:

	MHV	NMHV
$\mathcal{N} = 4$:	15	34
$\mathcal{N} = 8$:	186	919

counting $\leftrightarrow$ partitions of integers!

$\longrightarrow$ Simple relationship $Z_n^{\mathcal{N}=8} \propto Z_n^{\mathcal{N}=4} \times Z_n^{\mathcal{N}=4}$ (MHV)
clarifies SUSY and global symmetries in map
$[\mathcal{N} = 8] = [\mathcal{N} = 4]_L \otimes [\mathcal{N} = 4]_R$ of states
and KLT relations $M_n = \sum(k_n\, A_n\, A'_n)$.

$\longrightarrow$ Applications to intermediate state sums in loop amplitudes.

Outline

1. Motivation
2. MHV generating functions
   - $\rightarrow$ $\mathcal{N} = 4$ SYM
   - $\rightarrow$ $\mathcal{N} = 8$ SG
3. Spin factors as conformal correlators
4. Recursion relations $\leftrightarrow$ MHV vertex expansion
5. Next-to-MHV generating functions
   - $\rightarrow$ $\mathcal{N} = 4$ SYM
   - $\rightarrow$ $\mathcal{N} = 8$ SG
6. Intermediate State Spin Sums
7. Outlook

2. MHV generating function — $\mathcal{N} = 4$ SYM

$$\begin{array}{ccc} \text{States } X_i & \leftrightarrow & \text{differential operators } D_{X_i} \\ \downarrow & & \downarrow \\ \text{Amplitude } A_n(X_1\, X_2 \ldots X_n) & = & D_{X_1}\, D_{X_2} \cdots D_{X_n}\, Z_n \end{array}$$

First need (state $\leftrightarrow$ diff op) correspondence.

$\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM has 2^4 massless states: $a, b = 1, 2, 3, 4 \in SU(4)$

1 gluon B^- , B_+

4 gluini F_a^- , F_+^a

6 self-dual scalars $B^{ab} = \frac{1}{2}\epsilon^{abcd} B_{cd}$

4 supercharges $\tilde{Q}_a = \epsilon_{\dot{\alpha}} \tilde{Q}_a^{\dot{\alpha}}$ and $Q^a = \tilde{Q}_a^*$ act on annihilation operators:

$$
\begin{aligned}
[\tilde{Q}_a, B_+(p)] &= 0\,, \\
[\tilde{Q}_a, F_+^b(p)] &= \langle \epsilon\, p \rangle\, \delta_a^b\, B_+(p)\,, \\
[\tilde{Q}_a, B^{bc}(p)] &= \langle \epsilon\, p \rangle \left(\delta_a^b\, F_+^c(p) - \delta_a^c\, F_+^b(p) \right), \\
[\tilde{Q}_a, B_{bc}(p)] &= \langle \epsilon\, p \rangle\, \epsilon_{abcd}\, F_+^d(p)\,, \\
[\tilde{Q}_a, F_b^-(p)] &= \langle \epsilon\, p \rangle\, B_{ab}(p)\,, \\
[\tilde{Q}_a, B^-(p)] &= -\langle \epsilon\, p \rangle\, F_a^-(p)
\end{aligned}
$$

(consistent with crossing sym. and self − duality)

$\mathcal{N} = 4$ SYM (state $\leftrightarrow$ diff op) correspondence

Introduce auxiliary Grassman variable η_{ia}
i momentum label p_i, $a = 1, \dots, 4$ is $SU(4)$ index.

Associate to each state Grassman diff ops $\partial_i^a = \frac{\partial}{\partial \eta_{ia}}$:

$$B_+(p_i) \quad \leftrightarrow \quad 1$$

$$F_+^a(p_i) \quad \leftrightarrow \quad \partial_i^a$$

$$B_+^{ab}(p_i) \quad \leftrightarrow \quad \partial_i^a \, \partial_i^b$$

$$F_a^-(p_i) \quad \leftrightarrow \quad -\frac{1}{3!} \, \epsilon_{abcd} \, \partial_i^b \, \partial_i^c \, \partial_i^d$$

$$B^-(p_i) \quad \leftrightarrow \quad \partial_i^1 \, \partial_i^2 \, \partial_i^3 \, \partial_i^4$$

This is our (state $\leftrightarrow$ diff op) correspondence.

SUSY generators $\tilde{Q}_a = \sum_{i=1}^{n} \langle \epsilon i \rangle \, \eta_{ia}$ and $Q^a = \sum_{i=1}^{n} [i\, \epsilon] \, \frac{\partial}{\partial \eta_{ia}}$ give correct SUSY algebra

$$[Q^a, \tilde{Q}_b] = \delta^a_b \sum_{i=1}^{n} [\epsilon_1 i] \langle i \epsilon_2 \rangle = \delta^a_b \sum_{i=1}^{n} \epsilon_1^{\alpha} \, p_{i\,\alpha\dot{\beta}} \, \tilde{\epsilon}_2^{\dot{\beta}} \to 0 \quad \text{(mom. cons.)},$$

and

$$[\tilde{Q}, \text{diff op}] = \langle \epsilon p \rangle (\text{diff op})'$$

produces correct algebra on states.

The MHV generating function is

$$Z_n^{\mathcal{N}=4}(\eta_{ia}) = \frac{A_n(1^-,2^-,3^+,\dots,n^+)}{\langle 12\rangle^4}\,\delta^{(8)}\Big(\sum_i |i\rangle\eta_{ia}\Big)\,,$$

where $\delta^{(8)}\big(\sum_i |i\rangle\eta_{ia}\big) = 2^{-4}\prod_{a=1}^4 \sum_{i,j=1}^n \langle i\, j\rangle\, \eta_{ia}\,\eta_{ja}\,.$

[Nair (1988)] [GGK (2004)]

(δ-function of Grassman variables θ_a is $\prod \theta_a$)

η_{ia}	—	auxilliary Grassman variables
$a = 1,2,3,4$	—	$SU(4)$ indices
$i,j = 1,2,\dots,n$	—	momentum labels

Claim: any 8th order derivative operator built from (state $\leftrightarrow$ diff op) correspondence gives an MHV amplitude when applied to $Z_n^{\mathcal{N}=4}$:

$$A_n^{\rm MHV}(X_1,\dots,X_n) = D_{X_1}\cdots D_{X_n}\,Z_n^{\mathcal{N}=4}\,.$$

Let's prove this!

Proof: $Z_n^{\mathcal{N}=4}(\eta_{ia}) = \frac{A_n(1^-,2^-,3^+,\ldots,n^+)}{\langle 12\rangle^4}\ \delta^{(8)}\big(\sum_i |i\rangle\eta_{ia}\big)$

- $Z_n^{\mathcal{N}=4}$ reproduces pure MHV gluon amplitude $A_n(1^-, 2^-, 3^+, \ldots, n^+)$ correctly:

$$(\partial_1^1\partial_1^2\partial_1^3\partial_1^4)(\partial_2^1\partial_2^2\partial_2^3\partial_2^4)\,\delta^{(8)}\big(\sum_i |i\rangle\eta_{ia}\big)$$
$$= (\partial_1^1\partial_1^2\partial_1^3\partial_1^4)(\partial_2^1\partial_2^2\partial_2^3\partial_2^4)\,\big(2^{-4}\prod_{a=1}^4\sum_{i,j=1}^n\langle i\,j\rangle\,\eta_{ia}\,\eta_{ja}\big)$$
$$= \langle 12\rangle^4.$$

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- $\tilde{Q}_a\, Z_n^{\mathcal{N}=4} \propto \big(\sum_{i=1}^n |i\rangle\,\eta_{ia}\big)\,\delta^{(8)}\big(\sum_i |i\rangle\eta_{ia}\big) = 0.$

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- $[\tilde{Q}_a, D^{(9)}]\,Z_n^{\mathcal{N}=4} = 0$

  encode the MHV SUSY Ward identities:

$$0 = [\tilde{Q}_a, D^{(9)}]\,Z_n^{\mathcal{N}=4} = \textstyle\sum_t D_{X_1}\cdots[\tilde{Q}_a, D_{X_t}]\cdots D_{X_n}\,Z_n^{\mathcal{N}=4},$$

$$0 = \langle[\tilde{Q}_a, X_1\dots X_n]\rangle = \textstyle\sum_t\langle X_1\dots[\tilde{Q}_a, X_t]\dots X_n\rangle\,.$$

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- MHV SUSY Ward identities have *unique* solutions.

Proof: $Z_n^{\mathcal{N}=4}(\eta_{ia}) = \frac{A_n(1^-,2^-,3^+,\dots,n^+)}{\langle 12\rangle^4}\;\delta^{(8)}\big(\sum_i |i\rangle\eta_{ia}\big)$

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$$0 = \langle[\tilde{Q}_a, X_1\dots X_n]\rangle = \textstyle\sum_t\langle X_1\dots[\tilde{Q}_a, X_t]\dots X_n\rangle\,.$$

- MHV SUSY Ward identities have *unique* solutions.

$\Rightarrow$ $Z_n^{\mathcal{N}=4}$ produces all MHV amplitudes correctly.

Characterizing amplitudes in the MHV sector of $\mathcal{N}=4$ SYM:

$$D^{(8)}\, Z_n^{\mathcal{N}=4} = \text{MHV amplitude}$$

hence

$$\#\ \text{MHV amplitudes} \;=\; \#\ \text{partitions of 8 with } n_{\max} = 4.$$

MHV amplitudes:

$$\begin{aligned} 8 &= 4+4 && \leftrightarrow \langle B^- B^- B_+ \dots B_+ \rangle \\ &= 4+3+1 && \leftrightarrow \langle B^- F_a^- F_+^a B_+ \dots B_+ \rangle \\ &\dots \\ &= 1+\dots+1 && \leftrightarrow \langle F_+^{a_1} \dots F_+^{a_8} B_+ \dots B_+ \rangle \end{aligned}$$

Total of 15 MHV amplitudes in $\mathcal{N}=4$ SYM.

Example:

Calculate $\langle B^-(p_1)\, F^1_+(p_2)\, F^2_+(p_3)\, F^3_+(p_4)\, F^4_+(p_5)\, B^+(p_6)\rangle$

$$(\partial^1_1\partial^2_1\partial^3_1\partial^4_1)(\partial^1_2)(\partial^2_3)(\partial^3_4)(\partial^4_5)\ \delta^{(8)}\Big(\sum_i |i\rangle\eta_{ia}\Big)$$
$$= (\partial^1_1\partial^1_2)(\partial^2_2\partial^2_3)(\partial^3_1\partial^3_4)(\partial^4_1\partial^4_5)\ \delta^{(8)}\Big(\sum_i |i\rangle\eta_{ia}\Big)$$
$$= \langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 15\rangle$$

using $\delta^{(8)}\big(\sum_i |i\rangle\eta_{ia}\big) = \big(2^{-4}\prod_{a=1}^4 \sum_{i,j=1}^n \langle i\,j\rangle\,\eta_{ia}\,\eta_{ja}\big)$,

so

$$\langle B^-(p_1)\, F^1_+(p_2)\, F^2_+(p_3)\, F^3_+(p_4)\, F^4_+(p_5)\, B^+(p_6)\rangle$$
$$= \frac{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 15\rangle}{\langle 12\rangle^4} A_n(1^-, 2^-, 3^+, 4^+, 5^+, 6^+).$$

2. MHV generating function — $\mathcal{N} = 8$ SG

Completely analogous setup: $A, B, \dots = 1, \dots, 8 \in SU(8)$

1 graviton	$b_+(p_i)$	$\leftrightarrow$	1
8 gravitino	$f_+^A(p_i)$	$\leftrightarrow$	∂_i^A
28 gravi – photons	$b_+^{AB}(p_i)$	$\leftrightarrow$	$\partial_i^A\,\partial_i^B$
56 gravi – photinos	$f_+^{ABC}(p_i)$	$\leftrightarrow$	$\partial_i^A\,\partial_i^B\,\partial_i^C$
70 self – dual scalars	$b^{ABCD}(p_i)$	$\leftrightarrow$	$\partial_i^A\,\partial_i^B\,\partial_i^C\,\partial_i^D$
56 gravi – photinos	$f_{ABC}^-(p_i)$	$\leftrightarrow$	$-\frac{1}{5!}\epsilon_{ABCDEFGH}\,\partial_i^D\cdots\partial_i^H$
28 gravi – photons	$b_{AB}^-(p_i)$	$\leftrightarrow$	$\frac{1}{6!}\epsilon_{ABCDEFGH}\,\partial_i^C\cdots\partial_i^H$
8 gravitino	$f_+^A(p_i)$	$\leftrightarrow$	$-\frac{1}{7!}\epsilon_{ABCDEFGH}\,\partial_i^B\cdots\partial_i^H$
1 graviton	$b^-(p_i)$	$\leftrightarrow$	$\partial_i^1\cdots\partial_i^8$

Total of $256 = 2^8$ massless states.

8 supercharges $\tilde{Q}_A = \epsilon_{\dot\alpha}\,\tilde{Q}_A^{\dot\alpha}$ and $Q^A = \tilde{Q}_A^*$.

The **MHV generating function** for $\mathcal{N}=8$ SG is

$$Z_n^{\mathcal{N}=8}(\eta_{iA}) = \frac{M_n(1^-,2^-,3^+,\ldots,n^+)}{\langle 12\rangle^8}\,\delta^{(16)}\Big(\sum_i |i\rangle\eta_{iA}\Big)$$

with $\delta^{(16)}\big(\sum_i |i\rangle\eta_{iA}\big) = 2^{-8}\prod_{A=1}^{8}\sum_{i,j=1}^{n}\langle i\,j\rangle\,\eta_{iA}\,\eta_{jA}$

Any 16th order derivative operator built from (state $\leftrightarrow$ diff op) correspondence gives an MHV amplitude when applied to $Z_n^{\mathcal{N}=8}$.

$\mathcal{N} = 8$ supergravity:

$\#$ MHV amplitudes $=$ $\#$ partitions of 16 with $n_{\max} = 8$.

MHV amplitudes:

$$
\begin{aligned}
16 &= 8+8 && \leftrightarrow && \langle b^- \, b^- \, b_+ \dots b_+ \rangle \\
&= 8+7+1 && \leftrightarrow && \langle b^- \, f_A^- \, f_+^A \, b_+ \dots b_+ \rangle \\
\dots \\
&= 1+\dots+1 && \leftrightarrow && \langle f_+^{A_1} \dots f_+^{A_{16}} \, b_+ \dots b_+ \rangle
\end{aligned}
$$

Total of 186 MHV amplitudes in $\mathcal{N} = 8$ SYM.

Factorization

- Spectrum $[\mathcal{N}=8\ \mathrm{SG}] \ = \ [\mathcal{N}=4\ \mathrm{SYM}]_L \otimes [\mathcal{N}=4\ \mathrm{SYM}]_R$

  e.g. $b^- = B^- \otimes \tilde{B}^- \quad (2 = 1 \otimes 1)$.

- Also, supergravity amplitudes factor in to (sums of) products of SYM amplitudes (KLT relations)

$$M_n = \sum k_n \, A_n \, A'_n \,,$$

  with k_n kinematic factors.

For MHV this works because

- Diff operators factorize $\quad D^{\mathcal{N}=8} = D^{\mathcal{N}=4} \times D^{\mathcal{N}=4}$
- MHV generating function factorizes $\quad Z_n^{\mathcal{N}=8} \propto Z_{n\ (1234)}^{\mathcal{N}=4} \times Z_{n\ (5678)}^{\mathcal{N}=4}$

  $\Rightarrow$ dependence on external states factorizes

  $\Rightarrow$ $SU(8) \leftrightarrow SU(4)_L \times SU(4)_R$ naturally implemented.

MHV generating functions — summarized

- Simple encoding of external states.
- Clean and efficient way to calculate amplitudes.
- Factorization illuminates $[\mathcal{N}=8] \;=\; [\mathcal{N}=4]_L \otimes [\mathcal{N}=4]_R$ and $SU(8) \leftrightarrow SU(4) \times SU(4)$.
- Applications to intermediate spin sums in loop calculations (later).
- Fun conformal analogy (next).

3. Spin factors as conformal correlators

$\mathcal{N} = 4$ SYM: (similarly for gravity)

Define

$$\text{spin factor} \equiv D^{(8)}\delta^{(8)}(I),$$

so that

$$\text{MHV amplitude} = (\text{spin factor}) \times \frac{A_n(-,-,+,\ldots,+)}{\langle 12\rangle^4}.$$

For n-point amplitudes:

spin factor = product of 4 of $\binom{n}{2}$ independent $\langle ij\rangle$'s.

Example: (8=3+3+1+1)

$$\langle F_1^- \, F_2^- \, F_+^2 \, F_+^1\rangle = (-\partial_1^2\partial_1^3\partial_1^4)(\partial_2^1\partial_2^3\partial_2^4)(\partial_3^2)(\partial_4^1)\,\delta^{(8)}(I) = -\langle 12\rangle^2\langle 13\rangle\langle 24\rangle$$

- Case $n = 3$:

  $A_3(X_1(p_1)X_2(p_2)X_3(p_3))$

  with "weights" r_i = order of diff op for particle X_i. *Then*

  $$\text{spin factor } = D_1^{(r_1)} D_2^{(r_2)} D_3^{(r_3)}\, \delta^{(8)}(I) = \langle 12\rangle^{\nu_{12}} \langle 23\rangle^{\nu_{23}} \langle 31\rangle^{\nu_{31}},$$

  where

  $$\nu_{12} + \nu_{31} = r_1, \quad \nu_{23} + \nu_{12} = r_2, \quad \nu_{31} + \nu_{23} = r_3 \;.$$

  Solve to find $\nu_{ij} = \frac{1}{2}(r_i + r_j - r_k)$.

  $\rightarrow$ just like 3-point CFT correlator with primary operators of scale dimensions $(r_i, 0)$,

  $$\langle O_1(z_1)\, O_2(z_2)\, O_3(z_3)\rangle = c_{123} \frac{1}{z_{12}^{\nu_{12}} z_{23}^{\nu_{23}} z_{31}^{\nu_{31}}} \;.$$

- What about $n = 4$?
  - spin factor $= \langle 12\rangle^{\nu_{12}} \langle 13\rangle^{\nu_{13}} \langle 14\rangle^{\nu_{14}} \langle 23\rangle^{\nu_{23}} \langle 24\rangle^{\nu_{24}} \langle 34\rangle^{\nu_{34}}$,

  but $\nu_{ij} \geq 0$ only constrained by 4 equations.
  - Leaves freedom of multiplying by *cross-ratio* $\zeta = \frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle}$.
  - If $\bar{\nu}_{ij}$ is one solution, then so is

  $$\text{spin factor} = f(\zeta)\, \langle 12\rangle^{\nu_{12}} \langle 13\rangle^{\bar{\nu}_{13}} \langle 14\rangle^{\bar{\nu}_{14}} \langle 23\rangle^{\bar{\nu}_{23}} \langle 24\rangle^{\bar{\nu}_{24}} \langle 34\rangle^{\bar{\nu}_{34}} \quad ,$$

  where f is any function such that powers of $\langle ..\rangle$ remain positive and r_i are integers.

  The freedom to choose f corresponds to the distinct choices of $SU(4)$ indices on the external states.

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  where f is any function such that powers of $\langle ..\rangle$ remain positive and r_i are integers.

  The freedom to choose f corresponds to the distinct choices of $SU(4)$ indices on the external states.

Example: (8=3+3+1+1)

$\langle F_1^- \, F_2^- \, F_+^2 \, F_+^1\rangle = \langle 12\rangle^2\langle 13\rangle\langle 24\rangle,$

$\langle F_1^- \, F_2^- \, F_+^1 \, F_+^2\rangle = \langle 12\rangle^2\langle 14\rangle\langle 23\rangle = (1-\zeta)\langle 12\rangle^2\langle 13\rangle\langle 24\rangle.$

using the Schouten identity $\langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 42\rangle + \langle 14\rangle\langle 23\rangle = 0$.

- Note that

$$|i\rangle \;\rightarrow\; \binom{1}{z_i} \qquad \rightarrow \qquad \langle ij\rangle = z_i - z_j = z_{ij}$$

makes the conformal analogy precise.

- General n:

$n-3$ independent cross-ratios.

Outline

4. Recursion relations $\leftrightarrow$ MHV vertex expansion

- **Recursion relations**: express on-shell n-point amplitude in terms of k-point on-shell sub-amplitudes with $k < n$.
- Even better if sub-amplitudes are MHV
  $\rightarrow$ **MHV vertex expansion**.

For gluons:

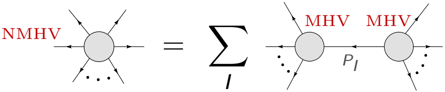
[Cachazo, Svrcek, Witten (2004)] [Risager (2005)]

For gravitons, $n = 6, 7$:

[Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager (2005)]

Our strategy

- Use recursion relations to expand NMHV amplitudes in terms of MHV vertex diagrams



- Apply MHV generating functions to MHV vertices
  $\longrightarrow$ generating function $\Omega_{n,I}$ for each diagram I in MHV vertex expansion.
- NMHV generating function is $\Omega_n = \sum_I \Omega_{n,I}$

3-line shift recursion relations

- Analytically continue amplitudes to complex values by *shifts* of 3 external momenta:

$$p_i^\mu \to \hat{p}_i^\mu = p_i^\mu + z\, q_i^\mu\,, \qquad \text{for} \quad i = 1, 2, 3.$$

  where

$$q_1^\mu + q_2^\mu + q_3^\mu = 0 \qquad \leftrightarrow \qquad \text{momentum conservation}$$

$$q_i^2 = 0 = q_i \cdot p_i \qquad \leftrightarrow \qquad \text{on} - \text{shell} \quad \hat{p}_i^2 = 0.$$

  Achieved by $|1] \to |\hat{1}] = |1] + z\langle 23\rangle |X]$ (+ cyclic) with $|X]$ arbitrary "reference spinor".

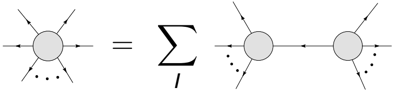
- The tree amplitude $A_n(z)$ has only simple poles, so **if** $A_n(z) \to 0$ for $z \to \infty$, then

$$0 = \oint \frac{A_n(z)}{z} \qquad \to \qquad A_n(0) = -\sum_{z \neq 0} \text{Res} \frac{A_n(z)}{z}$$

- Result is on-shell recursion relation

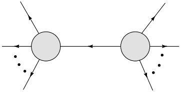
$$A_n(0) = \sum_I A_{n_1} \frac{1}{P_I^2} A_{n_2}\,, \qquad n_1 + n_2 = n + 2$$

The special 3-line shift ensures that the sub-amplitudes are both MHV if A_n is NMHV. [Risager (2005)]



→ Now use this to get NMHV gen func.

5. Next-to-MHV generating functions — $\mathcal{N}=4$ SYM

- Consider a single MHV vertex diagram: 

- Apply MHV gen func to each vertex to derive (details omitted)

$$\Omega_{n,I}^{\mathcal{N}=4} \quad = \quad \frac{A_{n,I}^{\text{gluons}}}{\langle m_1 P_I\rangle^4 \langle m_2 m_3\rangle^4}\,\delta^{(8)}(L_a + R_a)\prod_{a=1}^{4}\langle P_I\, L_a\rangle$$

where $L_a = \sum_{i\in L} |i\rangle\eta_{ia}$ and $R_a = \sum_{j\in R} |j\rangle\eta_{ja}$.

[Georgio, Glover and Khoze (2004)]

- Each term in $\Omega_{n,I}^{\mathcal{N}=4}$ is order 12 in η_{ia}'s.
- Value of diagram is $D^{(12)}\,\Omega_{n,I}^{\mathcal{N}=4}$ with $D^{(12)}$ built from the external states.
- Sum all diagram gen func's to get full NMHV gen func:

$$\Omega_n^{\mathcal{N}=4} = \sum_I \Omega_{n,I}^{\mathcal{N}=4}$$

Example:
NMHV gluon amplitude

$$A_n(1^-, 2^-, 3^-, 4^+, \ldots, n^+) = D_1^{(4)} D_2^{(4)} D_3^{(4)}\ \Omega_n^{\mathcal{N}=4}$$

Partition $12 = 4+4+4$.

$\mathcal{N} = 4$ SYM:

$$\#\ \text{NMHV amplitudes} \ = \ \#\ \text{partitions of 12 with } n_{\max} = 4.$$

Total of 34.

5. Next-to-MHV generating functions — $\mathcal{N} = 8$ SG

Repeat construction in $\mathcal{N} = 8$ SG $\rightarrow$ $\Omega_n^{\mathcal{N}=8} \leftarrow$ order 24 in η'_{iA}s.

Example: NMHV graviton amplitude

$$M_n(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) = D_1^{(8)} D_2^{(8)} D_3^{(8)}\ \Omega_n^{\mathcal{N}=8}$$

Partition $24 = 8+8+8$.

$\mathcal{N} = 8$ SG:

NMHV amplitudes = # partitions of 24 with $n_{\max} = 8$.

Total of 919.

- Now sum over more diagrams, since not color-ordered.
  For n=6 there are up to 21 diagrams.
- Spin factors factorize, but only diagram-by-diagram

$$\Omega_{n,I}^{\mathcal{N}=8} \propto \Omega_{n,I\ (1234)}^{\mathcal{N}=4} \times \Omega_{n,I\ (5678)}^{\mathcal{N}=4}.$$

But. . .

We used MHV vertex expansion from 3-line shift recursion relations, which *assumed*

$$A_n(z) \rightarrow 0 \quad \text{for} \quad z \rightarrow \infty.$$

Is this OK?

- In $\mathcal{N} = 4$ SYM we have shown that one can always choose 3 lines such that under a subsequent shift of these 3 lines each *diagram* in the corresponding MHV vertex expansion falls off at least as $1/z$ for large z.

  $\rightarrow$ so only "bad" large z behavior could come from a term "at infinity" missed by Cauchy's thm.

  $\rightarrow$ Have not seen any signs of such trouble.

Note

- Complex shifts included an arbitrary "reference spinor" $|X]$
  NB: $|1] \to |\hat{1}] = |1] + z\langle 23\rangle|X]$ and cyclic(123) copies.

- If $A_n(z; |X]) \to 0$ as $z \to \infty$ for *all* $|X]$, then the recursion sum of MHV vertex diagrams must be *independent* of $|X]$.

  Note: Generally each MHV vertex diagram depends on $|X]$, but sum of all diagrams must be $|X]$-independent.

- Indep of $|X]$ is very useful check of correctness of amplitude calculation.

- In $\mathcal{N} = 8$ SG we encounter for 6-point NMHV amplitudes:
  - "Good" amplitudes: $A_n(z) \to 0$ as $z \to \infty$

    Ex. $\langle b^{1234}\, b^{1234}\, b^{1234}\, b^{5678}\, b^{5678}\, b^{5678} \rangle$

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  - "Bad" amplitudes: $A_n(z) \to O(1)$ as $z \to \infty$

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    NMHV generating function valid for special $|X]$'s such that $O(1)_X = 0$.
  - "Very bad" amplitudes: $A_n(z) \to O(z)$ as $z \to \infty$

    2 such amplitudes

    No choice of $|X]$ makes $O(z)_X \to O(1/z)$.

    These 2 amplitudes can be determined by SUSY WI in terms of other 6-point NMHV amplitudes.

Graviton n-point amplitude

Large z for pure graviton n-point amplitude:

$$M_n(\hat{1}^-, \hat{2}^-, \hat{3}^-, 4^+, \ldots, n^+) \to z^{n-12} \quad \text{for} \quad z \to \infty$$

Numerically verified for $n = 5, \ldots, 11$.

How:

1. Calculate M_n with MHV vertex expansion.
   Test $|X]$-independence of sum of $3(2^{n-3} - 1)$ diagrams.
2. Calculate M_n using 2-line shift recursion relations $[-, -\rangle$.
   Test numerically agreement with M_n from MHV vertex expansion.
3. Perform $|1, 2, 3]$-shift on M_n and expand for large z with numerical values of all momentum spinors.

Also numerical test that the sum of 1533 MHV vertex diagrams for $n = 12$ is $|X]$-dependent.

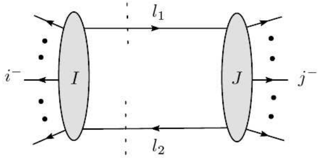
Expect the MHV vertex expansion to *fail* for $n \geq 12$.

NMHV generating functions — summarized

- *When it is valid*, the NMHV generating function provides very effective method for calculating NMHV amplitudes.
  - Easy to automate.
  - Useful checks of indep of reference spinor.
- Evidence that NMHV generating function valid for all *n*-point NMHV amplitudes of $\mathcal{N} = 4$ SYM.
- Examples, and a general analysis, shows that NMHV generating function is valid for a large set of NMHV amplitudes of $\mathcal{N} = 8$ SG, BUT *not* for all due to failure of MHV vertex expansion.
  - Must be careful in applications.

6. Intermediate state sum

Example: One-loop MHV amplitude



Use MHV generating function to compute intermediate state sum of unitarity cut:

$$D_{l_1}^{(4)}\, D_{l_2}^{(4)}\, (D_i^{(4)} \delta^{(8)}(I))(D_j^{(4)} \delta^{(8)}(J))$$

D_{l_1} and D_{l_2} distribute themselves between $\delta^{(8)}(I)$ and $\delta^{(8)}(J)$. This automatically takes care of the intermediate state sum.

Have done 1- and 2-loop sums with NMHV generating function, but care is needed to avoid "bad" shifts, especially in SG.

7. Outlook

Role of $E_{7,7}$?

- 70 scalars of $\mathcal{N} = 8$ SG are Goldstone bosons of spontaneously broken $E_{7,7} \to SU(8)$.
- How will $E_{7,7}$ reveal itself?
  $\to$ soft-scalar limits of amplitudes
  (analogous to soft-pion low-energy theorems of Adler).
- We find that 1-soft-"pion" limits of $\mathcal{N} = 8$ tree amplitudes *vanish*.
- Note that in pion physics 1-pion soft limits do not necessarily vanish, even in models with pions and nucleons both massless.

Loops in $\mathcal{N} = 8$ supergravity

Is there are connection between "bad" large z behavior in supergravity tree amplitudes and potential UV divergencies?