From Amplitudes to Wilson Loops

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based on work done in collaboration with:

Heslop & Travaglini 0707.1153
Heslop, Nasti, Spence & Travaglini 0805.2763

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Outline

- Motivation
- MHV amplitudes in N=4 SYM & Wilson loops at weak coupling
- Iterative structures of loop amplitudes in N=8 SUGRA and Wilson loops
- Conclusions

Motivation

- Why are amplitudes so simple and how can we make use of this observation?
  - Geometry in twistor space (Witten 2003)
  - Iterative structures in S-matrix of gauge theory & gravity
- Simplicity hidden by standard Feynman rules
  - no manifest gauge symmetry
  - unphysical poles
- (Generalised) Unitarity & Twistor inspired methods
  - only gauge invariant, on-shell quantities enter at intermediate steps
  - apply also in non-supersymmetric theories

- In theories with maximal supersymmetry amplitudes are particularly simple ⇒ Ideal laboratory to test new ideas

- N=4 SYM: colour ordered & planar limit (leading in $1/N$)

  - all one-loop amplitudes are linear combination of box functions (Bern-Dixon-Dunbar-Kosower), coefficients from generalised unitarity (Britto-Cachazo-Feng)

  - Recursive structures in higher loop splitting amplitudes and MHV amplitudes (Anastasiou-Bern-Dixon-Kosower, Bern-Dixon-Smirnov)

    - Splitting amplitudes: universal, govern collinear limits

    - MHV: gluon helicities are permutation of $--+++\ldots+$

- Surprising relation to lightlike Wilson loops:
  strong coupling: (Alday-Maldacena) Alday's talk
  weak coupling: (Drummond-Korchemsky-Sokatchev+Henn,AB-Heslop-Travaglini)

- Dual conformal symmetry

  - integral functions in planar amplitudes
    (Drummond-Henn-Smirnov-Sokatchev)

  - Wilson loops (Alday-Maldacena, Drummond-Henn-Korchemsky-Sokatchev)

- Maximal transcendentality

- Today, consider MHV amplitudes in N=4 SYM and N=8 SUGRA
- Some common features of N=4 & N=8
  - Tree level recursion relations, good UV behaviour under complex shifts (Bedford, AB, Spence, Travaglini; Cachazo-Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed, Kaplan; Bianchi-Elvang-Freedman)
  - One-loop: "No Triangle Hypothesis" (Bern, Dixon, Perelstein, Rozowsky; Bern, Bjerrum-Bohr, Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Bjerrum-Bohr, Vanhove)
    - Both are important for possible finiteness of N=8 SUGRA (Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Bern, Dixon, Roiban; Green, Russo, Vanhove; Bern, Carrasco, Dixon, Johansson, Kosower, Roiban)
  - Transcendentality

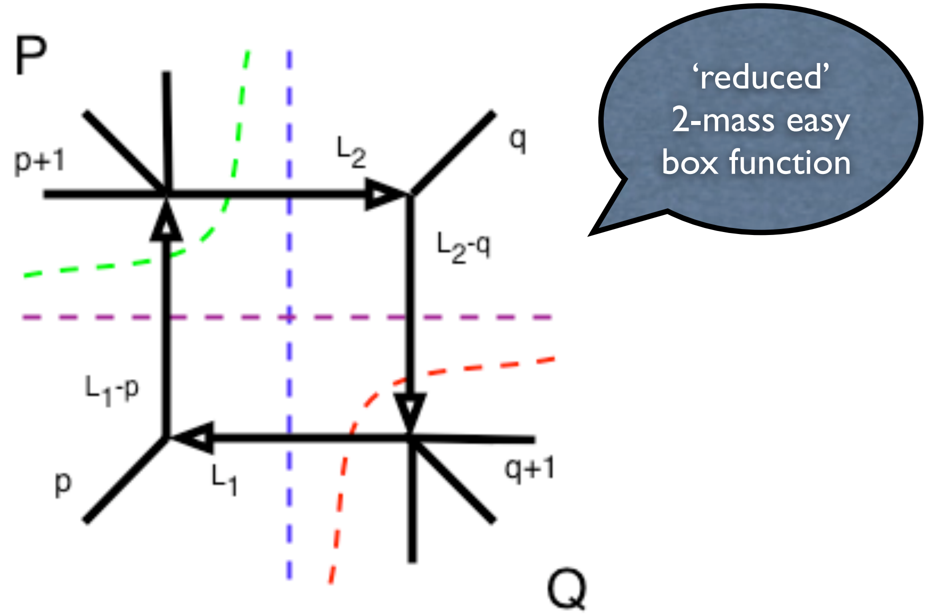
Goals for the rest of the talk

- MHV amplitudes in N=4 SYM
  - iterative structures in perturbative expansion
    (Korchemsky's talk)
  - relate one-loop n-gluon amplitudes to Wilson loops
    (AB-Heslop-Travaglini)
- 4-graviton MHV amplitude in N=8 SUGRA
  - look for iterative structures (similar to N=4)
  - try to find relation to Wilson loops

N=4 SYM

- Simplest one-loop amplitude is the n-point MHV amplitude in N=4 SYM at one loop (colour-ordered, partial amplitude):

$$\mathcal{A}_{MHV}^{1-loop} = \mathcal{A}_{MHV}^{tree} \sum_{p,q} 1 \times$$



- Calculated using unitarity in 1994 (Bern-Dixon-Dunbar-Kosower)
- Rederived from MHV diagrams in 2004 (AB-Spence-Travaglini)
- From Wilson loop in 2007 (AB-Heslop-Travaglini)

Suprising iterative structure at two loops...

- n-point MHV amplitude in N=4: $\mathcal{A}_n^{(L)} = \mathcal{A}_n^{tree}\mathcal{M}_n^{(L)}$
- First observed for 4 gluon scattering in planar N=4 SYM at 2 loops (Anastasiou-Bern-Dixon-Kosower)

$$\mathcal{M}_n^{(2)}(\epsilon) - \frac{1}{2}\left(\mathcal{M}_n^{(1)}(\epsilon)\right)^2 = f^{(2)}(\epsilon)\mathcal{M}_n^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon)$$

contains anomalous dimension of twist two operators at large spin

- Requires knowledge of one-loop amplitude to higher, positive orders in ϵ , $D = 4 - 2\epsilon$, in dimensional regularisaition

...and even higher loops

- In 2005 Bern-Dixon-Smirnov (BDS) found a similar iterative structure for n=4 at 3 loops and proposed an all-loop order formula for the MHV amplitudes in planar N=4 SYM.

$a \sim g_{\mathrm{YM}}^2 N/(8\pi^2)$

$$\mathcal{M}_n := 1 + \sum_{L=1}^{\infty} a^L \mathcal{M}_n^{(L)} \overset{?}{=} \exp\left[\sum_{L=1}^{\infty} a^L \left(f^{(L)}(\epsilon)\mathcal{M}_n^{(1)}(L\epsilon) + C^{(L)} + \mathcal{O}(\epsilon)\right)\right]$$

- $\mathcal{M}_n^{(1)}$ is the all orders in ϵ one-loop MHV amplitude
- In order to extract recursive relations order-by-order in a consider the log of this expression, e.g. for L=2 & 3

$$\mathcal{M}_n^{(2)} = \tfrac{1}{2}\left(\mathcal{M}_n^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\,\mathcal{M}_n^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)$$

$$\mathcal{M}_n^{(3)} = -\tfrac{1}{3}\left(\mathcal{M}_n^{(1)}(\epsilon)\right)^3 + \mathcal{M}_n^{(2)}(\epsilon)\mathcal{M}_n^{(1)}(\epsilon) + f^{(3)}(\epsilon)\,\mathcal{M}_n^{(1)}(3\epsilon) + \mathcal{O}(\epsilon)$$

Comments

- The exponential form is strongly motivated by the universal factorisation & exponentiation/resummation of IR divergences in gauge theories (not only N=4)
- The miracle in N=4 is that exponentiation also applies to the finite parts of the amplitude and the finite remainder becomes a constant independent of kinematics
- Confirmed by a recent strong coupling calculation using AdS/CFT by Alday-Maldacena (at least for n=4).

Test of the conjecture

- Two and three loops, $n=4$ (Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov)
- Two loops, $n=5$ (Bern, Czakon, Kosower, Roiban, Smirnov; Cachazo, Spradlin, Volovich)
- Problems for more gluons
    - $n \to \infty$ (Alday, Maldacena)
    - $n=6$ (Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich; Cachazo, Spradlin, Volovich)
- Exponent requires an additional kinematic dependent finite remainder function

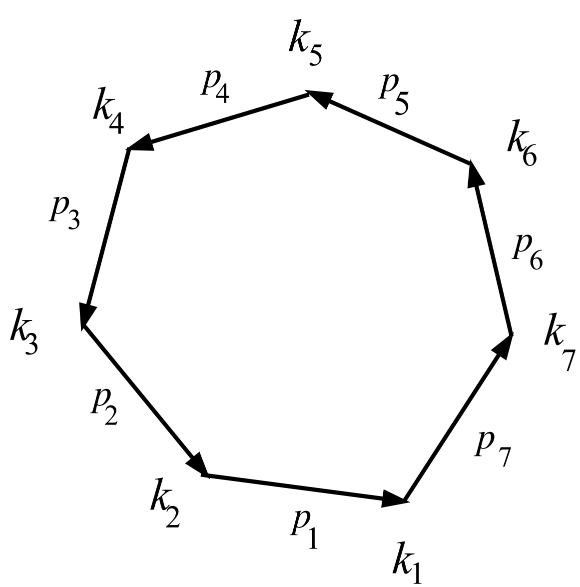
Amplitudes and Wilson Loops

(Drummond, Korchemsky, Sokatchev; AB, Heslop, Spence, Travaglini)

- MHV amplitudes in N=4 super Yang-Mills $\mathcal{A}_n^{(L)} = \mathcal{A}_n^{tree} \mathcal{M}_n^{(L)}$
- Surprisingly, $\mathcal{M}_n$ appears in a very different context!

Wilson loop calculation

- Lightlike Contour in dual momentum space ⇒ fixed by on-shell momenta of gluons (colour-ordered, helicity-blind)



$p_i = k_i - k_{i+1}$ k's are (T-)dual momenta

$\sum_{i=1}^{n} p_i = 0$ ⇒ Contour is closed

Amplitudes and Wilson Loops

- Calculate $< W[C] >$ at weak coupling

$$W[C] := \mathrm{Tr}\,\mathrm{P}\exp\left[ig\oint_C d\tau\Big(A_\mu(x(\tau))\dot{x}^\mu(\tau)+\phi_i(x(\tau))\dot{y}^i(\tau)\Big)\right]$$

- Contour C of previous page is the same as in the strong coupling calculation of Alday-Maldacena using AdS/CFT
- When $\dot{x}^2=\dot{y}^2$ Wilson loop is locally supersymmetric
- Here we have $\dot{x}^2=0$ (lightlike momenta) and $\dot{y}=0$
- Locally Supersymmetric

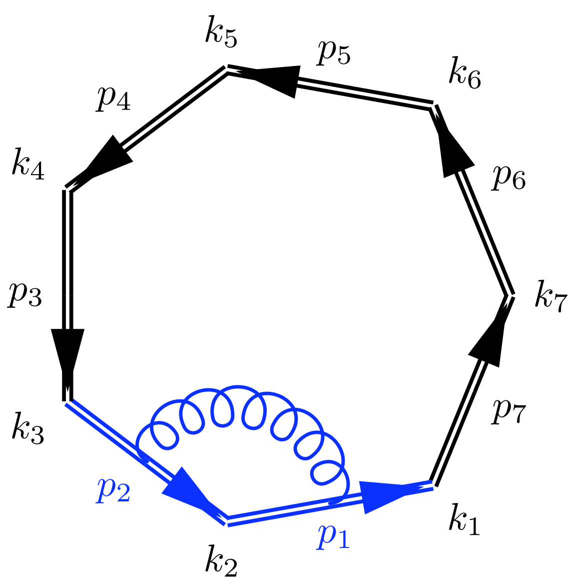
- **Motivation:** recent computation of gluon amplitudes at strong coupling (Alday-Maldacena)

  - scattering in AdS is at fixed angle, high energy ➡ similar to Gross-Mende calculation
  - ➪ exponential of classical string action $\mathcal{A} \sim e^{-S_{\rm cl}} = e^{-\sqrt{\lambda}/(2\pi)(\text{Area})_{\rm cl}}$
  - In T-dual variables the B.C.s of the string is a lightlike polygonal loop C embedded in the boundary of AdS
  - Finding the minimal area with these B.C.s is equivalent to the calculation of a lightlike Wilson loop in AdS/CFT (Maldacena; Rey-Yee)
  - Alday-Maldacena: confirmation of BDS conjecture at 4-points at strong coupling!

< $W[C]$ > and MHV amplitudes at 1-loop

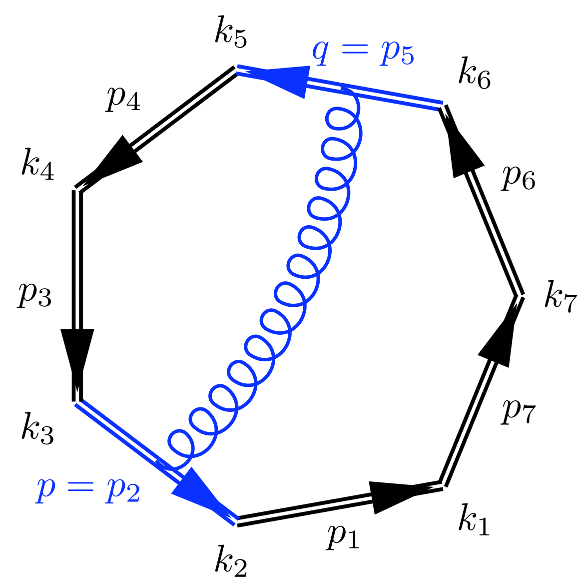
(AB, Heslop, Travaglini)

- Two classes of diagrams (Feynman gauge):



Gluon stretched between two segments meeting at a cusp

A. IR divergent



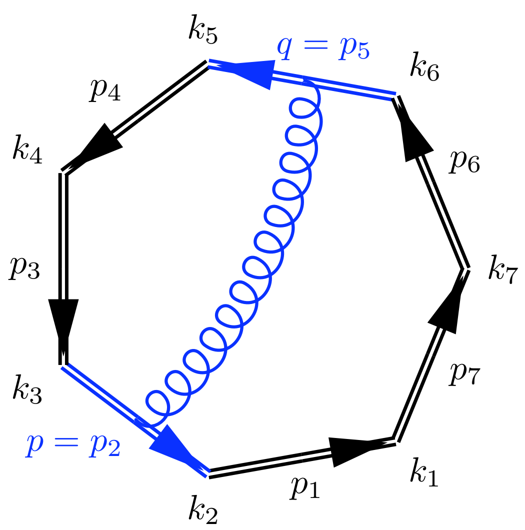
Gluon stretched between two non-adjacent segments

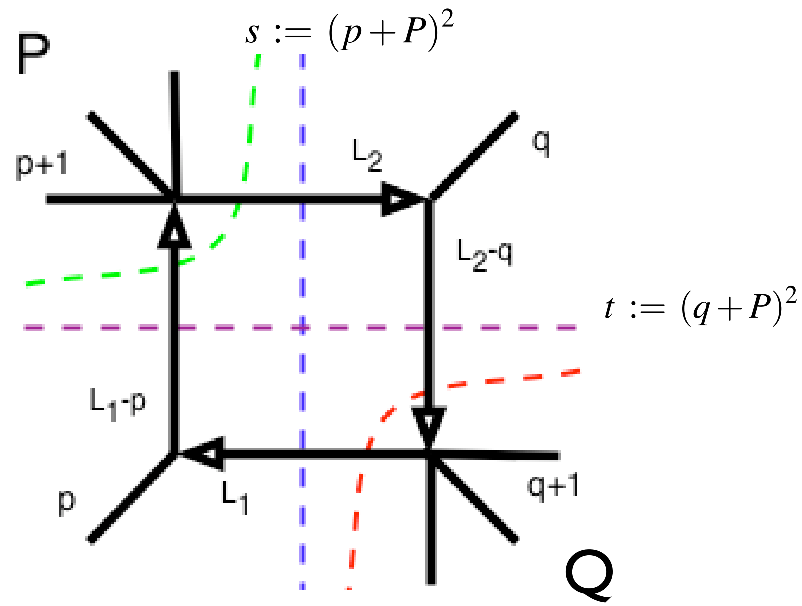
B. Finite

- Clean separation of IR divergent and Finite terms
- From diagrams in class A : $\mathcal{M}_n^{(1)}|_{IR} = -\frac{1}{\varepsilon^2}\sum_{i=1}^{n}\left(\frac{-s_{i,i+1}}{\mu^2}\right)^{-\varepsilon}$
  - $s_{i,i+1} = (p_i + p_{i+1})^2$ is the invariant formed from the momenta meeting at the cusp
- Diagrams in class B give rise to the following integral

$$\mathcal{F}_\varepsilon(s,t,P,Q) = \int_0^1 d\tau_p d\tau_q \frac{P^2+Q^2-s-t}{[-(P^2+(s-P^2)\tau_p+(t-P^2)\tau_q+(-s-t+P^2+Q^2)\tau_p\tau_q)]^{1+\varepsilon}}$$

  - equal to the finite part of 2-mass easy box function!
  - Comment: this integral is directly related to the Feynman parameter integral of the 2-mass easy box function





- In the example: $p = p_2 \quad q = p_5$

  $P = p_3 + p_4\,, \quad Q = p_6 + p_7 + p_1$

- One-to-one correspondence between Wilson loop diagrams and finite parts of 2-mass easy box functions

- "Explains" why box functions appear with coefficient = **1** in the one-loop N=4 MHV amplitude

- Explicit calculation gives at $\varepsilon \to 0$:

$$a := \frac{2(pq)}{P^2 Q^2 - st}$$

$$\mathcal{F}_{\varepsilon=0} = -\mathrm{Li}_2(1-as) - \mathrm{Li}_2(1-at) + \mathrm{Li}_2(1-aP^2) + \mathrm{Li}_2(1-aQ^2)$$

  - finite part of the box function appearing in the one-loop MHV amplitude in N=4 SYM!

- At 4 points, find the correct all-orders in ε result (terms up to O(ε) agree with result of Drummond-Korchemsky-Sokatchev):

$$\mathcal{M}_4^{(1)}(\varepsilon) = -\frac{2}{\varepsilon^2}\left[\left(\frac{-s}{\mu^2}\right)^{-\varepsilon} {}_2F_1\left(1,-\varepsilon,1-\varepsilon,1+\frac{s}{t}\right) + \left(\frac{-t}{\mu^2}\right)^{-\varepsilon} {}_2F_1\left(1,-\varepsilon,1-\varepsilon,1+\frac{t}{s}\right)\right]$$

- For $n > 4$, missing topologies (vanish as $\varepsilon \to 0$)

  - E.g. for n=5 amplitude contains parity odd term (pentagon integral). Wilson loop does not capture that!

Comment: "conformal gauge"

- Consider gluon propagator $\sim \left(1+\frac{1}{\epsilon}\right)\frac{(\eta_{\mu\nu}-2x_\mu x_\nu/x^2)}{(-x^2+i\varepsilon)^{1-\epsilon}}$
- Type A diagrams vanish!
- Type B diagrams are in one-to-one correspondence with complete 2-mass easy box functions incl. IR-div. terms
- would be interesting to investigate this further for higher loops
- We will come back to a similar gauge later when we consider Wilson loops for amplitudes in N=8 SUGRA

$< W[C] >$ at higher loops

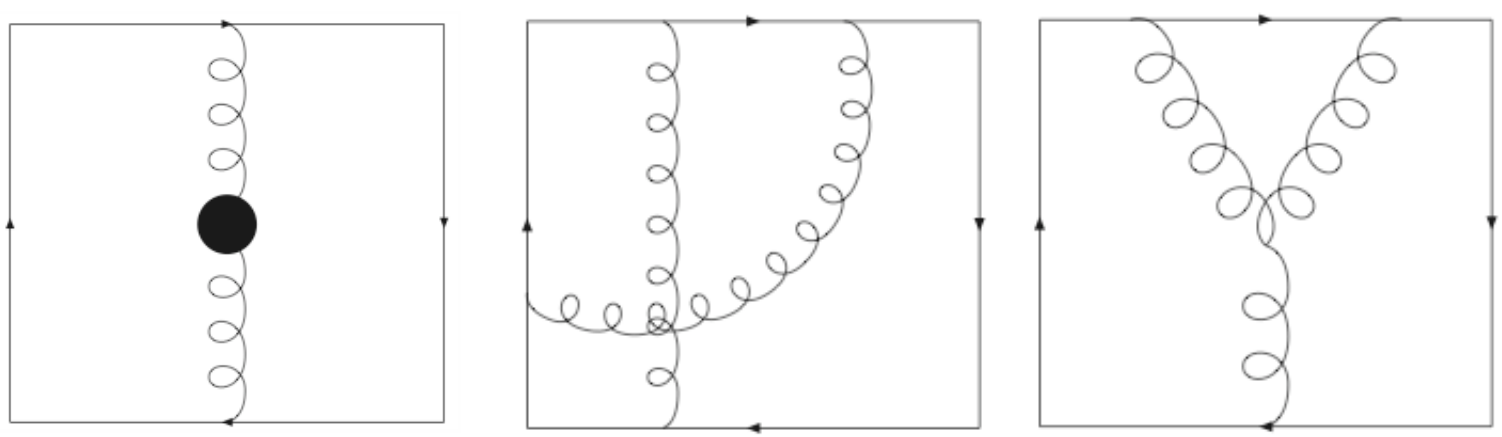
(Drummond, Henn, Korchemsky, Sokatchev)

- Key result: non-abelian exponentiation theorem
(Gatheral; Frenkel-Taylor)

$$\langle W[C] \rangle := 1 + \sum_{L=1}^{\infty} a^L W^{(L)} = \exp \sum_{L=1}^{\infty} a^L w^{(L)}$$

- w's are calculated by keeping only the subset of diagrams containing maximal non-abelian colour factor

- Also: exponential form of the answer is automatic



Two conjectures

- the Wilson loop and BDS conjecture can be written as

$$\mathcal{M}_n = 1 + \sum_{L=1}^{\infty} \mathcal{M}_n^{(L)}(\epsilon) = exp\left(\sum_{L=1}^{\infty} m_n^{(L)} + O(\epsilon)\right)$$

- it's more illuminating to write the log of this; expanding to e.g. 3 loop order $\Rightarrow$

$$\begin{aligned} \mathcal{M}_n^{(1)} &= m_n^{(1)} + O(\epsilon) \\ \mathcal{M}_n^{(2)} - \frac{1}{2}\left(\mathcal{M}_n^{(1)}\right)^2 &= m_n^{(2)} + O(\epsilon) \\ \mathcal{M}_n^{(3)} + \frac{1}{3}\left(\mathcal{M}_n^{(1)}\right)^3 - \mathcal{M}_n^{(1)}\mathcal{M}_n^{(2)} &= m_n^{(3)} + O(\epsilon) \end{aligned}$$

Note: RHS is parity even, hence the parity odd terms on LHS must cancel to order $O(\epsilon)$!

Checks of the conjectures

- BDS conjecture: $m_n^{(L)} = a^L[f^{(L)}(\epsilon)\mathcal{M}^{(1)}(L\epsilon) + C^{(L)}] + O(\epsilon)$
- Wilson loop conj.: $m_n^{(L)} = a^L w_n^{(L)} + O(\epsilon)$

- Checks of BDS conjecture:

  ✓ n=4 up to L=3 (BDS)

  ✓ n=5 up to L=2 (Cachazo-Spradlin-Volovich, Bern-Czakon-Kosower-Roiban-Smirnov)

  * Problems starting at n=6 at L=2, finite remainder
    (Bern-Dixon-Kosower-Roiban-Spradlin-Vergu-Volovich, Cachazo-Spradlin-Volovich)

- Checks of Wilson loop conjecture:

  ✓ all n at L=1 (Drummond-Korchemsky-Sokatchev, AB-Heslop-Travaglini)

  ✓ n=4, 5, 6 at L=2 (Drummond-Henn-Korchemsky-Sokatchev)

N=8 Supergravity

- We wish to address 2 questions:
  - Do amplitudes in N=8 SUGRA exhibit iterative structures as in N=4 SYM? (Naculich-Nastase-Schnitzer, AB-Heslop-Nasti-Spence-Travaglini)
  - Is there an analogous Wilson loop/Amplitude duality for MHV amplitudes? (AB-Heslop-Nasti-Spence-Travaglini)
- Focus on four graviton amplitudes
  - tree-level amplitude factors out

$$\mathcal{A}_4^{(L)} = \mathcal{A}_4^{\rm tree}\,\mathcal{M}_4^{(L)}$$

Looking for iterative structures

- As in N=4 SYM write:

$$\mathcal{M}_4 = 1 + \sum_{L=1}^{\infty} \mathcal{M}_4^{(L)} = \exp\left[\sum_{L=1}^{\infty} m_4^{(L)}\right]$$

$$m_4^{(1)} = \mathcal{M}_4^{(1)}, \;\; m_4^{(2)} = \mathcal{M}_4^{(2)} - \frac{1}{2}\left(\mathcal{M}_4^{(1)}\right)^2$$

- We want to find $m_4^{(2)}$

- Recall: in N=4 SYM this term was proportional to the one-loop amplitude for 4 and 5 gluons and hence IR divergent

Amplitudes in N=8 SUGRA

- Tree-level:
  - KLT (Kawai-Lewellen-Tye)
  - On-shell Recursions (Bedford-AB-Spence-Travaglini, Cachazo-Svrcek). Suprisingly good UV behaviour under complex shifts
- One-loop: sum of box functions ➪ “no-triangle hypothesis”
  - MHV amplitudes: 4 point (Green-Schwarz-Brink, Dunbar-Norridge); general case from unitarity (Bern-Dixon-Perelstein-Rozowsky). MHV-Amplitude = $\langle ij \rangle^8$ x (helicity blind function)
  - non-MHV amplitudes: many examples from generalised unitarity (Bern, Bjerrum-Bohr, Dunbar, Ita)
- 2-loop, 4 point (Bern-Dunbar-Dixon-Perelstein-Rozowsky)
  3-loop, 4 point (Bern-Carrasco-Dixon-Johansson-Kosower-Roiban)

IR divergences

- One-loop IR divergences known to exponentiate, similar to QED. Weinberg's proof used eikonal approximation
- IR behaviour is softer compared to YM. At one loop only $\frac{1}{\epsilon}$
  - E.g. for 4 points at one loop (Dunbar, Norridge)

$$\mathcal{M}^{(1)}\Big|_{\mathrm{IR}} = c_\Gamma \left(\frac{\kappa}{2}\right)^2 \frac{2}{\epsilon} \Big(s\,\log(-s) + t\,\log(-t) + u\,\log(-u)\Big)$$

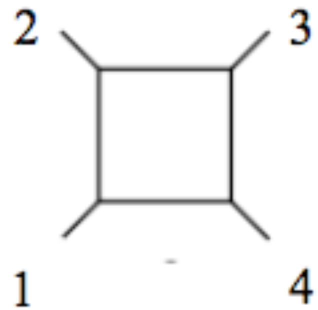
- Absence of colour ordering $\mathcal{M}\Big|_{\mathrm{IR}} = \prod_{i<j} \mathcal{M}_{\mathrm{div}}(s_{ij})$
- Also, soft and collinear amplitudes tree level exact (Bern, Dunbar, Dixon, Perelstein, Rozowsky)

One- and two-loop 4-point amplitudes

- One-loop (Green-Schwarz-Brink, Dunbar-Norridge)
  - no colour ordering ⇒ answer involves sum over permutations (1234), (1423), (1342)

$$\mathcal{M}_4^{(1)} = -i\, s\, t\, u \left(\frac{\kappa}{2}\right)^2 \Big[\mathcal{I}_4^{(1)}(s,t) + \mathcal{I}_4^{(1)}(s,u) + \mathcal{I}_4^{(1)}(u,t)\Big]$$

$$\mathcal{I}_4^{(1)}(s,t) := \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 (l-p_1)^2 (l-p_1-p_2)^2 (l+p_4)^2}$$

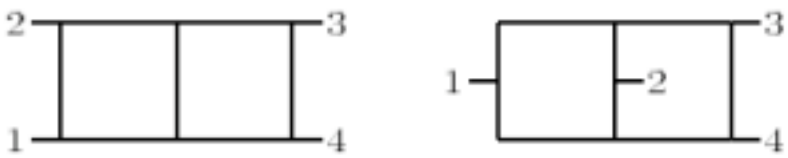


zero-mass box function

- Two-loop: (Bern-Dunbar-Dixon-Perelstein-Rozowsky)

$$\mathcal{M}_4^{(2)} = \left(\frac{\kappa}{2}\right)^4 stu \left[s^2 \mathcal{I}_4^{(2),\mathrm{P}}(s,t) + s^2 \mathcal{I}_4^{(2),\mathrm{P}}(s,u) + s^2 \mathcal{I}_4^{(2),\mathrm{NP}}(s,t) + s^2 \mathcal{I}_4^{(2),\mathrm{NP}}(s,u) \;+\; \text{cyclic}\right]$$

- Where $\mathcal{I}_4^{(2),\mathrm{P}}$, $\mathcal{I}_4^{(2),\mathrm{NP}}$ are the planar and non-planar double boxes



- Calculated analytically in DR by Smirnov and Tausk
  - Note: the non-planar integral is not transcendental
- Starting point to study possible iterations

Iterative Structure

- Main result: $$\mathcal{M}_4^{(2)} - \frac{1}{2}\left(\mathcal{M}_4^{(1)}\right)^2 = \text{finite} + \mathcal{O}(\epsilon)$$
- Finite remainder has uniform transcendentality
  - Specific combination of NP boxes is transcendental
  - Does this persist to higher loops?
- Remainder is not related to one-loop amplitude (unlike 4 point N=4 SYM amplitude) and contains logarithms and (Nielsen) polylogs.
- Answer is in agreement with the expected exponentiation of the one loop IR divergences, i.e. the remainder function is finite

- the full answer is

$$\mathcal{M}_4^{(2)} - \frac{1}{2}(\mathcal{M}_4^{(1)})^2 \;=\; -\left(\tfrac{\kappa}{8\pi}\right)^4 \Big[u^2\big[k(y)+k(1/y)\big] \;+\; s^2\big[k(1-y)+k(1/(1-y))\big]$$
$$+t^2\big[k(y/(y-1))+k(1-1/y)\big]\Big] + O(\epsilon)$$

where

$$k(y) \;:=\; \frac{L^4}{6} + \frac{\pi^2 L^2}{2} - 4S_{1,2}(y)L + \frac{1}{6}\log^4(1-y) + 4\; S_{2,2}(y) - \frac{19\pi^4}{90}$$
$$+i\;\left(-\frac{2}{3}\pi\log^3(1-y) - \frac{4}{3}\pi^3\log(1-y) - 4L\pi\;\mathrm{Li}_2(y) + 4\pi\mathrm{Li}_3(y) - 4\pi\zeta(3)\right)$$

and $\quad y = -s/t\,,\; L := \log(s/t)$

Wilson loops for gravity amplitudes

- Properties of candidate Wilson loop:
  - contour fixed by momenta of gravitons
  - invariant under diffeos
  - same symmetries as scattering amplitude
- As in eikonal approximation we do not expect to capture the helicity dependence

Holonomy

- Natural starting point would be the holonomy of the Christoffel connection Γ, $\langle \mathrm{Tr}\, \mathcal{U}(C) \rangle$ with

$$\mathcal{U}^{\alpha}_{\beta}(C) \;:=\; \mathcal{P}\exp\left[i\kappa \oint_C dy^{\mu} \Gamma^{\alpha}_{\mu\beta}(y) \right]$$

- Studied by Modanese in perturbation theory
  - Invariant under diffeos ...
  - ... but answer has nothing to do with an amplitude.

$$\kappa^2 \oint_C dx^{\mu} dy^{\nu} \, \langle \Gamma^{\alpha}_{\mu\beta}(x) \Gamma^{\beta}_{\nu\alpha}(y) \rangle \sim \kappa^2 \oint_C \; dx_{\mu} dy^{\mu} \delta^{(D)}(x-y)$$

Eikonal Wilson loop

- Try an expression that has been used in the past for calculations of amplitudes involving gravitons in the eikonal approximation (Kabat-Ortin, Fabbrichesi-Pettorino-Veneziano-Vilkovisky)
- In linearised approximation $g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)$

$$W[C] \ := \ \left\langle \mathcal{P} \exp \left[i\kappa \oint_C d\tau \ h_{\mu\nu}(x(\tau)) \dot{x}^\mu(\tau) \dot{x}^\nu(\tau) \right] \right\rangle$$

- The exponent can be written as $\int d^D x \mathcal{T}^{\mu\nu}(x) h_{\mu\nu}(x)$, where the EM-tensor is that of a free point particle
- However, if the contour C has cusps, then the loop is not diffeomorphism invariant!

- Try anyway!

- First, in order to implement the symmetries of the amplitude we propose to consider

$$W \;:=\; W[C_{1234}]\, W[C_{1423}]\, W[C_{1342}]$$

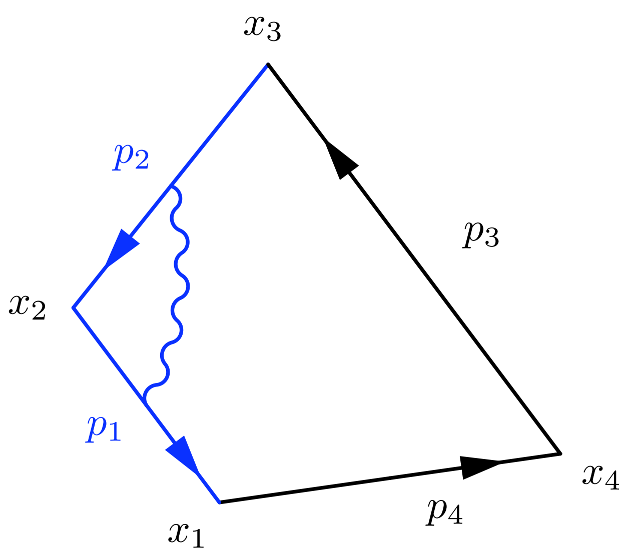
- At one loop this becomes

$$W^{(1)} \;:=\; W^{(1)}[C_{1234}] + W^{(1)}[C_{1423}] + W^{(1)}[C_{1342}]$$

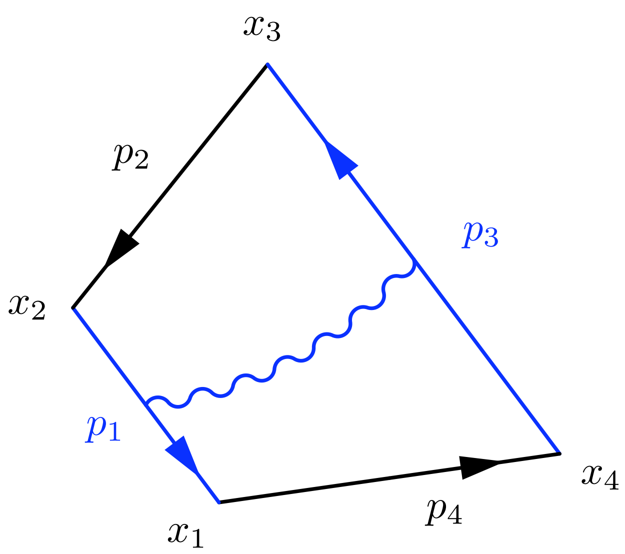
where the contours C_{ijkl} are constructed by connecting the momenta in the prescribed order

<W> at one loop

- There are two classes of diagrams as in N=4 SYM. (A sum over cyclic permutations in (234) is understood)



A. IR divergent



B. Finite

- From diagrams in Class A we get:

$$\kappa^2 \frac{c(\epsilon)}{\epsilon^2}\left[(-s)^{1+\epsilon} + (-t)^{1+\epsilon} + (-u)^{1+\epsilon}\right]$$

  - The leading divergence cancels since $s + t + u = 0$.
  - Subleading terms as expected

- From diagrams in Class B we get:

$$c(\epsilon)\,\frac{u}{2}\,\frac{1}{4}\left[\log^2\left(\frac{s}{t}\right) + \pi^2\right]$$

  - This is the finite part of a zero mass box function. Sum over perms reproduces the finite part of amplitude

Summary of Results

- Tree level factor missing (as in N=4 SYM)
- Relative normalisation betwee IR divergent and finite terms is incorrect by a factor of (-2)
    - a factor of 2 can be accounted for by an effective overcounting of cusp contributions in W; the minus sign is harder to explain
- The result is gauge dependent (so far we were using de Donder gauge), but close to the correct answer...

Conformal Gauge

- Defined as the gauge where the cusp diagrams vanish
  - have illustrated that earlier for Yang-Mills, where Wilson loop is gauge invariant
- Get the correct N=8 SUGRA amplitude !
- This gauge is a special case of de Donder gauge with an unusual value for the gauge fixing parameter $\alpha = -\frac{2\epsilon}{1+\epsilon}$

$$\mathcal{L}^{(\mathrm{gf})} = \frac{\alpha}{2}\left(\partial_\nu h^\nu_\mu - \frac{1}{2}\partial_\mu h^\alpha_\alpha\right)^2$$

Note: in usual de Donder gauge $\alpha = -2$

- Graviton propagator in x-space, conf. gauge

$$D_{\mu\nu,\mu'\nu'}(x) \sim \frac{\epsilon-1}{\epsilon}\left[\frac{1}{(-x^2)^{1-\epsilon}}\left(\eta_{\mu'(\mu}\eta_{\nu)\nu'} + \frac{\epsilon}{2(\epsilon-1)^2}\eta_{\mu\nu}\eta_{\mu'\nu'}\right) + 2\frac{1}{(-x^2)^{2-\epsilon}}x_{(\mu}\eta_{\nu)(\nu'}x_{\mu')}\right]$$

- Gluon propagator in x-space, conf. gauge

$$\Delta^{\rm conf}_{\mu\nu}(x) \sim \frac{1-\epsilon}{\epsilon}\frac{1}{(-x^2+i\varepsilon)^{1-\epsilon}}\underbrace{\left[\eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}\right]}_{\text{Inversion Tensor}}$$

Conclusions

- Mysterious relation between planar MHV amplitudes in N=4 SYM and light-like Wilson loops
- Why does this work? Dual conformal symmetry is insufficient to explain this, are there other symmetries?
- Unitarity for Wilson loop?
- Possible relations to world line formalism?
- What about other theories/non-MHV amplitudes?
    - 1-loop: Wilson loops insensitive to matter content of theory
    - 2 loops: Wilson loops in any SCFT identical
    - in N=1 SYM $\mathcal{M}_n^{(1)}$ depends on helicities

Conclusions cont'd

- Iterative structure in N=8 SUGRA amplitudes
    - IR divergences iterate completely
    - relatively simple finite remainder with uniform transcendentality
- Wilson loop reproduces almost the one-loop amplitude
- IR divergent and finite parts come out correctly
- cusps break the gauge invariance; can this be fixed?
- Conformal gauge gives the complete amplitude