Gauging WONDERS.

$\mathcal{N}=4$ SYM helping QCD:

Whether. How. When.

Yuri Dokshitzer

kicking-off the discussion in a provocative manner

Saclay
26.06 2008

exploring one idea

exploring one idea

Inheritance

exploring one idea

Inheritance:

✗ Higher loops inherit complexity from lower orders

exploring one idea

Inheritance:

- ✗ Higher loops inherit complexity from lower orders
- ✗ QCD inherits from SYM-4

3rd loop non-singlet a.d.

$$P^{(2)+}_{\rm ns}(x) = 16 C_A C_F n_f \Bigg(\frac{1}{6} p_{\rm qq}(x)\Bigg[\frac{10}{3}\zeta_2 - \frac{209}{36} - 9\zeta_3 - \frac{167}{18}\mathrm{H}_0 + 2\mathrm{H}_0\zeta_2 - 7\mathrm{H}_0$$

$$+3\mathrm{H}_{1,0,0} - \mathrm{H}_3\Bigg] + \frac{1}{3} p_{\rm qq}(-x)\Bigg[\frac{3}{2}\zeta_3 - \frac{5}{3}\zeta_2 - \mathrm{H}_{-2,0} - 2\mathrm{H}_{-1}\zeta_2 - \frac{10}{3}\mathrm{H}_{-1,0} - \mathrm{H}_-$$

$$+2\mathrm{H}_{-1,2} + \frac{1}{2}\mathrm{H}_0\zeta_2 + \frac{5}{3}\mathrm{H}_{0,0} + \mathrm{H}_{0,0,0} - \mathrm{H}_3\Bigg] + (1-x)\Bigg[\frac{1}{6}\zeta_2 - \frac{257}{54} - \frac{43}{18}\mathrm{H}_0 - \frac{1}{6}$$

$$-(1+x)\Bigg[\frac{2}{3}\mathrm{H}_{-1,0} + \frac{1}{2}\mathrm{H}_2\Bigg] + \frac{1}{3}\zeta_2 + \mathrm{H}_0 + \frac{1}{6}\mathrm{H}_{0,0} + \delta(1-x)\Bigg[\frac{5}{4} - \frac{167}{54}\zeta_2 + \frac{1}{20}\zeta_2$$

$$+16\, C_A {C_F}^2 \Bigg(p_{\rm qq}(x)\Bigg[\frac{5}{6}\zeta_3 - \frac{69}{20}{\zeta_2}^2 - \mathrm{H}_{-3,0} - 3\mathrm{H}_{-2}\zeta_2 - 14\mathrm{H}_{-2,-1,0} + 3\mathrm{H}_{-2,0}$$

$$-4\mathrm{H}_{-2,2} - \frac{151}{48}\mathrm{H}_0 + \frac{41}{12}\mathrm{H}_0\zeta_2 - \frac{17}{2}\mathrm{H}_0\zeta_3 - \frac{13}{4}\mathrm{H}_{0,0} - 4\mathrm{H}_{0,0}\zeta_2 - \frac{23}{12}\mathrm{H}_{0,0,0} + 5\mathrm{H}$$

$$-24\mathrm{H}_1\zeta_3 - 16\mathrm{H}_{1,-2,0} + \frac{67}{9}\mathrm{H}_{1,0} - 2\mathrm{H}_{1,0}\zeta_2 + \frac{31}{3}\mathrm{H}_{1,0,0} + 11\mathrm{H}_{1,0,0,0} + 8\mathrm{H}_{1,1,0,0}$$

3rd loop, more

$$+\frac{67}{9}\mathrm{H}_2 - 2\mathrm{H}_2\zeta_2 + \frac{11}{3}\mathrm{H}_{2,0} + 5\mathrm{H}_{2,0,0} + \mathrm{H}_{3,0}\Big] + p_{\mathrm{qq}}(-x)\Big[\frac{1}{4}\zeta_2{}^2 - \frac{67}{9}\zeta_2 + \frac{31}{4}\zeta_3$$

$$-32\mathrm{H}_{-2}\zeta_2 - 4\mathrm{H}_{-2,-1,0} - \frac{31}{6}\mathrm{H}_{-2,0} + 21\mathrm{H}_{-2,0,0} + 30\mathrm{H}_{-2,2} - \frac{31}{3}\mathrm{H}_{-1}\zeta_2 - 42\mathrm{H}$$

$$-4\mathrm{H}_{-1,-2,0} + 56\mathrm{H}_{-1,-1}\zeta_2 - 36\mathrm{H}_{-1,-1,0,0} - 56\mathrm{H}_{-1,-1,2} - \frac{134}{9}\mathrm{H}_{-1,0} - 42\mathrm{H}_{-1}$$

$$+32\mathrm{H}_{-1,3} - \frac{31}{6}\mathrm{H}_{-1,0,0} + 17\mathrm{H}_{-1,0,0,0} + \frac{31}{3}\mathrm{H}_{-1,2} + 2\mathrm{H}_{-1,2,0} + \frac{13}{12}\mathrm{H}_0\zeta_2 + \frac{29}{2}\mathrm{H}$$

$$+13\mathrm{H}_{0,0}\zeta_2 + \frac{89}{12}\mathrm{H}_{0,0,0} - 5\mathrm{H}_{0,0,0,0} - 7\mathrm{H}_2\zeta_2 - \frac{31}{6}\mathrm{H}_3 - 10\mathrm{H}_4\Big] + (1-x)\Big[\frac{133}{36} +$$

$$-\frac{167}{4}\zeta_3 - 2\mathrm{H}_0\zeta_3 - 2\mathrm{H}_{-3,0} + \mathrm{H}_{-2}\zeta_2 + 2\mathrm{H}_{-2,-1,0} - 3\mathrm{H}_{-2,0,0} + \frac{77}{4}\mathrm{H}_{0,0,0} - \frac{20}{6}$$

$$+4\mathrm{H}_{1,0,0} + \frac{14}{3}\mathrm{H}_{1,0}\Big] + (1+x)\Big[\frac{43}{2}\zeta_2 - 3\zeta_2{}^2 + \frac{25}{2}\mathrm{H}_{-2,0} - 31\mathrm{H}_{-1}\zeta_2 - 14\mathrm{H}_{-1,-}$$

$$+24\mathrm{H}_{-1,2} + 23\mathrm{H}_{-1,0,0} + \frac{55}{2}\mathrm{H}_0\zeta_2 + 5\mathrm{H}_{0,0}\zeta_2 + \frac{1457}{48}\mathrm{H}_0 - \frac{1025}{36}\mathrm{H}_{0,0} - \frac{155}{6}\mathrm{H}_2$$

3rd loop, and more

$$+2\mathrm{H}_{2,0,0} - 3\mathrm{H}_4\Big] - 5\zeta_2 - \frac{1}{2}{\zeta_2}^2 + 50\zeta_3 - 2\mathrm{H}_{-3,0} - 7\mathrm{H}_{-2,0} - \mathrm{H}_0\zeta_3 - \frac{37}{2}\mathrm{H}_0\zeta_2$$

$$-2\mathrm{H}_{0,0}\zeta_2 + \frac{185}{6}\mathrm{H}_{0,0} - 22\mathrm{H}_{0,0,0} - 4\mathrm{H}_{0,0,0,0} + \frac{28}{3}\mathrm{H}_2 + 6\mathrm{H}_3 + \delta(1-x)\Big[\frac{151}{64} +$$

$$-\frac{247}{60}{\zeta_2}^2 + \frac{211}{12}\zeta_3 + \frac{15}{2}\zeta_5\Big]\Big) + 16\, {C_A}^2 C_F \Big(p_{\mathrm{qq}}(x)\Big[\frac{245}{48} - \frac{67}{18}\zeta_2 + \frac{12}{5}{\zeta_2}^2 + \frac{1}{2}$$

$$+\mathrm{H}_{-3,0} + 4\mathrm{H}_{-2,-1,0} - \frac{3}{2}\mathrm{H}_{-2,0} - \mathrm{H}_{-2,0,0} + 2\mathrm{H}_{-2,2} - \frac{31}{12}\mathrm{H}_0\zeta_2 + 4\mathrm{H}_0\zeta_3 + \frac{389}{72}$$

$$-\mathrm{H}_{0,0,0,0} + 9\mathrm{H}_1\zeta_3 + 6\mathrm{H}_{1,-2,0} - \mathrm{H}_{1,0}\zeta_2 - \frac{11}{4}\mathrm{H}_{1,0,0} - 3\mathrm{H}_{1,0,0,0} - 4\mathrm{H}_{1,1,0,0} + 4$$

$$+\frac{11}{12}\mathrm{H}_3 + \mathrm{H}_4\Big] + p_{\mathrm{qq}}(-x)\Big[\frac{67}{18}\zeta_2 - {\zeta_2}^2 - \frac{11}{4}\zeta_3 - \mathrm{H}_{-3,0} + 8\mathrm{H}_{-2}\zeta_2 + \frac{11}{6}\mathrm{H}_{-2,0}$$

$$-3\mathrm{H}_{-1,0,0,0} + \frac{11}{3}\mathrm{H}_{-1}\zeta_2 + 12\mathrm{H}_{-1}\zeta_3 - 16\mathrm{H}_{-1,-1}\zeta_2 + 8\mathrm{H}_{-1,-1,0,0} + 16\mathrm{H}_{-1,-1,2}$$

$$-8\mathrm{H}_{-2,2} + 11\mathrm{H}_{-1,0}\zeta_2 + \frac{11}{6}\mathrm{H}_{-1,0,0} - \frac{11}{3}\mathrm{H}_{-1,2} - 8\mathrm{H}_{-1,3} - \frac{3}{4}\mathrm{H}_0 - \frac{1}{6}\mathrm{H}_0\zeta_2 - 4$$

3rd loop, and again

$$-3\mathrm{H}_{0,0}\zeta_2 - \frac{31}{12}\mathrm{H}_{0,0,0} + \mathrm{H}_{0,0,0,0} + 2\mathrm{H}_2\zeta_2 + \frac{11}{6}\mathrm{H}_3 + 2\mathrm{H}_4\Big] + (1-x)\Big[\frac{1883}{108} - \frac{1}{2}$$

$$-\mathrm{H}_{-2,-1,0} + \frac{1}{2}\mathrm{H}_{-3,0} - \frac{1}{2}\mathrm{H}_{-2}\zeta_2 + \frac{1}{2}\mathrm{H}_{-2,0,0} + \frac{523}{36}\mathrm{H}_0 + \mathrm{H}_0\zeta_3 - \frac{13}{3}\mathrm{H}_{0,0} - \frac{5}{2}\mathrm{H}$$

$$-2\mathrm{H}_{1,0,0}\Big] + (1+x)\Big[8\mathrm{H}_{-1}\zeta_2 + 4\mathrm{H}_{-1,-1,0} + \frac{8}{3}\mathrm{H}_{-1,0} - 5\mathrm{H}_{-1,0,0} - 6\mathrm{H}_{-1,2} - \frac{13}{3}$$

$$-\frac{43}{4}\zeta_3 - \frac{5}{2}\mathrm{H}_{-2,0} - \frac{11}{2}\mathrm{H}_0\zeta_2 - \frac{1}{2}\mathrm{H}_2\zeta_2 - \frac{5}{4}\mathrm{H}_{0,0}\zeta_2 + 7\mathrm{H}_2 - \frac{1}{4}\mathrm{H}_{2,0,0} + 3\mathrm{H}_3 + \frac{3}{4}$$

$$+\frac{1}{4}\zeta_2^{\,2} - \frac{8}{3}\zeta_2 + \frac{17}{2}\zeta_3 + \mathrm{H}_{-2,0} - \frac{19}{2}\mathrm{H}_0 + \frac{5}{2}\mathrm{H}_0\zeta_2 - \mathrm{H}_0\zeta_3 + \frac{13}{3}\mathrm{H}_{0,0} + \frac{5}{2}\mathrm{H}_{0,0,0}$$

$$-\delta(1-x)\Big[\frac{1657}{576} - \frac{281}{27}\zeta_2 + \frac{1}{8}\zeta_2^{\,2} + \frac{97}{9}\zeta_3 - \frac{5}{2}\zeta_5\Big]\Big) + 16\, C_F n_f^2\Big(\frac{1}{18}p_{\mathrm{qq}}(x)\Big[\mathrm{H}_{0,}$$

$$+(1-x)\Big[\frac{13}{54} + \frac{1}{9}\mathrm{H}_0\Big] - \delta(1-x)\Big[\frac{17}{144} - \frac{5}{27}\zeta_2 + \frac{1}{9}\zeta_3\Big]\Big) + 16\, C_F^{\,2} n_f\Big(\frac{1}{3}p_{\mathrm{qq}}(x)\Big[$$

3rd loop, and still some more

$$-\frac{55}{16}+\frac{5}{8}\mathrm{H}_0+\mathrm{H}_0\zeta_2+\frac{3}{2}\mathrm{H}_{0,0}-\mathrm{H}_{0,0,0}-\frac{10}{3}\mathrm{H}_{1,0}-\frac{10}{3}\mathrm{H}_2-2\mathrm{H}_{2,0}-2\mathrm{H}_3\Big]+\frac{2}{3}$$

$$-\frac{3}{2}\zeta_3+\mathrm{H}_{-2,0}+2\mathrm{H}_{-1}\zeta_2+\frac{10}{3}\mathrm{H}_{-1,0}+\mathrm{H}_{-1,0,0}-2\mathrm{H}_{-1,2}-\frac{1}{2}\mathrm{H}_0\zeta_2-\frac{5}{3}\mathrm{H}_{0,0}-$$

$$-(1-x)\Big[\frac{10}{9}+\frac{19}{18}\mathrm{H}_{0,0}-\frac{4}{3}\mathrm{H}_1+\frac{2}{3}\mathrm{H}_{1,0}+\frac{4}{3}\mathrm{H}_2\Big]+(1+x)\Big[\frac{4}{3}\mathrm{H}_{-1,0}-\frac{25}{24}\mathrm{H}_0+$$

$$+\frac{7}{9}\mathrm{H}_{0,0}+\frac{4}{3}\mathrm{H}_2-\delta(1-x)\Big[\frac{23}{16}-\frac{5}{12}\zeta_2-\frac{29}{30}\zeta_2{}^2+\frac{17}{6}\zeta_3\Big]\Big)+16\,{C_F}^3\Big(p_{\mathrm{qq}}(x)\Big[$$

$$+6\mathrm{H}_{-2}\zeta_2+12\mathrm{H}_{-2,-1,0}-6\mathrm{H}_{-2,0,0}-\frac{3}{16}\mathrm{H}_0-\frac{3}{2}\mathrm{H}_0\zeta_2+\mathrm{H}_0\zeta_3+\frac{13}{8}\mathrm{H}_{0,0}-2\mathrm{H}_0$$

$$+12\mathrm{H}_1\zeta_3+8\mathrm{H}_{1,-2,0}-6\mathrm{H}_{1,0,0}-4\mathrm{H}_{1,0,0,0}+4\mathrm{H}_{1,2,0}-3\mathrm{H}_{2,0}+2\mathrm{H}_{2,0,0}+4\mathrm{H}_{2,1}$$

$$+4\mathrm{H}_{3,0}+4\mathrm{H}_{3,1}+2\mathrm{H}_4\Big]+p_{\mathrm{qq}}(-x)\Big[\frac{7}{2}\zeta_2{}^2-\frac{9}{2}\zeta_3-6\mathrm{H}_{-3,0}+32\mathrm{H}_{-2}\zeta_2+8\mathrm{H}_{-2}$$

$$-26\mathrm{H}_{-2,0,0}-28\mathrm{H}_{-2,2}+6\mathrm{H}_{-1}\zeta_2+36\mathrm{H}_{-1}\zeta_3+8\mathrm{H}_{-1,-2,0}-48\mathrm{H}_{-1,-1}\zeta_2+40\mathrm{I}$$

3rd loop, and UFF

$$+48H_{-1,-1,2} + 40H_{-1,0}\zeta_2 + 3H_{-1,0,0} - 22H_{-1,0,0,0} - 6H_{-1,2} - 4H_{-1,2,0} - 32$$

$$-\frac{3}{2}H_0\zeta_2 - 13H_0\zeta_3 - 14H_{0,0}\zeta_2 - \frac{9}{2}H_{0,0,0} + 6H_{0,0,0,0} + 6H_2\zeta_2 + 3H_3 + 2H_{3,0}$$

$$+(1-x)\left[2H_{-3,0} - \frac{31}{8} + 4H_{-2,0,0} + H_{0,0}\zeta_2 - 3H_{0,0,0,0} + 35H_1 + 6H_1\zeta_2 - H_{1,}\right.$$

$$+(1+x)\left[\frac{37}{10}{\zeta_2}^2 - \frac{93}{4}\zeta_2 - \frac{81}{2}\zeta_3 - 15H_{-2,0} + 30H_{-1}\zeta_2 + 12H_{-1,-1,0} - 2H_{-1,0}\right.$$

$$-24H_{-1,2} - \frac{539}{16}H_0 - 28H_0\zeta_2 + \frac{191}{8}H_{0,0} + 20H_{0,0,0} + \frac{85}{4}H_2 - 3H_{2,0,0} - 2H_3$$

$$\left.-H_4\right] + 4\zeta_2 + 33\zeta_3 + 4H_{-3,0} + 10H_{-2,0} + \frac{67}{2}H_0 + 6H_0\zeta_3 + 19H_0\zeta_2 - 25H_{0,0}$$

$$-2H_2 - H_{2,0} - 4H_3 + \delta(1-x)\left[\frac{29}{32} - 2\zeta_2\zeta_3 + \frac{9}{8}\zeta_2 + \frac{18}{5}{\zeta_2}^2 + \frac{17}{4}\zeta_3 - 15\zeta_5\right]\Bigg)$$

music of the spheres

2×2 anomalous dimension matrix occupies

1 st loop: $1/10$ page

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3 rd loop: 100 pages (200 K asci)

Moch, Vermaseren and Vogt

[waterfall of results launched
March 2004, and counting]

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$$V \sim \begin{cases} 10^{\frac{N(N-1)}{2}-1} \\ 10^{2^{N-1}-2} \end{cases}$$

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not too encouraging a trend ...

Fighting complexity

How to reduce complexity ?

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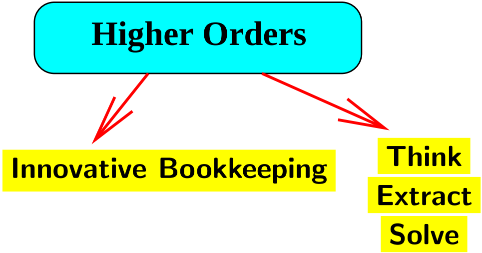
Guidelines



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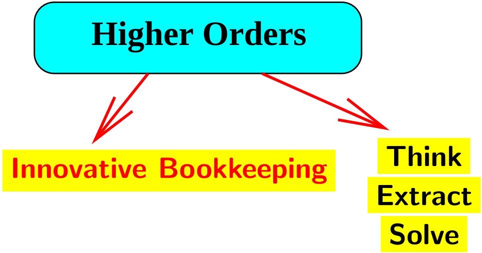

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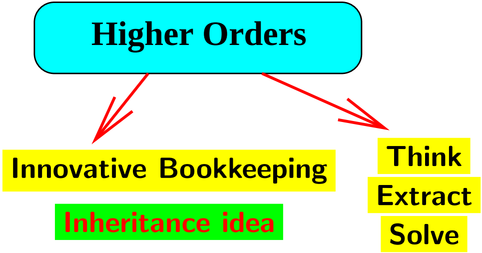

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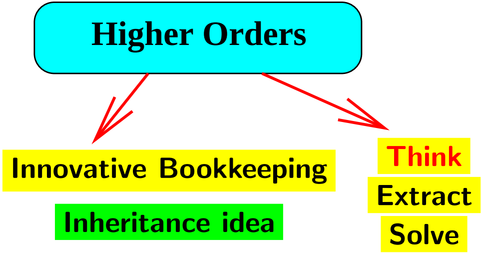

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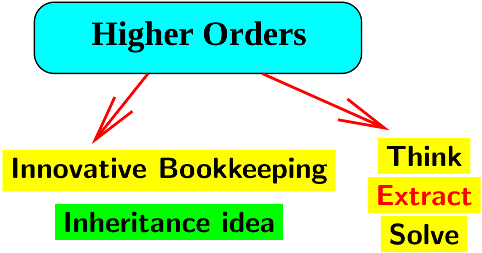

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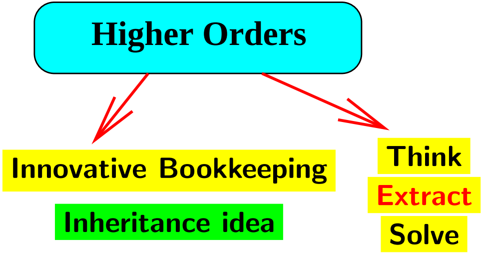

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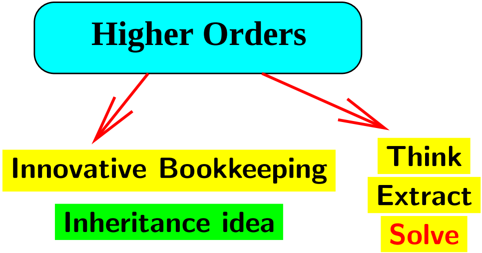

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Higher Orders

→ **Innovative Bookkeeping**

Inheritance idea

→ **Think**
Extract
Solve

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➥ A playing ground for theoretical theory: SUSY, AdS/CFT, . . .

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inherited from previous loops !

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What is so special about $\mathcal{N}=4$ SYM ?

This QFT has a good chance to be *solvable* — “integrable”.
Dynamics can be fully integrated if the system possesses a sufficient (infinite!) number of conservation laws, — integrals of motion.

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Maximally super-symmetric $\mathcal{N}=4$ YM allows for a compact analytic solution of the GLR problem in 3 loops ($\forall N$) D-r & Marchesini (2006)

Moreover, in $\mathcal{N}=4$ YM
✗ GLR holds for twist 3, in $3+4$ loops Matteo Beccaria et al. (2007)

What is so special about $\mathcal{N}=4$ SYM ?

This QFT has a good chance to be *solvable* — "integrable".
Dynamics can be fully integrated if the system possesses a sufficient (infinite!) number of conservation laws, — integrals of motion.

from Bookkeeping to Solving

The integrability feature manifests itself already in *certain sectors* of QCD, in specific problems where one can *identify* QCD with SUSY-QCD :

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Recall the diagonal first loop anomalous dimensions:

$$\tilde{\gamma}_{q\to q(x)+g} = \frac{C_F\alpha_s}{\pi}\left[\frac{x}{1-x} + (1-x)\cdot\frac{1}{2}\right],$$
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Let us look at the rôles these animals play on the QCD stage

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Parke–Taylor (1986) = Bassetto–Ciafaloni–Marchesini (1983)

N=4 SYM and LBK

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Matter content = 4 Majorana fermions, 6 scalars;
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Employ $\mathcal{N} = 4$ SYM to simplify the essential part of the QCD dynamics

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If this is true, the goal would be

to derive a one-line-all-orders expression for γ from $\gamma^{(1)}$ in $\mathcal{N} = 4$ SYM
and then to export it into QCD,
to cover "90%" of the small-distance parton dynamics

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Extras

gluons in-between-jets

Another hidden message: QCD Radiophysics

2- and 3-prong colour antennae are sort of "trivial": coherence being taken care of, the answers turned out to be essentially additive

The case of $2 \to 2$ hard parton scattering is more involved (4 emitters), especially so for gluon–gluon scattering.
Here one encounters 6 (5 for $SU(3)$) colour channels that mix with each other under soft gluon radiation

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Three "ain't-so-simple" ones were found to satisfy the cubic equation:

$$\left[E_i - \frac{4}{3}\right]^3 - \frac{(1+3b^2)(1+3x^2)}{3}\left[E_i - \frac{4}{3}\right] - \frac{2(1-9b^2)(1-9x^2)}{27} = 0,$$

where

$$x = \frac{1}{N}, \qquad b \equiv \frac{\ln(t/s) - \ln(u/s)}{\ln(t/s) + \ln(u/s)}$$

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Three "ain't-so-simple" ones were found to satisfy the cubic equation:

$$\left[E_i - \frac{4}{3}\right]^3 - \frac{(1+3b^2)(1+3x^2)}{3}\left[E_i - \frac{4}{3}\right] - \frac{2(1-9b^2)(1-9x^2)}{27} = 0,$$

where

$$x = \frac{1}{N}, \qquad b \equiv \frac{\ln(t/s) - \ln(u/s)}{\ln(t/s) + \ln(u/s)}$$

Mark the *mysterious symmetry* w.r.t. to $x \to b$: interchanging internal (group rank) and external (scattering angle) variables of the problem ...

Physical coupling

$$A = \sum_{1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n A_n, \qquad \frac{A^{(g)}}{C_A} = \frac{A^{(q)}}{C_F} \qquad P_{a\to a[x]+g}(x) = \frac{A(\alpha_s)}{1-x}$$

$$\begin{aligned}
\frac{A_1}{C} &= 4 \\
\frac{A_2}{C} &= 8\left[\left(\frac{67}{18} - \zeta_2\right) C_A - \frac{5}{9}\, n_f\right] \\
\frac{A_3}{C} &= 16 C_A^2 \left(\frac{245}{24} - \frac{67}{9}\,\zeta_2 + \frac{11}{6}\,\zeta_3 + \frac{11}{5}\,\zeta_2^2\right) \\
&\quad + 16 C_F n_f \left(-\frac{55}{24} + 2\,\zeta_3\right) \\
&\quad + 16 C_A n_f \left(-\frac{209}{108} + \frac{10}{9}\,\zeta_2 - \frac{7}{3}\,\zeta_3\right) + 16 n_f^2 \left(-\frac{1}{27}\right).
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= *universal* magnitude of double-log enhanced contributions.

Enters in :

large-N asymptotics of anomalous dimensions *and* coefficient functions,

Sudakov quark and gluon form factors,

quark and gluon Regge trajectories,

threshold resummation,

singular ($x \to 1$) part of the Drell–Yan K-factor,

distributions of jet event shapes in the near-to-two-jet kinematics,

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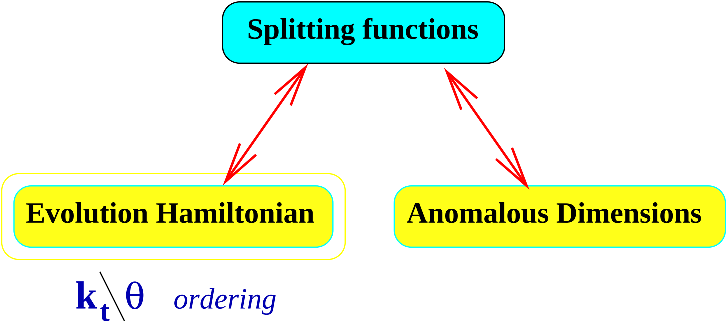
Innovative Bookkeeping

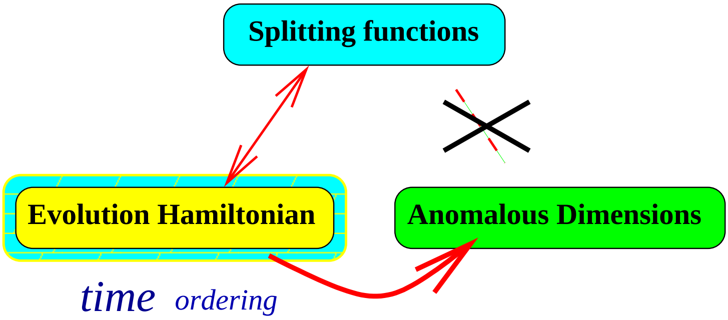
In the standard approach,



- parton splitting functions are equated with anomalous dimensions;
- they are different for DIS and e^+e^- evolution;
- "clever evolution variables" are different too

Innovative Bookkeeping

In the new approach,



- splitting functions are disconnected from the anomalous dimensions;
- the evolution kernel is identical for space- and time-like cascades (Gribov–Lipatov reciprocity relation true in all orders);
- unique evolution variable — parton fluctuation time

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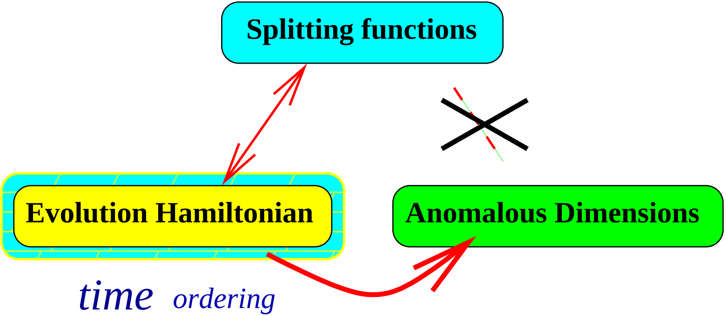

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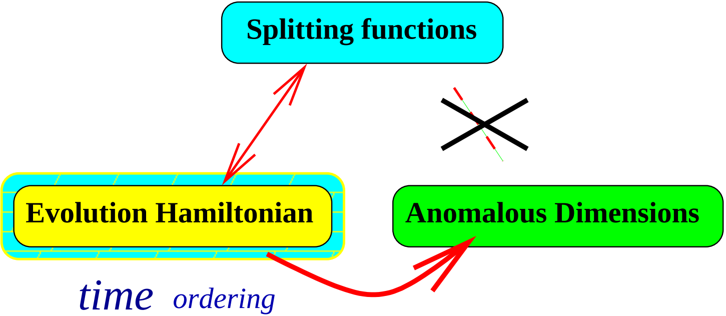

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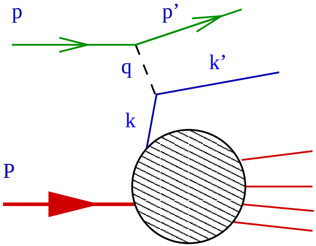

Kinematics of the parton splitting $A \to B + C$

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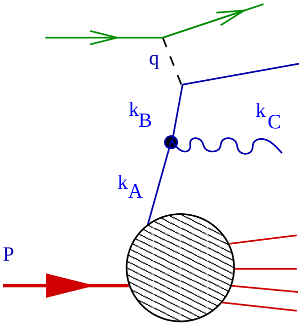

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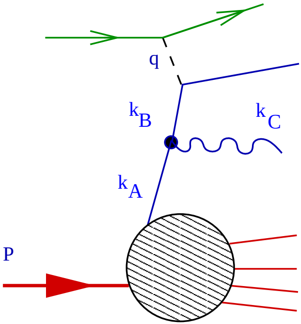

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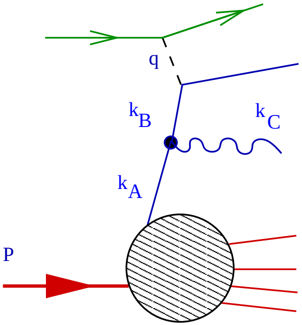

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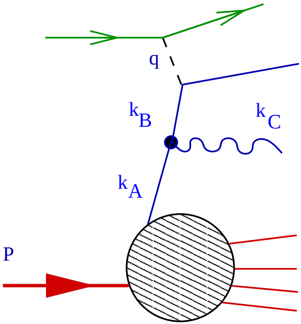

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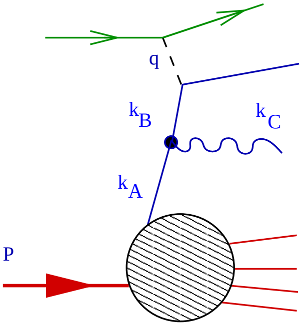

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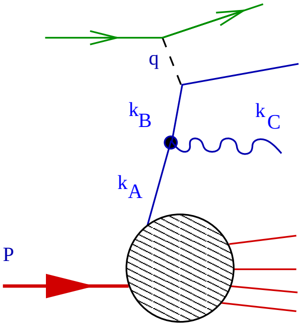

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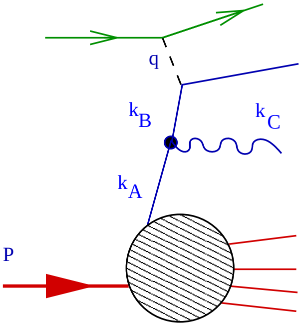

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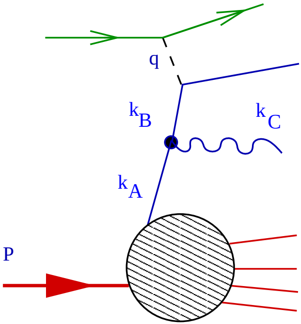

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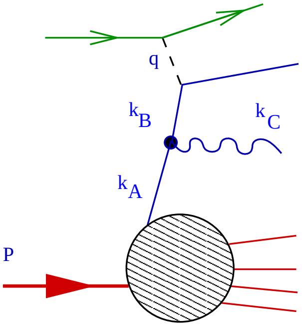

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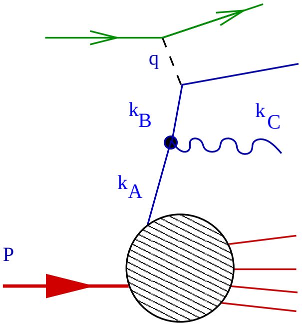

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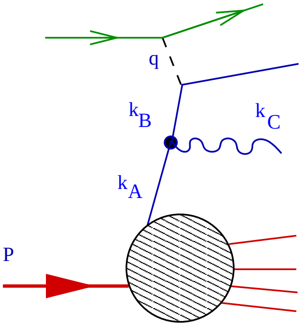

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strongly ordered *lifetimes* of successive parton fluctuations !

How to Order parton splittings?

Beyond the 1st loop, it starts to matter how does one order successive parton splittings that is, what one chooses for "parton evolution time".

The "clever choices" had been established quite some time ago:

$$d\xi = d\ln\frac{k_\perp^2}{1} \quad \text{(space-like)}, \qquad d\xi = d\ln\frac{k_\perp^2}{z^2} \quad \text{(time-like)}.$$

Transverse momentum ordering vs. angular ordering.

Each of these two clever choices — consequence of taking into full consideration soft gluon coherence in order to prevent explosively large terms $(\alpha_s \ln^2 x)^n$ from appearing in higher loop anomalous dimensions.

A good *dynamical* move. But a lousy one *kinematically* :

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space- and time-like parton multiplication

Space-like parton evolution (S) vs. *time-like* fragmentation (T)

Drell–Levy–Yan relation

$$P_{BA}^{(T)}(x) = \mp x \cdot P_{AB}^{(S)}(x^{-1}).$$

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True in any QFT, it reflects the crossing and allows to link the two channels by analytic continuation, from $x < 1$ to $x > 1$:

Bukhvostov, Lipatov, Popov (1974)

Drell–Levy–Yan relation beyond leading log

Blümlein, Ravindran, W.L. van Neerven (2000)

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$$P_{BA}^{(T)}(x) = \mp x \cdot P_{AB}^{(S)}(x^{-1}).$$

True in any QFT, it reflects the crossing and allows to link the two channels by analytic continuation, from $x < 1$ to $x > 1$:

Bukhvostov, Lipatov, Popov (1974)

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$$P_{BA}^{(T)}(x_{\text{Feynman}}) = P_{BA}^{(S)}(x_{\text{Bjorken}}); \qquad x_B = \frac{-q^2}{2pq}, \quad x_F = \frac{2pq}{q^2}$$

Mark the different meaning of x in the two channels!

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GLR was found to be broken beyond the 1st loop. But *WHY* ?

Reciprocity Respecting Evolution

Fluctuation time ordering : D-r (HERA, 1993)

$$\frac{dD^A(x,Q^2)}{d\ln Q^2} = \int_0^1 \frac{dz}{z}\,\mathcal{P}_B^A(z;\alpha_s)\,D^B\left(\frac{x}{z}, z^\sigma Q^2\right)$$

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In the Mellin moment space,

$$P_N \equiv \int_0^1 \frac{dz}{z}\,P(z)\,z^N \qquad \Longrightarrow \qquad \gamma_N \cdot D_N(Q^2) = \mathcal{P}_{N+\sigma d} \cdot D_N(Q^2)$$

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Expanding, get an equation for the an.dim. γ, one for both channels

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GLR beyond the 1st loop

Examine the "reciprocity respecting equation" (RRE) by feeding in the one-loop parton "Hamiltonian", $\mathcal{P}(\alpha) \simeq \alpha P_1$:

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The difference between time- and space-like anomalous dimensions,

$$\tfrac{1}{2}\left[P^{(T)} - P^{(S)}\right] = \alpha^2 \cdot P_1 \dot{P}_1 + \mathcal{O}\left(\alpha^3\right),$$

in the x-space corresponds to the convolution

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responsible for GLR violation in the 2nd loop non-singlet quark anomalous dimension, as found by Curci, Furmanski & Petronzio (1980)

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$\Longrightarrow$ the genuine $\mathcal{P}_2$ does not contain σ, is GLR respecting

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More generally, a *renormalization scheme transformation* as a cure for/against GLR violation was proposed by Stratmann & Vogelsang (1996)

non-diagonal transitions

Second loop $G \to G$ [quark box] $(n_f T_R C_F)$

$$P_G^{(S)} = 8x - 16 + \frac{20}{3}x^2 + \frac{4}{3}x^{-1} - (6 + 10x)\ln x - 2(1 + x)\ln^2 x,$$

$$P_G^{(T)} = 12x - 4 - \frac{164}{9}x^2 + \frac{92}{9}x^{-1} + (10 + 14x + \frac{16}{3}[x^2 + x^{-1}])\ln x + 2(1 + x)\ln^2 x;$$

Non-singlet $F \to F$ [via 2 gluons] $(n_f T_R C_F)$

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large x

Another important aspect of the RREE is the "double nature" of the perturbative expansion — in α_{phys} and, at the same time, in $(1-x)$:

$$\gamma[\alpha] = \mathcal{P} + \dot{\mathcal{P}} \cdot \left(\sigma\gamma + \beta/\alpha\right) + \tfrac{1}{2}\ddot{\mathcal{P}} \cdot \left(\gamma^2 + \sigma(2\beta/\alpha\,\gamma + \beta\partial_\alpha\gamma) + \beta/\alpha\,\partial_\alpha\beta\right) + \ldots$$
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A gap between *classical radiation* (Low–Burnett–Kroll wisdom)

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and *quantum fluctuations*

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Generated: D-r, Marchesini & Salam (2005)

$C=-\sigma A^2$ — relation observed by MVV in 3 loops

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Generated: D-r, Marchesini & Salam (2005)

$C = -\sigma A^2$ — relation observed by MVV in 3 loops

$D = -\sigma A\,B + \mathcal{O}(\beta)$ — another all-order relation

Euler–Zagier harmonic sums

In spite of having many states ($s = 0, \frac{1}{2}, 1$), the SYM-4 parton dynamics is built of a single "universal" anomalous dimension:

$$\gamma_+(N+2) = \tilde{\gamma}_+(N+1) = \gamma_0(N) = \tilde{\gamma}_-(N-1) = \gamma_-(N-2) \equiv \gamma_{\text{uni}}(N)$$

with the 1st loop given by

$$\gamma^{(1)}_{\text{uni}}(N) = -S_1(N) \;=\; -\int_0^1 \frac{dx}{x}\left(x^N - 1\right)\cdot \frac{x}{x-1}$$

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as we as multiple indices — *nested sums*

$$S_{m,\vec{\rho}}(N) = \sum_{k=1}^N \frac{S_{\vec{\rho}}(k)}{k^m} \qquad (\vec{\rho} = (m_1, m_2, \ldots, m_i)),$$

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$(a) \leftrightarrow (b)$

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(direct calculation by Kotikov & Lipatov, 2000)

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generates positives and simplifies negatives.

RR evolution kernel

In terms of the perturbative expansion in the physical coupling,

$$a_{\rm ph} = a\left(1 - \tfrac{1}{2}\zeta_2 a + \tfrac{11}{20}\zeta_2^2 a^2 + \ldots\right),$$

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\mathcal{P}_1 &= -\ S_1; \\
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Twist-3

The $\mathfrak{sl}(2)$ sector of planar $\mathcal{N}=4$ SYM contains single trace states which are linear combinations of the basic operators

$$\mathrm{Tr}\left\{(\mathcal{D}^{s_1} Z) \cdots (\mathcal{D}^{s_L} Z)\right\}, \quad s_1 + \cdots + s_L = N,$$

where Z is one of the three complex scalar fields and $\mathcal{D}$ is a light-cone covariant derivative. The numbers $\{s_i\}$ are non-negative integers and N is the total spin. The number L of Z fields is the twist of the operator, *i.e.* the classical dimension minus spin.

The anomalous dimensions of these states are the eigenvalues $\gamma_L(N; g)$ of the dilatation operator — integrable Hamiltonian.

These values were obtained by solving numerically the Bethe Ansatz equations (BAE), order by order in g^2, and guessing the answer in terms of harmonic sums of transcedentality $\tau = 2n-1$, at n loops.

Since *wrapping problems*, delayed by supersymmetry, appear at $L+2$ loop order for twist-L operators, the BAE for twist-3 are reliable up to *four loops* (including, at the fourth loop, the dressing factor).

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Since *wrapping problems*, delayed by supersymmetry, appear at $L+2$ loop order for twist-L operators, the BAE for twist-3 are reliable up to *four loops* (including, at the fourth loop, the dressing factor).

Twist-3

The $\mathfrak{sl}(2)$ sector of planar $\mathcal{N}=4$ SYM contains single trace states which are linear combinations of the basic operators

$$\mathrm{Tr}\left\{(\mathcal{D}^{s_1} Z) \cdots (\mathcal{D}^{s_L} Z)\right\}, \quad s_1 + \cdots + s_L = N,$$

where Z is one of the three complex scalar fields and $\mathcal{D}$ is a light-cone covariant derivative. The numbers $\{s_i\}$ are non-negative integers and N is the total spin. The number L of Z fields is the twist of the operator, *i.e.* the classical dimension minus spin.

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Twist-3 : Answer

$$
\begin{aligned}
\gamma_3^{(1)} &= 4\,S_1\,, \\
\gamma_3^{(2)} &= -2\,(S_3 + 2\,S_1 S_2) \\
\gamma_3^{(3)} &= 5\,S_5 + 6\,S_2\,S_3 - 8\,S_{3,1,1} + 4\,S_{4,1} - 4\,S_{2,3} + S_1(4\,S_2^2 + 2\,S_4 + 8\,S_{3,1}), \\
\gamma_3^{(4)} &= \tfrac{1}{2}\,S_7 + 7\,S_{1,6} + 15\,S_{2,5} - 5\,S_{3,4} - 29\,S_{4,3} - 21\,S_{5,2} - 5\,S_{6,1} \\
&\quad -40\,S_{1,1,5} - 32\,S_{1,2,4} + 24\,S_{1,3,3} + 32\,S_{1,4,2} - 32\,S_{2,1,4} + 20\,S_{2,2,3} \\
&\quad +40\,S_{2,3,2} + 4\,S_{2,4,1} + 24\,S_{3,1,3} + 44\,S_{3,2,2} + 24\,S_{3,3,1} + 36\,S_{4,1,2} \\
&\quad +36\,S_{4,2,1} + 24\,S_{5,1,1} + 80\,S_{1,1,1,4} - 16\,S_{1,1,3,2} + 32\,S_{1,1,4,1} \\
&\quad -24\,S_{1,2,2,2} + 16\,S_{1,2,3,1} - 24\,S_{1,3,1,2} - 24\,S_{1,3,2,1} - 24\,S_{1,4,1,1} \\
&\quad -24\,S_{2,1,2,2} + 16\,S_{2,1,3,1} - 24\,S_{2,2,1,2} - 24\,S_{2,2,2,1} - 24\,S_{2,3,1,1} \\
&\quad -24\,S_{3,1,1,2} - 24\,S_{3,1,2,1} - 24\,S_{3,2,1,1} - 24\,S_{4,1,1,1} - 64\,S_{1,1,1,3,1} \\
&\quad -8\,\beta\,S_1\,S_3.
\end{aligned}
$$

The last term, with $\beta = \zeta_3$, is the contribution from the dressing factor that appears in the BAE at the fourth loop.

Twist-3 : features

The twist-3 anomalous dimension has two characteristic features:

1. All harmonic functions $S_{\vec{a}}$ are evaluated at half the spin, $S_{\mathbf{a}} \equiv S_{\mathbf{a}}(N/2)$. On the integrability side, this does not look unwarranted, since only *even* N belong to the non-degenerate ground state of the magnet.
2. No negative indices appear at twist-3, while in the case of twist-2 negative index sums were present starting from the second loop.

At the $N \to \infty$ limit, the *minimal* anomalous dimension γ (corresponding to the ground state) must exhibit the universal (LBK-classical) ln N behaviour which depends neither on the twist, nor on the nature of fields under consideration. Computing analytically the large N asymptotics yields

$$\frac{\gamma_3(N)}{\ln N} = 4\,g^2 - \frac{2\pi^2}{3}\,g^4 + \frac{11\pi^4}{45}\,g^6 - \left(4\zeta_3^2 + \frac{73\pi^6}{630}\right)\,g^8 + \mathcal{O}\left(g^{10}\right),$$

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Twist-3 : Evolution Kernel (rough)

After processing thru $\gamma = \mathcal{P}(N + \frac{1}{2}\gamma)$, in series in $g^2 = \frac{N_c \alpha}{2\pi}$,

$$
\begin{aligned}
P^{(1)} &= 4\,S_1, \\
P^{(2)} &= -2\,S_3 - 4\,\zeta_2\,S_1, \\
P^{(3)} &= S_5 + 2\,\zeta_2\,S_3 + 4\,\left(S_{3,2} + S_{4,1} - 2\,S_{3,1,1}\right) \\
&\quad + 4\,S_1\,(2\,S_{3,1} - S_4 + 4\,\zeta_4) - 4\,S_1^2\,(S_3 - \zeta_3).
\end{aligned}
$$

The fourth loop kernel we split into two terms: $P^{(4)} = P_S^{(4)} + P_\zeta^{(4)}$.

$$
\begin{aligned}
P_S^{(4)} &= -8\left[S_{3,3} + S_{1,5} + 2S_{2,4} - 4(S_{2,1,3} + S_{1,2,3} + S_{1,1,4}) + 8S_{1,1,1,3}\right]S_1 \\
&+ \frac{3}{2}\,S_7 - 16\,\left(S_{1,6} + S_{4,3}\right) - 24\,\left(S_{2,5} + S_{3,4}\right) \\
&\quad + 48\,\left(S_{1,1,5} + S_{1,3,3} + S_{3,1,3}\right) + 64\,\left(S_{2,2,3} + S_{2,1,4} + S_{1,2,4}\right) \\
&\quad - 128\,\left(S_{1,1,1,4} + S_{2,1,1,3} + S_{1,2,1,3} + S_{1,1,2,3}\right) + 256\,S_{1,1,1,1,3}\,, \\
P_\zeta^{(4)} &= 8\zeta_4\,\mathcal{S}_1^3 - 4\left[\zeta_2\zeta_3 + 8\zeta_5\right]\mathcal{S}_1^2 - \left[4(\zeta_3 + 2\beta)\mathcal{S}_3 + 49\zeta_6\right]\mathcal{S}_1 \\
&\quad + (8\mathcal{S}_{1,1,3} - 4\mathcal{S}_{1,4} - 4\mathcal{S}_{2,3} - \mathcal{S}_5)\,\zeta_2 - 8\mathcal{S}_3\,\zeta_4.
\end{aligned}
$$

RR harmonic functions

Let $\vec{m} = \{m_1, m_2, \ldots, m_\ell\}$, and examine the recurrence relation

$$\tilde{\Phi}_{b,\vec{m}}(x) = -[\Gamma(b)]^{-1} \frac{x}{x-1} \int_x^1 \frac{dz\,(z+1)}{z^2} \ln^{b-1} \frac{z}{x} \cdot \tilde{\Phi}_{\vec{m}}(z),$$

where the single index function coincides with the image of the standard harmonic sum,

$$\tilde{\Phi}_a(x) = [\Gamma(a)]^{-1} \frac{x}{x-1} \ln^{a-1} \frac{1}{x} = \tilde{\mathcal{S}}_a(x).$$

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At the base of the recursion, we have (the *weight* $w \equiv \tau - \ell$)

$$\tilde{\Phi}_a(x) = \Big(-x\,\tilde{\Phi}_a(x^{-1}) \Big) \cdot (-1)^{a-1} \equiv \Big(-x\,\tilde{\Phi}_a(x^{-1}) \Big) \cdot (-1)^{w[a]}.$$

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An iteration increases transcedentality $\tau = \sum_{i=1}^{\ell} |m_i|$ of the function by b, and the length ℓ of the index vector by one, so that

$$w[\vec{m}] + b - 1 = w[b, \vec{m}].$$

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Twist-3 : Evolution Kernel (beautified)

Then, in terms of the physical coupling,

$\mathbf{g}_{\text{ph}}^2 \equiv \frac{N_c\,\alpha_{\text{ph}}}{2\pi} = g^2 - \zeta_2\, g^4 + \frac{11}{5}\zeta_2^2\, g^6 - \left(\frac{73}{10}\zeta_2^3 + \zeta_3^2\right) g^8 + \ldots,$

the perturbative series for the kernel, $\mathcal{P} = \sum_{n=1} \mathbf{g}_{\text{ph}}^{2n}\,\mathcal{P}_{\text{ph}}^{(n)}$, becomes

$$\begin{aligned}
\mathcal{P}_{\text{ph}}^{(1)} &= 4\,\mathcal{S}_1, \\
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\mathcal{P}_{\text{ph}}^{(3)} &= 3\,\mathcal{S}_5 - 2\,\Phi_{1,1,3} + \zeta_2 \cdot (-2\,\mathcal{S}_3), \\
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Twist-3 vis a vis Twist-2

This result can be compared with the evolution kernel that generates the twist-2 universal anomalous dimension :

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\mathcal{P}^{(1)}_{\rm ph} &= 4\,\mathcal{S}_1; \\
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Remark : in general, the GL parity is

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since

$$\frac{x}{x-1} \implies \frac{x}{x+1}$$

Logs in γ and $\mathcal{P}$

General structure of the RR Evolution Kernel

$$\mathcal{P}(N) = \mathcal{S}_1 \cdot \left(\alpha_{\rm ph} + \widehat{\mathcal{A}}\right) + \mathcal{B}, \qquad \widehat{\mathcal{A}} = \mathcal{O}\left(1/N^2\right).$$

This feature is in a marked contrast with the anomalous dimension *per se*, whose large N expansion includes growing powers of $\log N$:

$$\gamma(N) = a \ln N + \sum_{k=0}^{\infty} \frac{1}{N^k} \sum_{m=0}^{k} a_{k,m} \ln^m N.$$

Easy to see from

$$\gamma_\sigma = \mathcal{P}(N + \sigma\gamma) \quad \Longrightarrow \quad \gamma_\sigma(N) = \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sigma \frac{d}{dN}\right)^{k-1} \left[\mathcal{P}(N)\right]^k,$$

Physically, the reduction of singularity of the large N expansion shows that the tower of subleading logarithmic singularities in the anomalous dimension is actually *inherited* from the first loop — the LBK-classical $\gamma^{(1)} = \mathcal{P}^{(1)} \propto S_1$, and the RREE generates them automatically !

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Fresh results

- RRE as a natural consequence of the conformal invariance
  "Anomalous dimensions of high-spin operators beyond the leading order"
  Benjamin Basso & Gregory Korchemsky *hep-th/0612247*
- *"N=4 SUSY Yang-Mills: three loops made simple(r)"*
  D-r & Pino Marchesini *hep-th/0612248*
- *"Anomalous dimensions at twist-3 in the sl(2) sector of N=4 SYM"*
  Matteo Beccaria *0704.3570 [hep-th]*
- Bethe Ansatz fails ("maximally") at 4 loops for twist-2
  "Dressing and Wrapping"
  Kotikov, Lipatov, Rej, Staudacher & Velizhanin *0704.3586 [hep-th]*
- twist-3 gaugino = twist-2 "universal"
  "Universality of three gaugino anomalous dimensions in N=4 SYM"
  Beccaria *0705.0663 [hep-th]*
- *"Twist 3 of the sl(2) sector of N=4 SYM and reciprocity respecting evolution"*
  Beccaria, D-r & Marchesini

$\mathcal{N} = 4$ SYM serving QCD

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A deeper understanding of the $s \to u$ crossing ($x \to -x$ symmetry) should turn the "viability of" into the "power of"

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$$\frac{\text{clever 2nd loop}}{\text{clever 1st loop}} < 2\% \qquad \left(\begin{array}{c} \text{Heavy quark fragmentation} \\ \text{D-r, Khoze \& Troyan , PRD 1996} \end{array} \right)$$

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Employ $\mathcal{N} = 4$ SYM to simplify the essential part of the QCD dynamics