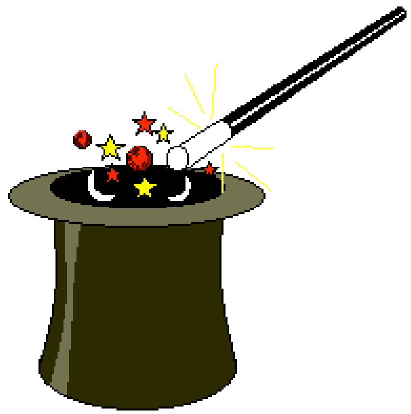
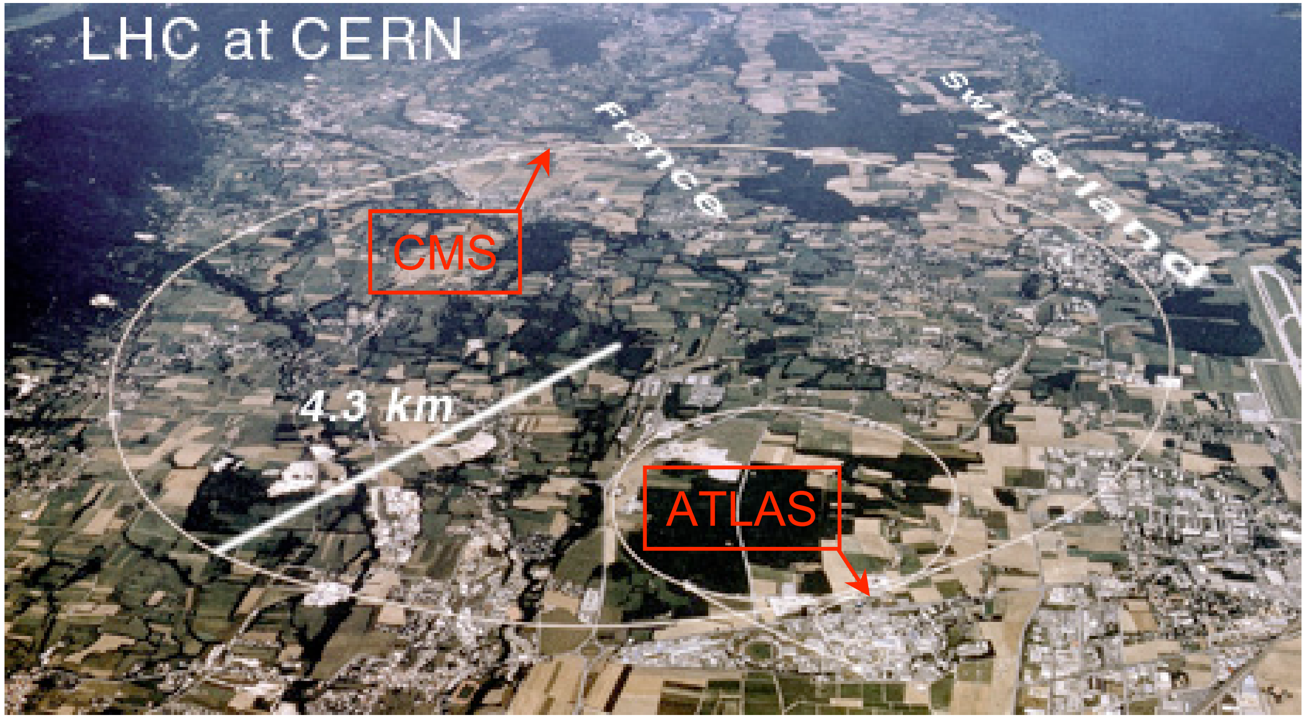
Numerical Implementation of Analytic Methods for One-loop QCD Amplitudes



C. Berger, Z. Bern, L.D., F. Febres Cordero,
D. Forde, H. Ita, D. Kosower, D. Maître
0803.4180 [hep-ph]

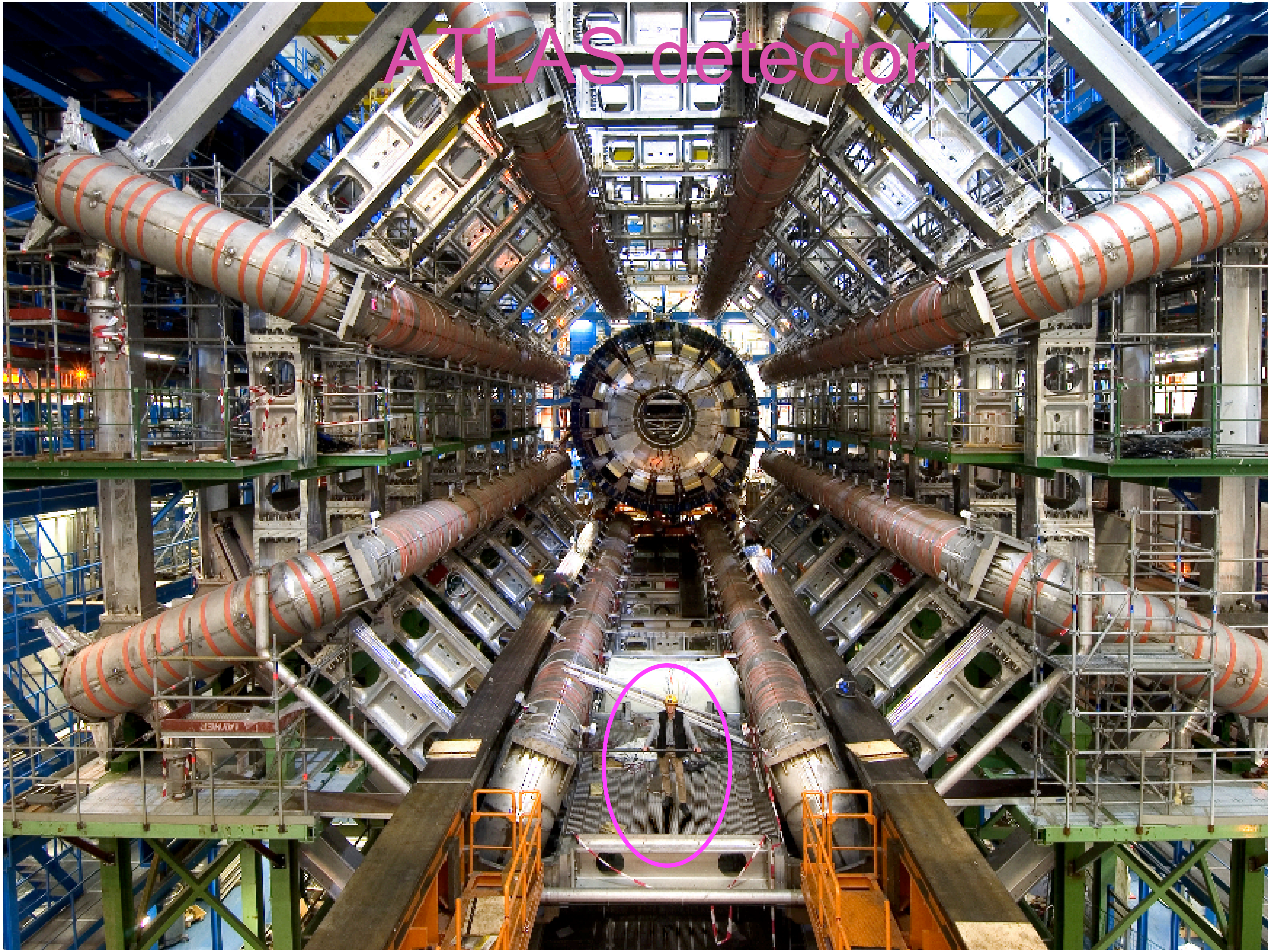
Wonders of Gauge Theory and Supergravity Workshop
Ecole Normale Supérieure and Saclay
26 June, 2008

The Large Hadron Collider

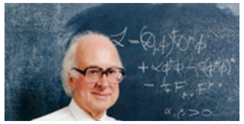


- Proton-proton collisions at **14 TeV** center-of-mass energy, 7 times greater than previous (Tevatron)
- Luminosity (collision rate) 10—100 times greater
- New window into physics at the shortest distances – opening this year

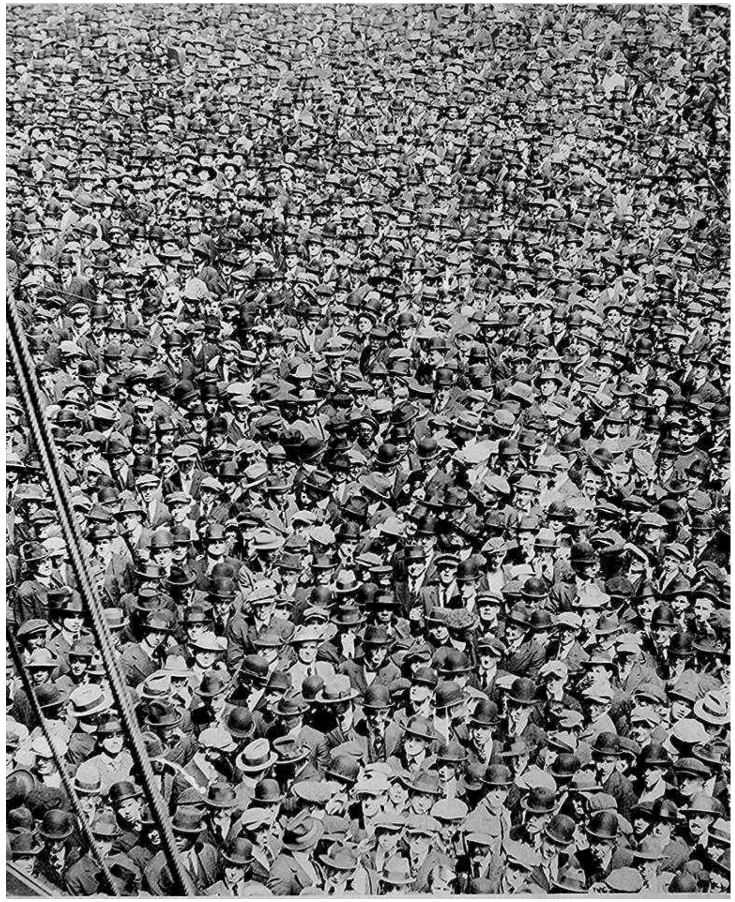
ATLAS detector



Signals vs. Backgrounds



vs.



electron-positron colliders
– small backgrounds

proton colliders
– large backgrounds

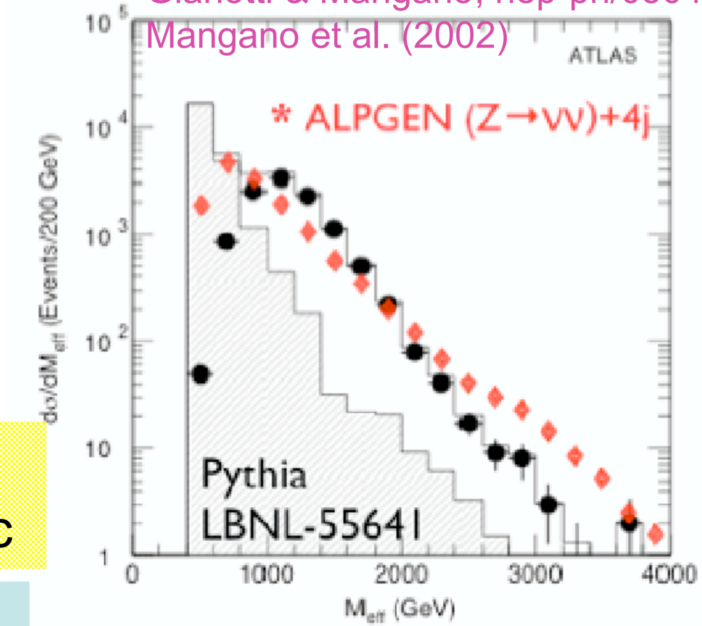
Example: Search for Supersymmetry

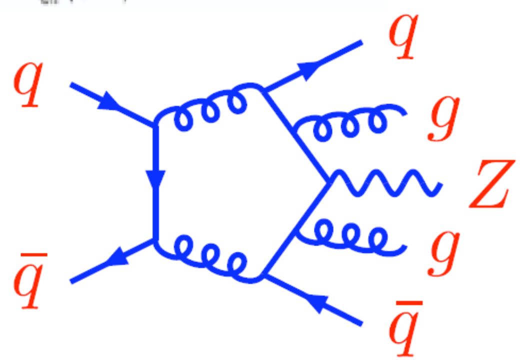
- **Supersymmetric cascade:** gluinos → squarks → neutralinos (dark matter, escapes detector)
- **Signal: missing energy + jets**
- SM background from Z + jets, when Z → neutrinos

Early studies using parton shower Monte Carlo PYTHIA overly optimistic

- ALPGEN based on **tree** amplitudes, much better than PYTHIA at modeling hard jets
- But even normalization of ALPGEN quite uncertain

Gianotti & Mangano, hep-ph/0504221
Mangano et al. (2002)





Need $pp \to Z + 4$ jets at NLO beyond state-of-art

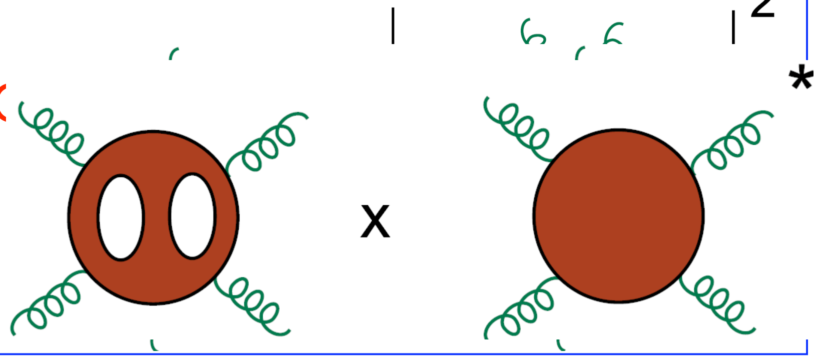
Why do we need to do better?

- Leading-order (LO), tree-level predictions only qualitative, due to **poor convergence** of expansion in strong coupling $\alpha_s(\mu) \sim 0.1$
- NLO corrections can be 30% - 80% of LO

$$\sigma(n \text{ jets}) = [\alpha_s(\mu)]^n \{A + \alpha_s(\mu) B + \alpha_s^2(\mu) C + \cdots\}$$

state of the art:

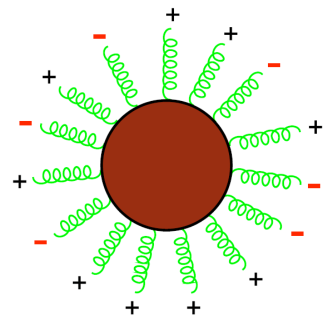
NNLO NLO = 2-loop x tree* + …



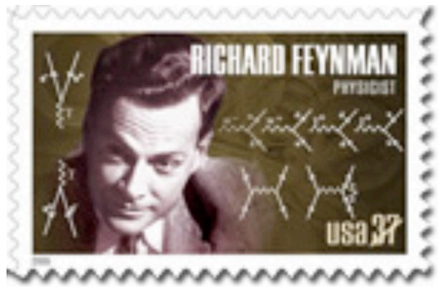
n=2 *n*=3

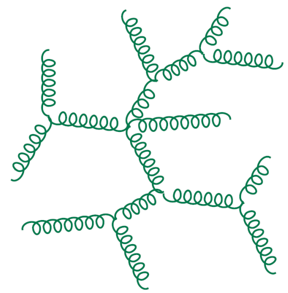
A better way to compute?

- Backgrounds (and many signals) require detailed understanding of scattering amplitudes for many ultra-relativistic ("massless") particles – especially quarks and gluons of QCD



- Feynman told us how to do this – in principle

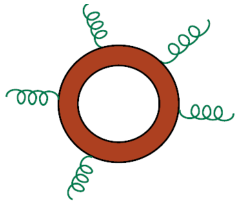
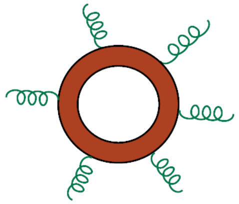
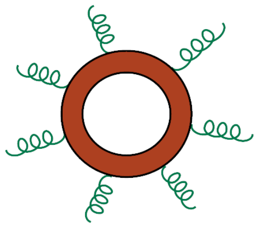
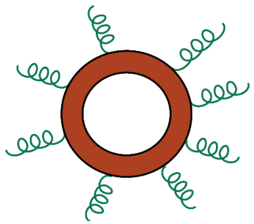




+ …

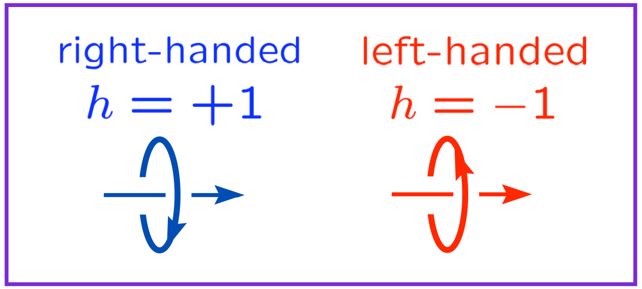
- Feynman rules, while very general and **wonderful**, are not optimized for these processes
- Can find more efficient methods, making use of analyticity + hidden symmetries (N=4 SUSY, twistor structure) of QCD

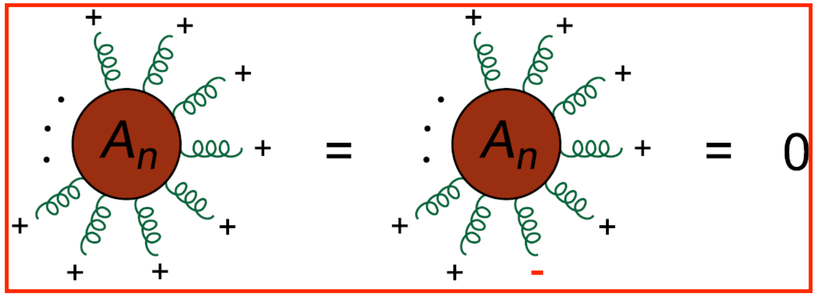
Strong growth in difficulty at one loop (NLO) with number of final-state objects

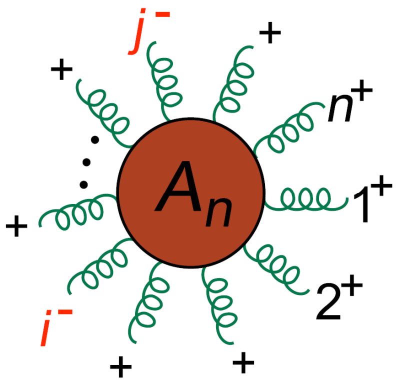
# of jets	# 1-loop Feynman diagrams (gluons only)	
3		810
4		10,860
5		168,925
6		3,017,490

Take advantage of tree-level simplicity

Many helicity amplitudes either vanish or are very short





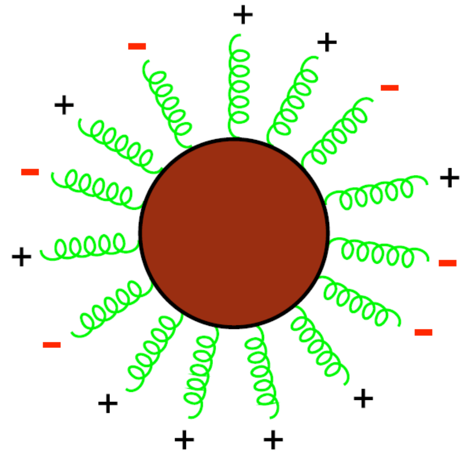


$$= \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\cdots\langle n\, 1\rangle}$$

Parke-Taylor formula (1986)

The right variables

Scattering amplitudes for massless plane waves of definite 4-momentum:
Lorentz vectors k_i^μ $k_i^2 = 0$



Natural to use Lorentz-invariant products (invariant masses): $s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$

But for particles with spin **there are better variables**

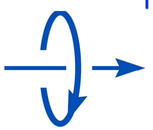
massless q,g,γ all have 2 helicities

Take "square root" of 4-vectors k_i^μ (spin 1)
use 2-component Dirac (Weyl) spinors $u_\alpha(k_i)$ (spin _)

$2 \times 2 = 4$

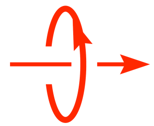
right-handed: $(\lambda_i)_\alpha = u_+(k_i)$

$h = +1/2$



left-handed: $(\tilde{\lambda}_i)_{\dot{\alpha}} = u_-(k_i)$

$h = -1/2$

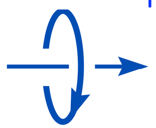


Spinors & spinor products

Instead of 4-vectors k_i^μ (spin 1) use Weyl spinors $u_\alpha(k_i)$ (spin _)

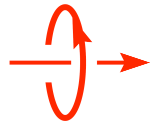
right-handed: $(\lambda_i)_\alpha = u_+(k_i)$

$h = +1/2$



left-handed: $(\tilde{\lambda}_i)_{\dot{\alpha}} = u_-(k_i)$

$h = -1/2$



Instead of Lorentz products:

$$s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$$

Use spinor products:

$$\varepsilon^{\alpha\beta}(\lambda_i)_\alpha(\lambda_j)_\beta = \langle i\, j\rangle$$

$$\varepsilon^{\dot{\alpha}\dot{\beta}}(\tilde{\lambda}_i)_{\dot{\alpha}}(\tilde{\lambda}_j)_{\dot{\beta}} = [i\, j]$$

Complex square roots of Lorentz products (if k_i real):

$$\langle i\, j\rangle = \sqrt{s_{ij}}\, e^{i\phi_{ij}} \qquad [j\, i] = \sqrt{s_{ij}}\, e^{-i\phi_{ij}}$$

Analyticity

"One of the most remarkable discoveries in elementary particle physics has been the existence of the complex plane"

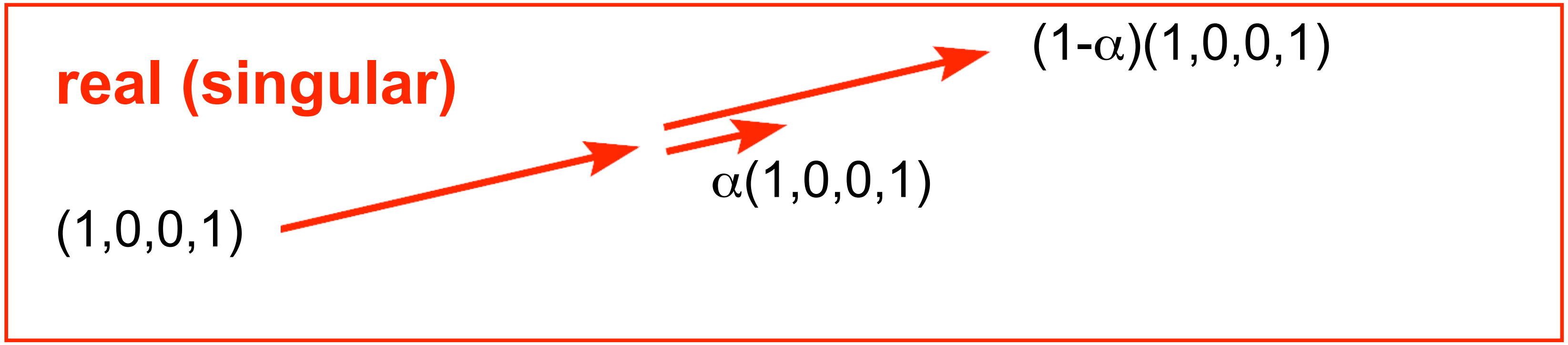
-J. Schwinger

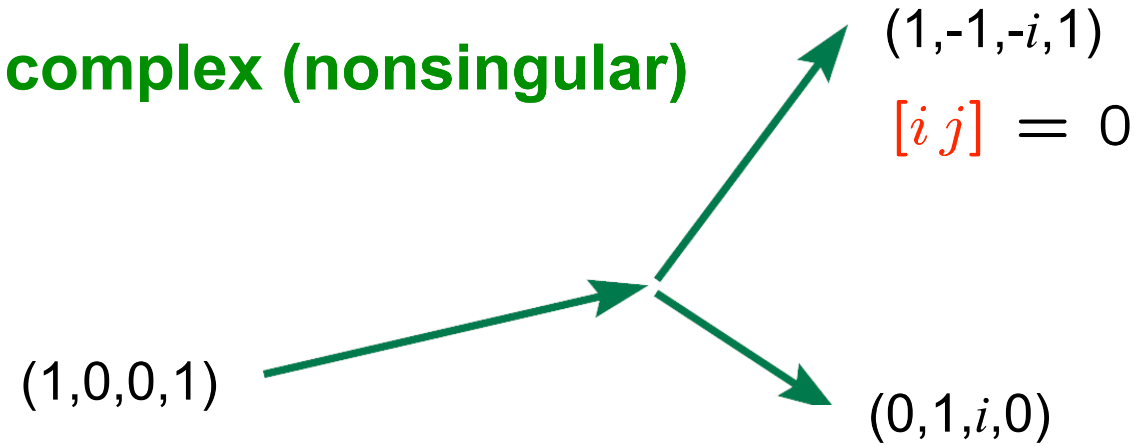
- **Treat λ_i and $\tilde{\lambda}_i$ as independent → momenta are complex**

(for real momenta λ_i and $\tilde{\lambda}_i$ are complex conjugates)

Virtues of complex momenta

- Makes sense of most basic process with all 3 particles massless





$[i\,j] = 0$ but $\langle i\,j\rangle \neq 0$

$$\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}$$

makes sense

For Efficient Computation

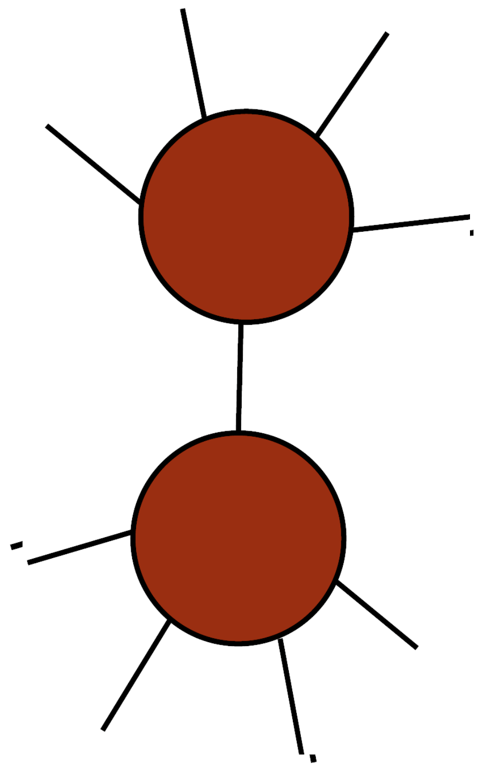
Reduce
the number of “diagrams”

Reuse
building blocks over & over

Recycle
lower-point (1-loop) & lower-loop (tree)
on-shell amplitudes

Factorization

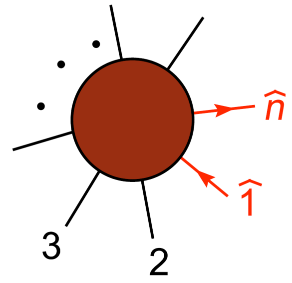
How amplitudes "fall apart" into simpler ones in special limits



Explore limits in complex plane

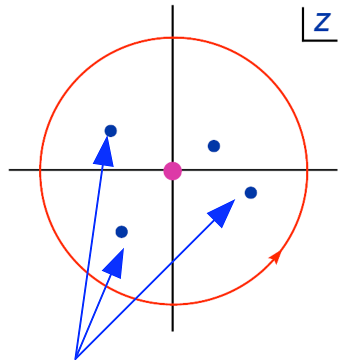
Britto, Cachazo, Feng, Witten, hep-th/0501052

Inject complex momentum at leg 1, remove it at leg *n*.



$$k_1(z) + k_n(z) \; = \; k_1 + k_n \qquad \Rightarrow \qquad A(0) \;\rightarrow\; A(z)$$

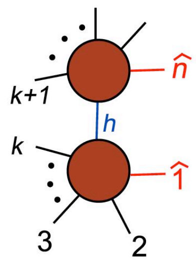
$$k_1^2(z) = k_n^2(z) = 0$$



special limits $\Leftrightarrow$ poles in z

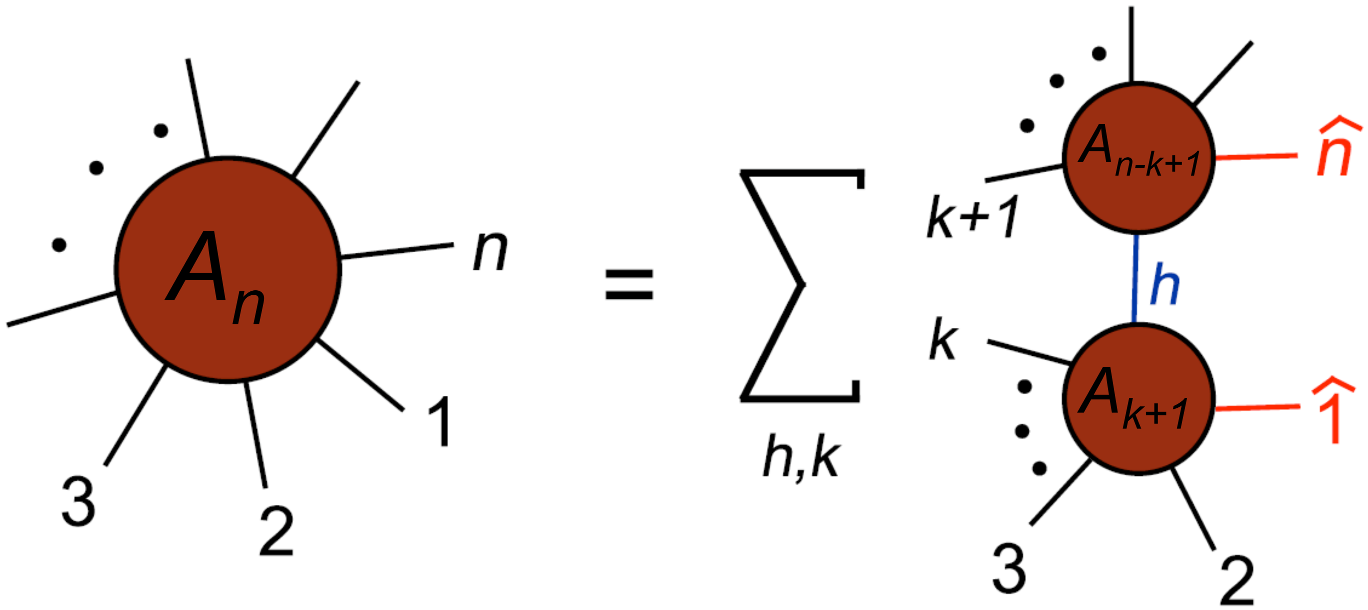
Cauchy: If $A(\infty) = 0$ then

residue at z_k = [k^{th} factorization limit] =



→ BCFW (on-shell) recursion relations

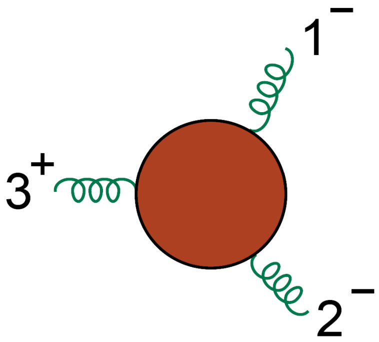
Britto, Cachazo, Feng, hep-th/0412308



A_{k+1} and A_{n-k+1} are **on-shell** tree amplitudes with **fewer** legs, and with momenta shifted by a **complex** amount

Trees recycled into trees

All gluon tree amplitudes built from:



$$= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}$$

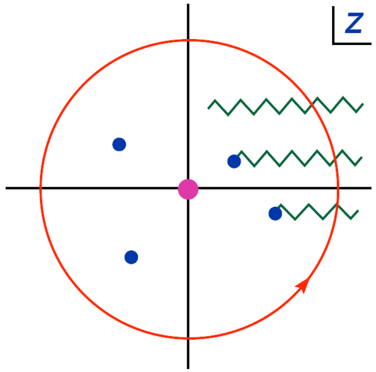
(In contrast to Feynman vertices, it is on-shell, gauge invariant.)

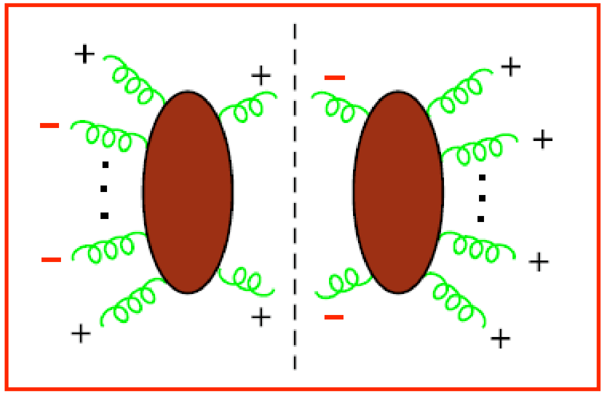
On-shell recursion at one loop

Bern, LD, Kosower, hep-th/0501240, hep-ph/0505055, hep-ph/0507005;
Berger, et al., hep-ph/0604195, hep-ph/0607014, 0803.4180 [hep-ph]

- **Same techniques** work for one-loop QCD amplitudes

- **New features** compared with tree case, especially **branch cuts**
- Determine cut terms efficiently using (generalized) unitarity





Trees recycled into loops!

One-loop amplitude decomposition

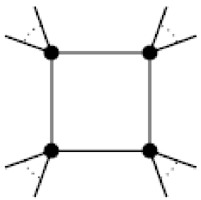
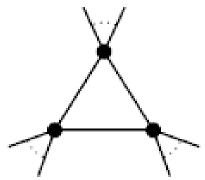
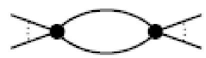
When all external momenta are in *D=4*, loop momenta in *D=4-2ε* (dimensional regularization), one can write: BDDK (1994)

rational part

cut part

known **scalar** one-loop integrals, same for all amplitudes

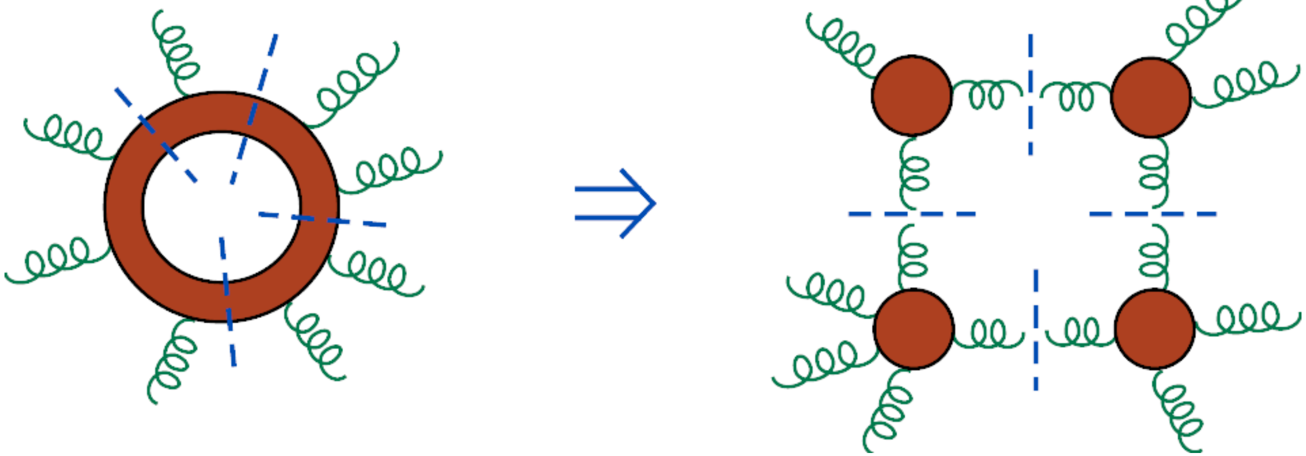
$$A = R + C + \mathcal{O}(\epsilon)$$

$$C = \sum_i d_i$$  $$+ \sum_i c_i$$  $$+ \sum_i b_i$$ 

determine coefficients – all rational functions – using (generalized) unitarity

Generalized unitarity for box coefficients b_i

Britto, Cachazo, Feng, hep-th/0412308

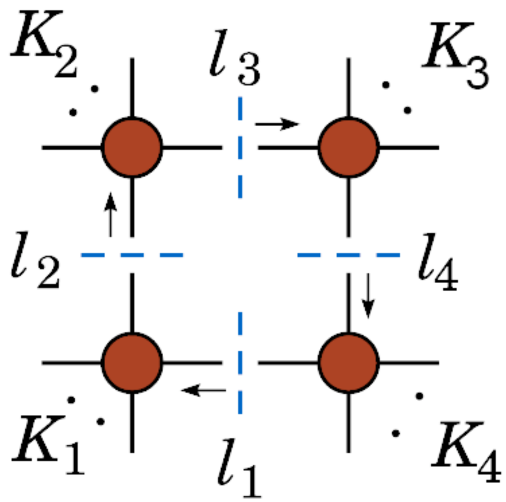


$$\int d^4\ell \; \delta(\ell_1^2 - m_1^2)\, \delta(\ell_2^2 - m_2^2) \times \delta(\ell_3^2 - m_3^2)\, \delta(\ell_4^2 - m_4^2) \times A^{1-\text{loop}}(\ell_i)$$
$$= A_1^{\text{tree}}(\ell_0) A_2^{\text{tree}}(\ell_0) A_3^{\text{tree}}(\ell_0) A_4^{\text{tree}}(\ell_0)$$
$$= d_i$$

no. of dimensions = 4 = no. of constraints → discrete solutions (2)

Box coefficients b_i (cont.)

For improved numerical stability, we use simplified solutions when all internal lines massless, at least one external line (K_1) massless:



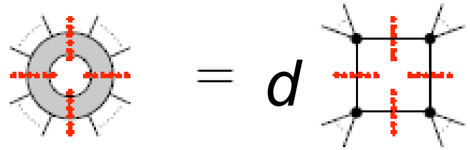
$$(l_1^{(\pm)})^\mu = \frac{\langle 1^\mp|\, \not{K}_2 \not{K}_3 \not{K}_4 \gamma^\mu \,|1^\pm\rangle}{2\,\langle 1^\mp|\, \not{K}_2 \not{K}_4 \,|1^\pm\rangle}, \qquad (l_2^{(\pm)})^\mu = -\frac{\langle 1^\mp|\, \gamma^\mu \not{K}_2 \not{K}_3 \not{K}_4 \,|1^\pm\rangle}{2\,\langle 1^\mp|\, \not{K}_2 \not{K}_4 \,|1^\pm\rangle},$$

$$(l_3^{(\pm)})^\mu = \frac{\langle 1^\mp|\, \not{K}_2 \gamma^\mu \not{K}_3 \not{K}_4 \,|1^\pm\rangle}{2\,\langle 1^\mp|\, \not{K}_2 \not{K}_4 \,|1^\pm\rangle}, \qquad (l_4^{(\pm)})^\mu = -\frac{\langle 1^\mp|\, \not{K}_2 \not{K}_3 \gamma^\mu \not{K}_4 \,|1^\pm\rangle}{2\,\langle 1^\mp|\, \not{K}_2 \not{K}_4 \,|1^\pm\rangle}.$$

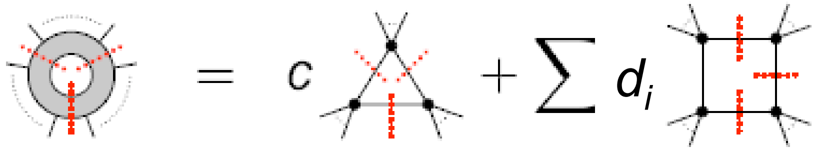
See also Risager (2008)

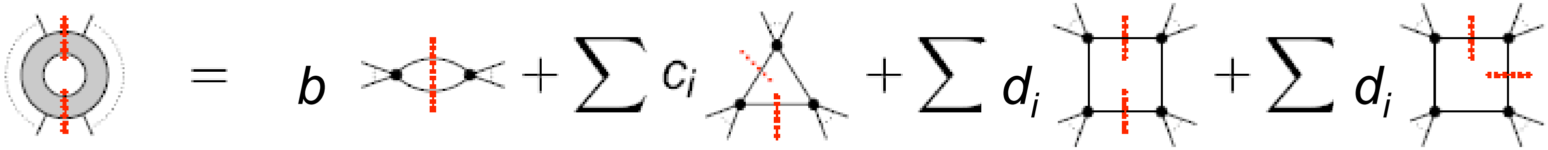
Generalized unitarity (cont.)

With a 4-ple cut we select one coefficient



Triangle and bubble coefficients are more complicated since a double or triple cut does not isolate a single coefficient.





Also, solutions to cut constraints are now continuous,
so there are multiple ways to solve and eliminate d_i , etc.

Britto et al. (2005,2006); Mastrolia (2006); Ossola, Papadopoulos, Pittau (2007); Ellis, Giele, Kunszt, 0708.2398[ph]; Forde (2007); …

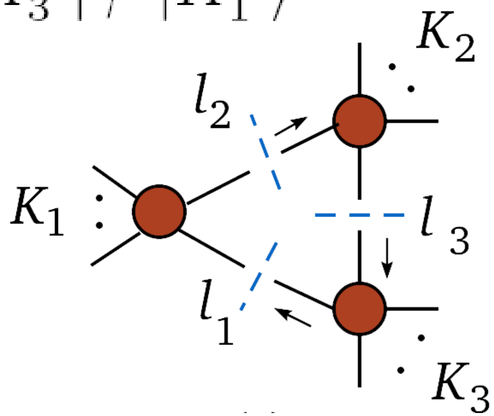
Triangle coefficients

Triple cut solution depends on one complex parameter, t

$$l_1^\mu(t) \;=\; \tilde{K}_1^\mu + \tilde{K}_3^\mu + \frac{t}{2}\langle \tilde{K}_1^- | \gamma^\mu | \tilde{K}_3^- \rangle + \frac{1}{2t}\langle \tilde{K}_3^- | \gamma^\mu | \tilde{K}_1^- \rangle$$

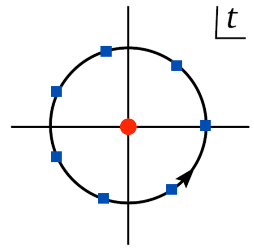
Triple cut

$$C_3(t) \equiv A_{(1)}^{\text{tree}} A_{(2)}^{\text{tree}} A_{(3)}^{\text{tree}} \Big|_{l_i = l_i(t)}$$



has poles at $t = t_i$ with residues
given by known box coefficients d_i^σ
Subtract them to get

$$T_3(t) \equiv C_3(t) - \sum_{\sigma=\pm} \sum_i \frac{d_i^\sigma}{\xi_i^\sigma (t - t_i^\sigma)} = \sum_{j=-p}^{p} c_j t^j$$



Extract desired triangle coefficient by Z_p orbifold projection
(discrete Fourier transform). Similarly for bubbles (2d transform)

Rational functions in loop amplitudes

Rational functions have no cuts – can't get from [D=4] unitarity
Can get using D=4-2ε unitarity:

Bern, Morgan (1996); Bern, LD, Kosower (1996);
Brandhuber, McNamara, Spence, Travaglini hep-th/0506068;
Anastasiou et al., hep-th/0609191, hep-th/0612277;
Britto, Feng, hep-ph/0612089, 0711.4284 [ph];
Giele, Kunszt, Melnikov, 0801.2237 [ph];
Britto, Feng, Mastrolia, 0803.1989 [ph];
Britto, Feng, Yang, 0803.3147 [ph];
Ossola, Papadopolous, Pittau, 0802.1876 [ph];
Mastrolia, Ossola, Papadopolous, Pittau, 0803.3964 [ph];
Giele, Kunszt, Melnikov (2008);
Giele, Zanderighi, 0805.2152 [ph];
Ellis, Giele, Kunszt, Melnikov, 0806.3467 [ph]

Rational functions via on-shell recursion

Already used to get infinite series of new QCD helicity amplitudes analytically:

- n-gluon MHV amplitudes at 1-loop $(- + \cdots + - + \cdots +)$
- n-gluon "split" helicity amplitudes $(- - \cdots - + + \cdots +)$
- "Higgs" + n-gluon MHV amplitudes $(\phi; - + \cdots + - + \cdots +)$

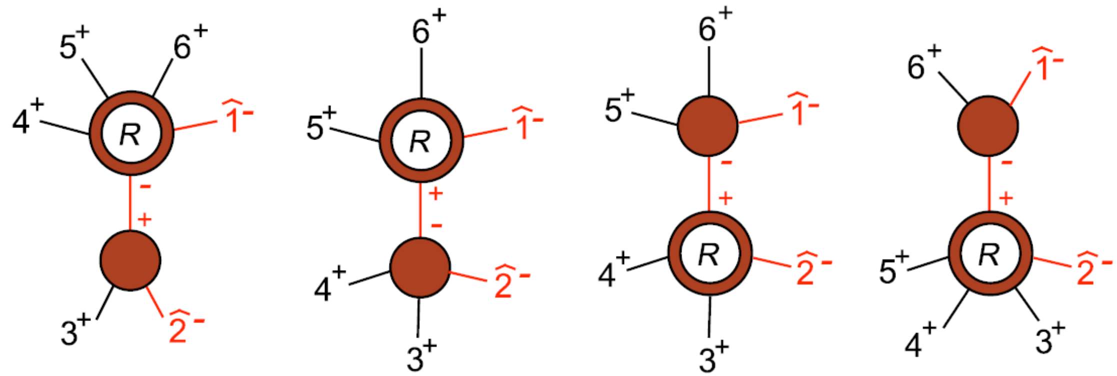
Forde, Kosower, hep-ph/0509358; Berger, Bern, LD, Forde, Kosower, hep-ph/0604195, hep-ph/0607014; Badger, Glover, Risager, 0704.3194 [ph]; Glover, Mastrolia, Williams, 0804.4149 [hep-ph]

Next step: Apply method(s) to all helicity configurations, and to generic processes [quarks, vector bosons, Higgs, …]

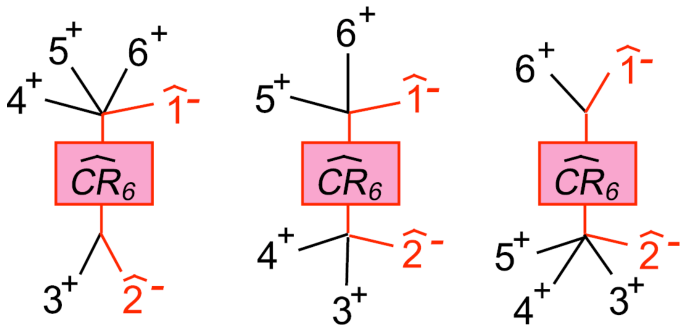
Example of recursive diagrams

For rational part of $A_6^{1-\text{loop}}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$

recursive:



overlap:



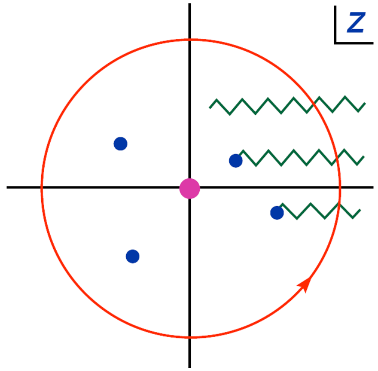
loops recycled into loops

Compared with 10,860 1-loop Feynman diagrams

Loop amplitudes with cuts

Generic analytic properties of shifted 1-loop amplitude, $A_n(z)$

Cuts and poles in *z*-plane:



$$\ln(s_{23}) \Rightarrow \ln[(\langle 2\,3\rangle + z\langle 1\,3\rangle)[3\,2]]$$

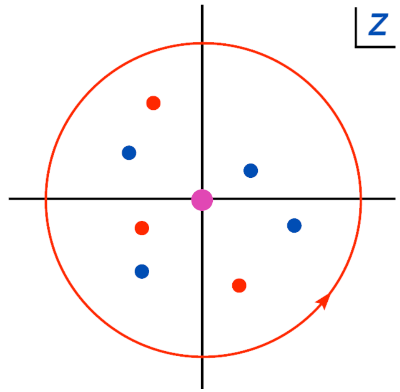
But if we know the cuts (*via* unitarity in D=4), we can subtract them: $R_n \equiv A_n - C_n$

rational part — full amplitude — cut-containing part

Shifted rational function $R_n(z) = A_n(z) - C_n(z)$

has no cuts, but has spurious poles in *z* because of C_n:

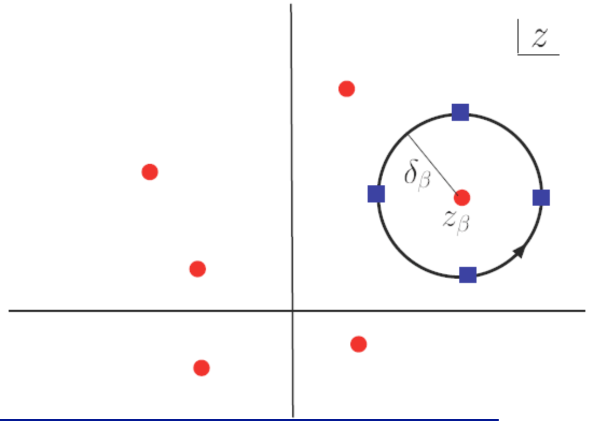
$$C_n \longrightarrow \frac{\ln(r) + 1 - r}{(1-r)^2} \longleftarrow R_n$$



Spurious poles

Locations are all known (dictated by Gram determinants associated with the various scalar integrals)

Residues can be determined from cut part (since they cancel in the full amplitude)



$$R_n^S(0) = -\sum_{\text{spur. poles }\beta} \operatorname*{Res}_{z=z_\beta} \frac{R_n(z)}{z} = \sum_{\text{spur. poles }\beta} \operatorname*{Res}_{z=z_\beta} \frac{C_n(z)}{z}$$

Loop integrals appear in C_n. We expand them around the spurious poles, keeping only rational parts. E.g. for 3-mass triangle integral:

$$I_3^{3\mathrm{m}}(s_1,s_2,s_3) \to -\frac{1}{2}\sum_{i=1}^{3}\ln(-s_i)\frac{s_i-s_{i+1}-s_{i-1}}{s_{i+1}s_{i-1}}\left[1-\frac{1}{6}\frac{\Delta_3}{s_{i+1}s_{i-1}}+\frac{1}{30}\left(\frac{\Delta_3}{s_{i+1}s_{i-1}}\right)^2\right]$$

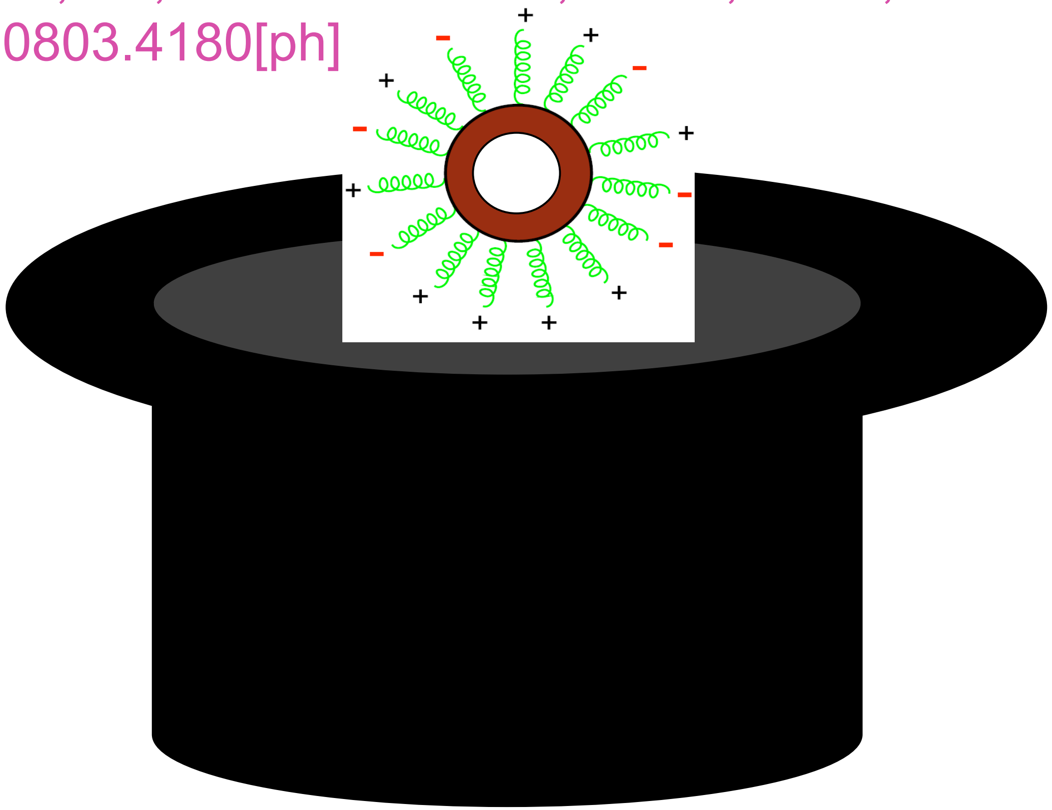
as $\Delta_3 \to 0$

$$+\frac{1}{6}\frac{\Delta_3}{s_1s_2s_3}-\frac{s_1+s_2+s_3}{120}\left(\frac{\Delta_3}{s_1s_2s_3}\right)^2+\cdots,$$

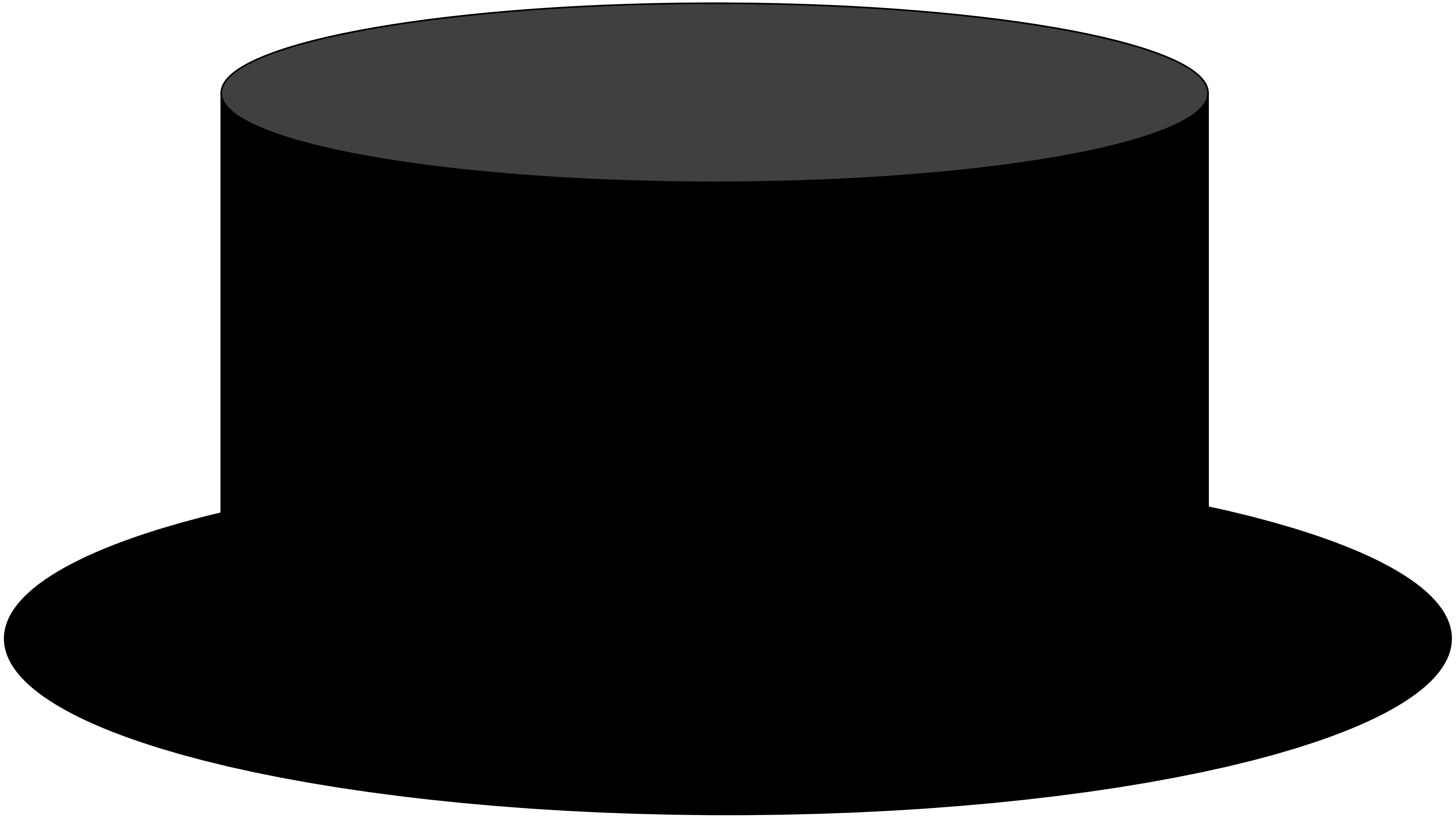
Automation required

BlackHat: C++ program

Berger, Bern, LD, Febres Cordero, Forde, H. Ita, D. Kosower, D. Maître, 0803.4180[ph]



Inside BlackHat



Practical issues

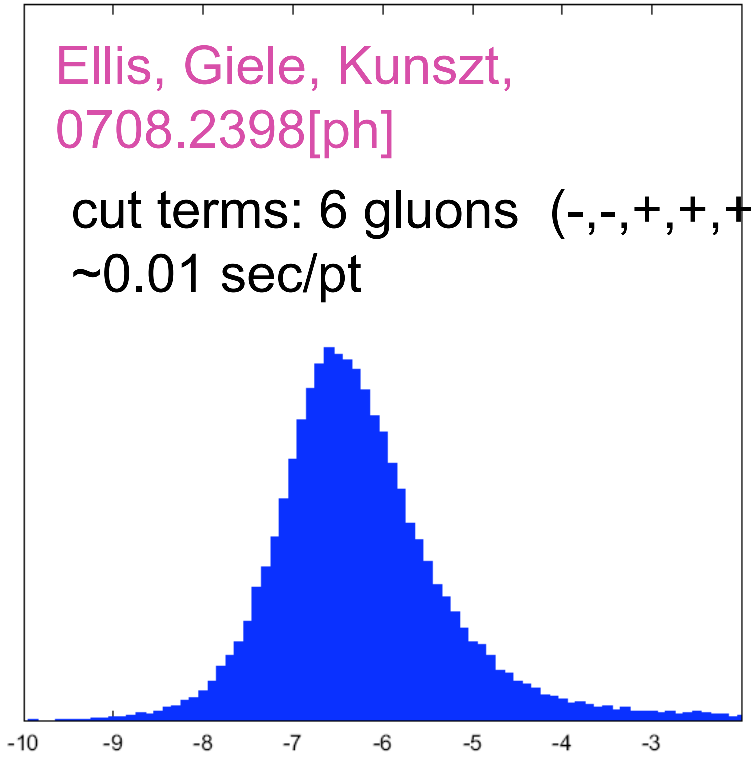
• **Evaluation time** (for Monte Carlo integration over phase space)

• **Numerical imprecision due to round-off errors** (can be large cancellations between different cut terms, and also against rational terms, in special phase space regions)

Ellis, Giele, Kunszt, 0708.2398[ph]

cut terms: 6 gluons (-,-,+,+,+,+)
~0.01 sec/pt

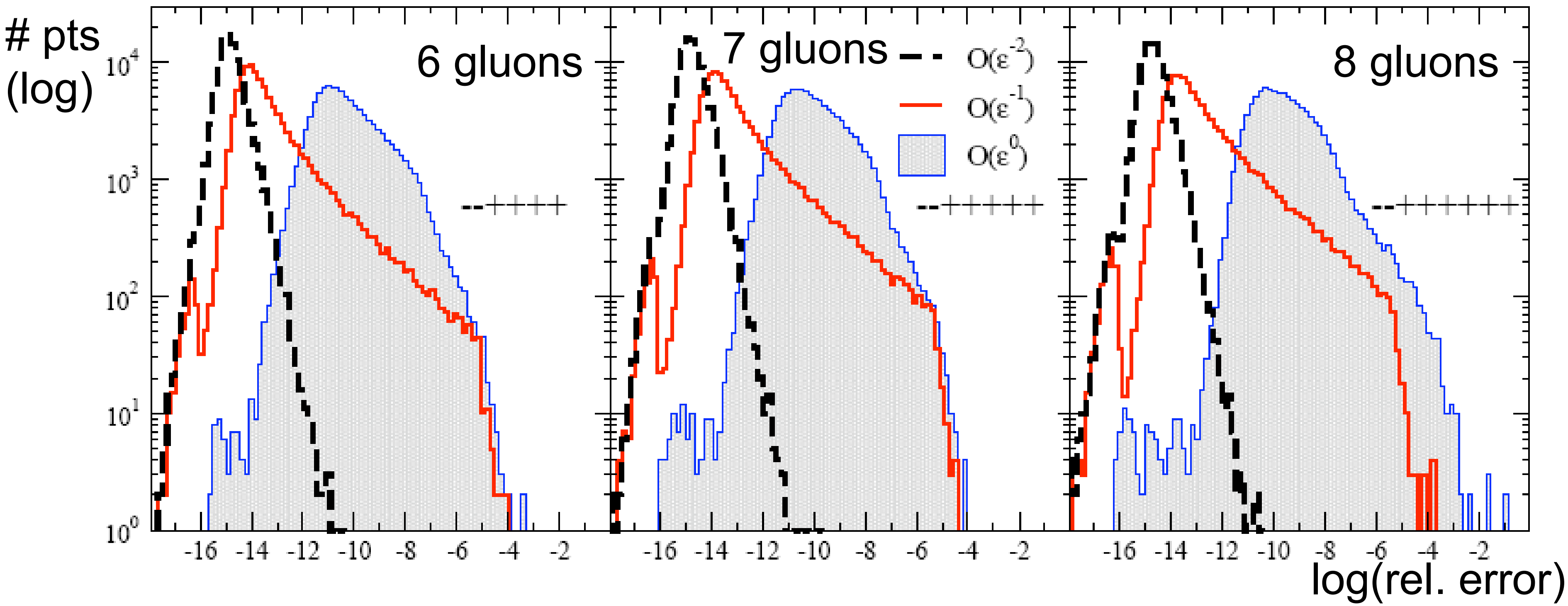
no. of pts (linear)



log(rel. error)

BlackHat results for n gluons

Berger, Bern, LD, Febres Cordero, Forde, Ita, Kosower, Maître, 0803.4180[ph]



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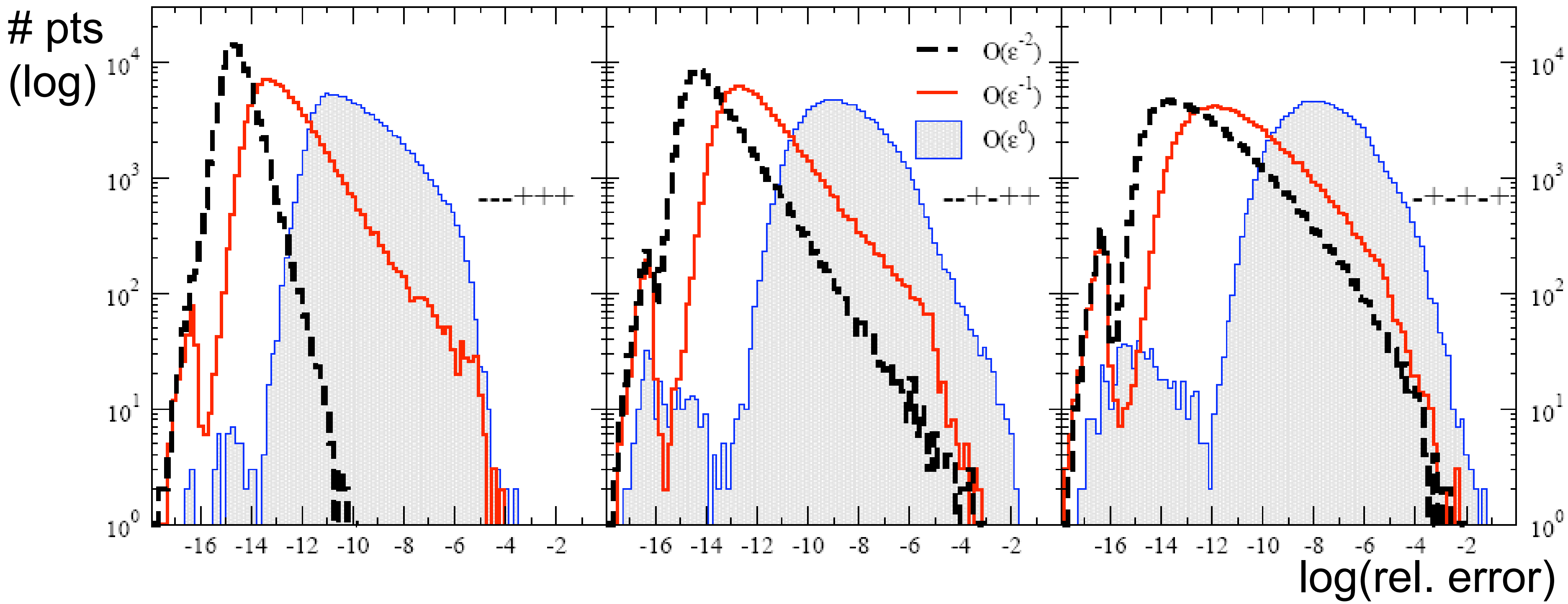
Most complex
6-gluon helicity amplitudes



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See also Giele, Zanderighi, 0805.2152 [ph];
(they are ~ 4 x slower for 6 gluons, but scale well to more gluons)

Conclusions

- New and efficient computational approaches to gauge theories are based on unitarity and factorization (inspired by twistor string developments).
- At first, practical spinoffs were mostly for trees,

  and loops in supersymmetric theories.
- But now, new loop amplitudes in full QCD – needed for collider applications – are beginning to fall to these techniques.
- More phenomenologically important processes than *n* gluons under construction.
- The pressing challenge now is to **automate everything at NLO, including also real radiation**

  Gleisberg, Krauss, 0709.2881 [hep-ph] ; Seymour, Tevlin, 0803.2231 [hep-ph]

  for wide classes of important LHC background processes.