Ultraviolet Properties of $\mathcal{N} = 8$ Supergravity at Three Loops and Beyond

Paris, June 25, 2008

Zvi Bern, UCLA

Based on following papers:

ZB, L. Dixon , R. Roiban, hep-th/0611086
ZB, J.J. Carrasco, L. Dixon, H. Johansson, D. Kosower, R. Roiban, hep-th/0702112
ZB, J.J. Carrasco, H. Johansson and D. Kosower, arXiv:0705.1864[hep-th]
ZB, J.J. Carrasco, D. Forde, H. Ita and H. Johansson, arXiv:0707.1035 [hep-th]
ZB, J.J. Carrasco, H. Johansson, arXiv:0805.3993 [hep-ph]

Outline

Will present concrete evidence for perturbative UV finiteness of $\mathcal{N} = 8$ supergravity.

- Review of conventional wisdom on UV divergences in quantum gravity.
- Surprising one-loop cancellations point to improved UV properties. Motivates multi-loop investigation.
- Calculational method – reduce gravity to gauge theory:
  - (a) Kawai-Lewellen-Tye tree-level relations.
  - (b) Unitarity method – maximal cuts.
- All-loop arguments for UV finiteness of $\mathcal{N} = 8$ supergravity.
- Explicit three-loop calculation and "superfinitness".
- Progress on four-loop calculation.
- Origin of cancellation -- generic to all gravity theories.

Finiteness of $N = 8$ Supergravity?

We are interested in UV finiteness of $N = 8$ supergravity because it would imply a new symmetry or non-trivial dynamical mechanism.

The discovery of either would have a fundamental impact on our understanding of gravity.

- **Here we only focus on order-by-order UV finiteness.**
- **Non-perturbative issues and viable models of Nature are *not* the goal for now.**

$N = 8$ Supergravity

The most supersymmetry allowed for maximum particle spin of 2 is $N = 8$. Eight times the susy of $N = 1$ theory of Ferrara, Freedman and van Nieuwenhuizen

We consider the $N = 8$ theory of Cremmer and Julia.

256 massless states

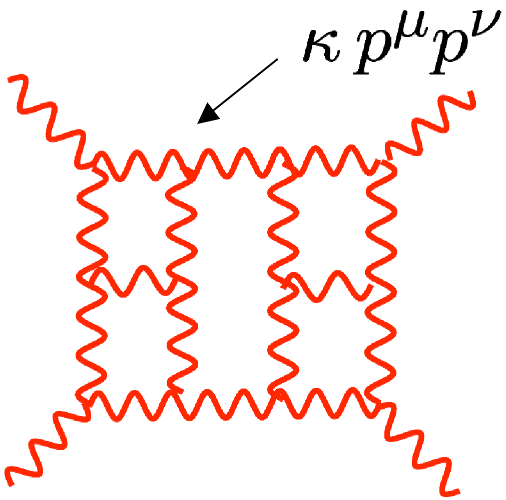
$$h^- \quad \psi_i^- \quad v_{ij}^- \quad \chi_{ijk}^- \quad s_{ijkl} \quad \chi_{ijk}^+ \quad v_{ij}^+ \quad \psi_i^+ \quad h^+$$

Reasons to focus on this theory:

- With more susy suspect better UV properties.
- High symmetry implies technical simplicity.

Power Counting at High Loop Orders

$$\kappa = \sqrt{32\pi G_N}$$ ← Dimensionful coupling



Gravity: $$\int \prod_{i=1}^{L} \frac{dp_i^D}{(2\pi)^D} \frac{(\kappa\, p_j^\mu p_j^\nu)\cdots}{\text{propagators}}$$

Gauge theory: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g\, p_j^\nu)\cdots}{\text{propagators}}$$

Extra powers of loop momenta in numerator means integrals are badly behaved in the UV

Much more sophisticated power counting in supersymmetric theories but this is the basic idea.

See Stelle's talk

Quantum Gravity at High Loop Orders

A key unsolved question is whether a finite point-like quantum gravity theory is possible.

- Gravity is non-renormalizable by power counting.

$$\kappa = \sqrt{32\pi G_N} \longleftarrow \text{Dimensionful coupling}$$

- Every loop gains $G_N \sim 1/M_{\rm Pl}^2$ mass dimension –2.
  At each loop order potential counterterm gains extra

$$R^{\mu}_{\nu\sigma\rho} \sim g^{\mu\kappa}\partial_\rho\partial_\nu g_{\kappa\sigma} \quad \text{or } D^2$$

- As loop order increases potential counterterms must have either more *R*'s or more derivatives

Divergences in Gravity

One loop: Vanish on shell

$R^2, \ R_{\mu\nu}^2, \ R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ ← vanishes by Gauss-Bonnet theorem

Pure gravity 1-loop finite (but not with matter)

't Hooft, Veltman (1974)

Two loop: Pure gravity counterterm has non-zero coefficient:

$$R^3 \equiv R^{\lambda\rho}{}_{\mu\nu} R^{\mu\nu}{}_{\sigma\tau} R^{\sigma\tau}{}_{\lambda\rho}$$

Goroff, Sagnotti (1986); van de Ven (1992)

Any supergravity:

R^3 is ***not*** a valid supersymmetric counterterm.

Produces a helicity amplitude $(-,+,+,+)$ forbidden by susy.

Grisaru (1977); Tomboulis (1977)

The first divergence in *any* supergravity theory can be no earlier than three loops.

R^4 **squared Bel-Robinson tensor expected counterterm**

Deser, Kay, Stelle (1977); Kaku, Townsend, van Nieuwenhuizen (1977), Ferrara, Zumino (1978)

Opinions from the 80's

If certain patterns that emerge should persist in the higher orders of perturbation theory, then … $N = 8$ supergravity in four dimensions would have ultraviolet divergences starting at three loops.

Green, Schwarz, Brink, (1982)

Unfortunately, in the absence of further mechanisms for cancellation, the analogous $N = 8$ $D = 4$ supergravity theory would seem set to diverge at the three-loop order.

Howe, Stelle (1984)

There are no miracles… It is therefore very likely that all supergravity theories will diverge at three loops in four dimensions. … The final word on these issues may have to await further explicit calculations.

Marcus, Sagnotti (1985)

The idea that *all* supergravity theories diverge at 3 loops has been widely accepted wisdom for over 20 years

Reasons to Reexamine This

1) **The number of *established* counterterms for *any* pure supergravity theory is zero.**

2) **Discovery of remarkable cancellations at 1 loop – the "no-triangle hypothesis".** ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr, Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager, Bjerrum-Bohr, Vanhove

3) ***Every* explicit loop calculation to date finds $N = 8$ supergravity has identical power counting as $N = 4$ super-Yang-Mills theory, which is UV finite.** Green, Schwarz and Brink; ZB, Dixon, Dunbar, Perelstein, Rozowsky; Bjerrum-Bohr, Dunbar, Ita, PerkinsRisager; ZB, Carrasco, Dixon, Johanson, Kosower, Roiban.

4) **Interesting hint from string dualities.** Chalmers; Green, Vanhove, Russo

   – Dualities restrict form of effective action. May prevent divergences from appearing in $D = 4$ supergravity, athough difficulties with decoupling of towers of massive states.

See Russo's talk

Gravity Feynman Rules

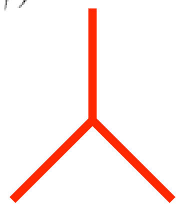
$$\mathcal{L} = \frac{2}{\kappa^2}\sqrt{g}R, \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

Propagator in de Donder gauge:

$$P_{\mu\nu;\alpha\beta}(k) = \frac{1}{2}\left[\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2}\eta_{\mu\alpha}\eta_{\nu\beta}\right]\frac{i}{k^2 + i\epsilon}$$

Three vertex:

$$\begin{aligned}
&G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1, k_2, k_3) = \\
&\quad \mathrm{sym}\Big[-\tfrac{1}{2}P_3(k_1 \cdot k_2 \eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1 \cdot k_2 \eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\
&\qquad + P_6(k_1 \cdot k_2 \eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\
&\qquad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\
&\qquad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1 \cdot k_2 \eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big]
\end{aligned}$$



About 100 terms in three vertex

An infinite number of other messy vertices.

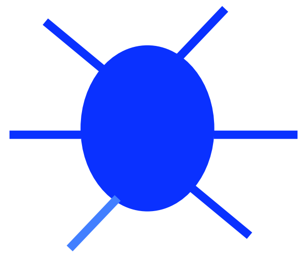
Naive conclusion: Gravity is a nasty mess.

Off-shell Formalisms

In graduate school you learned that scattering amplitudes need to be calculated using unphysical gauge dependent quantities: Off-shell Green functions

Standard machinery:

$$p^2 \neq m^2$$



- **Fadeev-Popov procedure for gauge fixing.**
- **Taylor-Slavnov Identities.**
- **BRST.**
- **Gauge fixed Feynman rules.**
- **Batalin-Fradkin-Vilkovisky quantization for gravity.**
- **Off-shell constrained superspaces.**

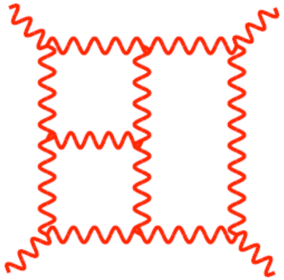
For all this machinery relatively few calculations in quantum gravity – very few checks of assertions on UV properties.

Explicit calculations from ‘t Hooft and Veltman;
Goroff and Sagnotti; van de Ven

Feynman Diagrams for Gravity

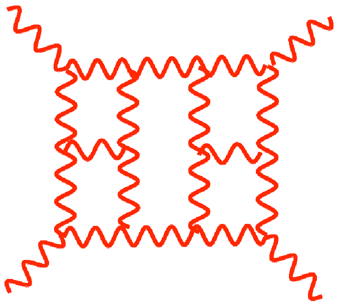
Suppose we want to put an end to the speculations by explicitly calculating to see what is true and what is false:

Suppose we wanted to check superspace claims with Feynman diagrams:



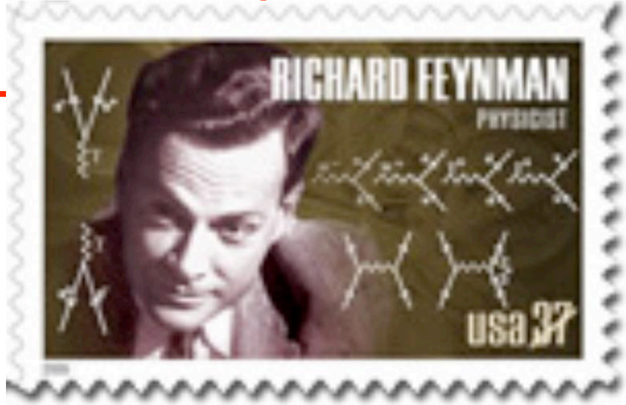
If we attack this directly get $\sim 10^{20}$ terms in diagram. There is a reason why this hasn't been evaluated using Feynman diagrams..

In 1998 we suggested that five loops is where the divergence is:

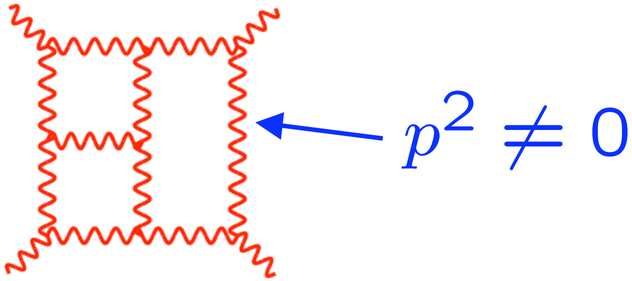


This single diagram has $\sim 10^{30}$ terms prior to evaluating any integrals. More terms than atoms in your brain!

Why are Feynman diagrams clumsy for high loop processes?



- Vertices and propagators involve gauge-dependent off-shell states. Origin of the complexity.

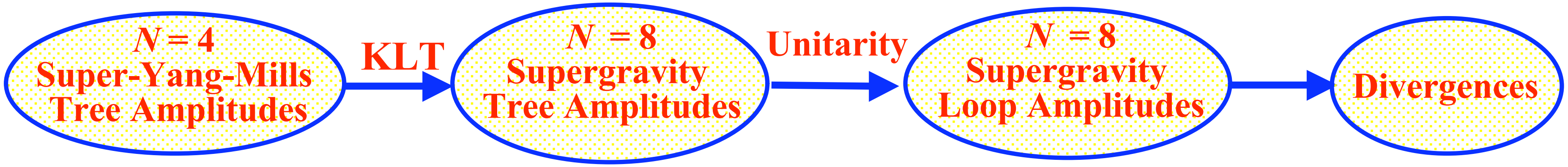


- To get at root cause of the trouble we need to do things differently.

• All steps should be in terms of gauge invariant on-shell states. $p^2 = m^2$ On shell formalism.
• Radical rewrite of quantum field theory needed.

Basic Strategy

ZB, Dixon, Dunbar, Perelstein and Rozowsky (1998)



- Kawai-Lewellen-Tye relations: sum of products of gauge theory tree amplitudes gives gravity tree amplitudes.
- Unitarity method: efficient formalism for perturbatively quantizing gauge and gravity theories. Loop amplitudes from tree amplitudes.

ZB, Dixon, Dunbar, Kosower (1994)

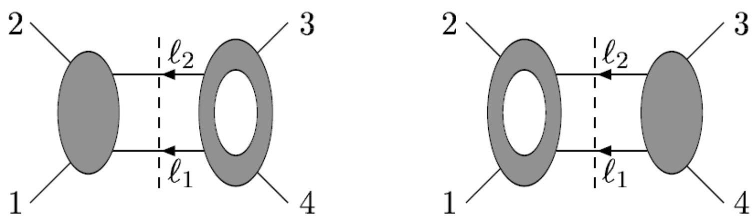
Key features of this approach:

- **Gravity calculations equivalent to two copies of much simpler gauge theory calculations.**
- **Only on-shell states appear.**

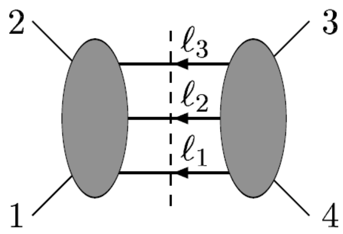
Bern, Dixon, Dunbar and Kosower

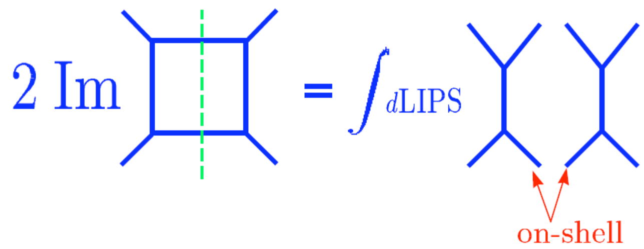
Unitarity Method

Two-particle cut:



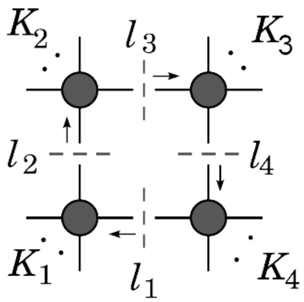
Three- particle cut:





Generalized unitarity:

Bern, Dixon and Kosower



Complex momenta very helpful.

Britto, Cachazo and Feng;
Buchbinder and Cachazo

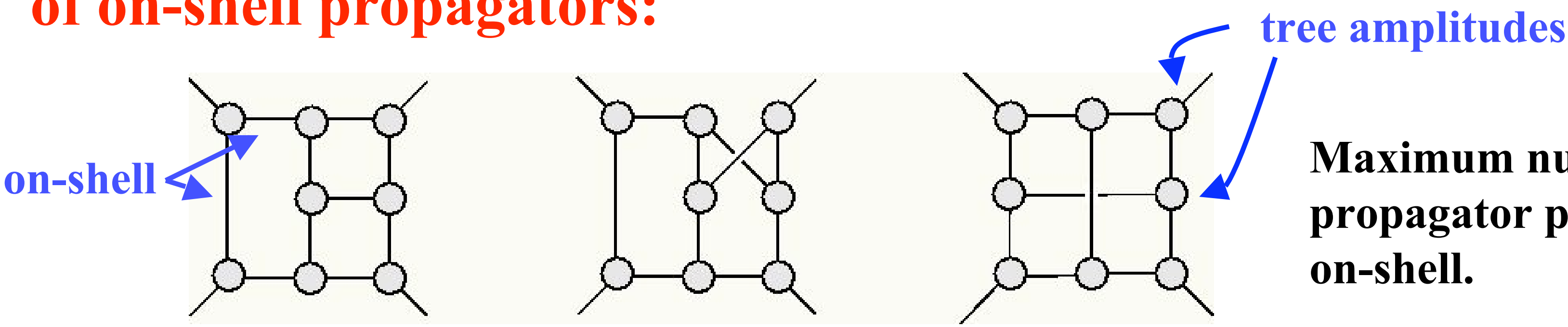
Apply decomposition of cut amplitudes in terms of product of tree amplitudes.

Method of Maximal Cuts

ZB,,Carrasco, Johansson, Kosower (2007), arXiv:0705.1864[hep-th]

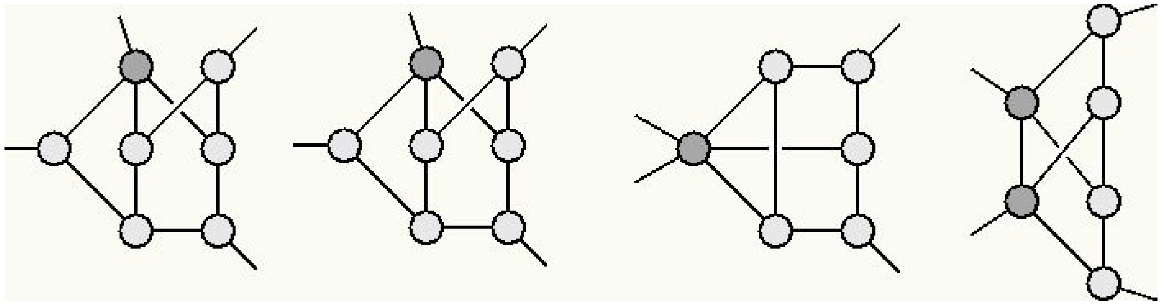
A very potent means of constructing complete higher- loop amplitudes is "Method of Maximal Cuts" .

To construct the amplitude we use cuts with maximum number of on-shell propagators:



Maximum number of propagator placed on-shell.

Then systematically release cut conditions to obtain contact terms:



Fewer propagators placed on-shell.

Related to more recent work from Cachazo and Skinner. A difference is we don't bother with hidden singularities.

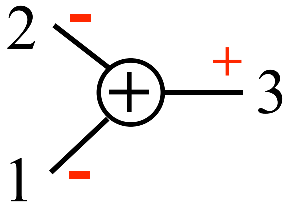
Cachazo and Skinner; Cachazo (2008)

Method of Maximal Cuts: Singlet Cuts

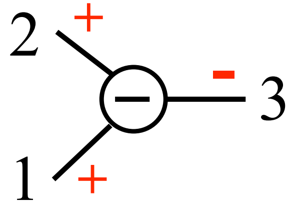
ZB,,Carrasco, Johansson, Kosower (2007)

Three vertices are special in that for given kinematics only holomorphic or anti-holomorphic

$$0 = k_3^2 = (k_1+k_2)^2 = \langle 12\rangle[21] \Rightarrow \langle 12\rangle = 0 \;\; or \;\; [21] = 0$$

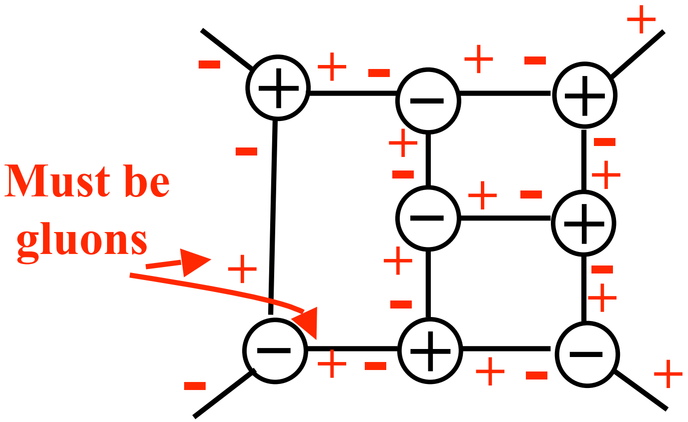


$$= i\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$



$$= -i\frac{[12]^3}{[23][31]}$$

Three vertices are either holomorphic or anti-holomorphic



Can exploit this to force only gluons into maximal cuts: "Singlet solution" for cut conditions.

Amazingly for $N = 4$ super-Yang-Mills singlets determine entire expression for the cut! Non-singlet solutions give same results.

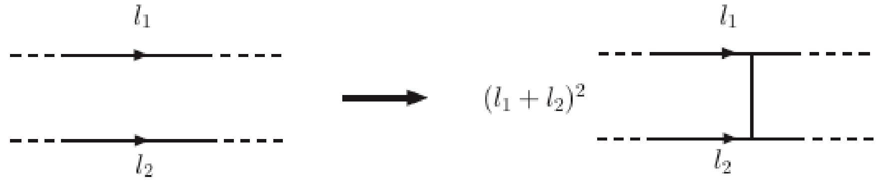
Extremely powerful to use singlet solutions.

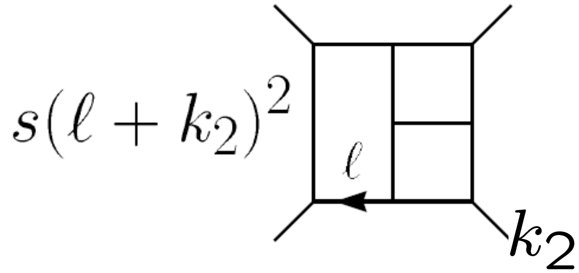
Pictorial Rules

Additional simplicity for maximal susy cases

- **Rung Rule:**

ZB, Yan Rozowsky (1997)

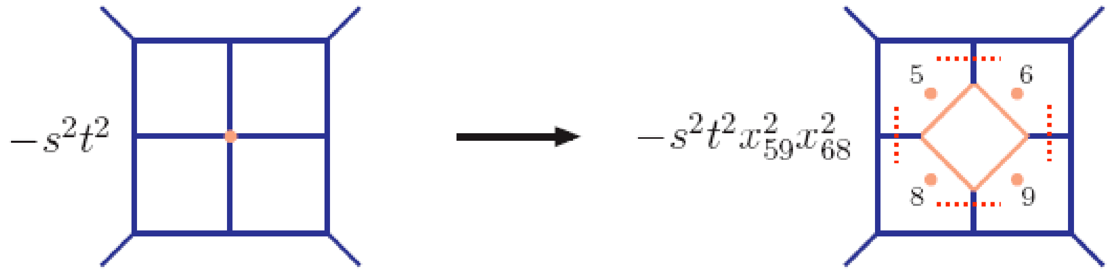




Derived from iterated 2-particle cuts

- **Box-cut substitution rule:**

ZB, Carrasco, Johansson, Kosower (2007)



***N* = 4 sYM rule**
***N* = 8 sugra similar**

Derived from generalized four-particle cuts.

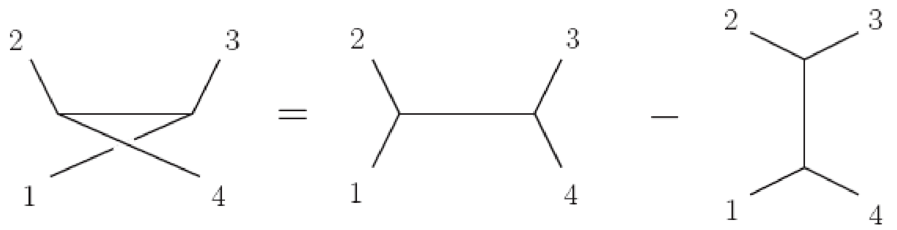
If box subdiagram present, contribution easily obtained!

Similar trickery recently also used by Cachazo and Skinner

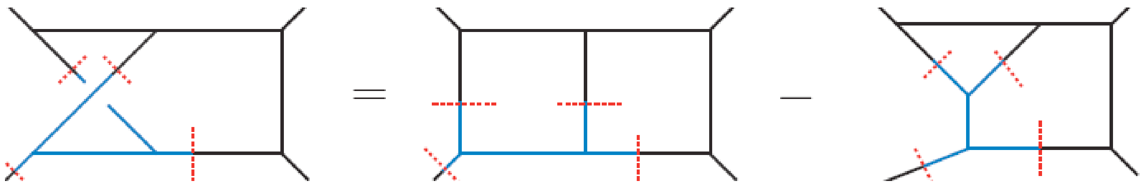
Relations Between Planar and Nonplanar

ZB, Carrasco, Johansson (2008)

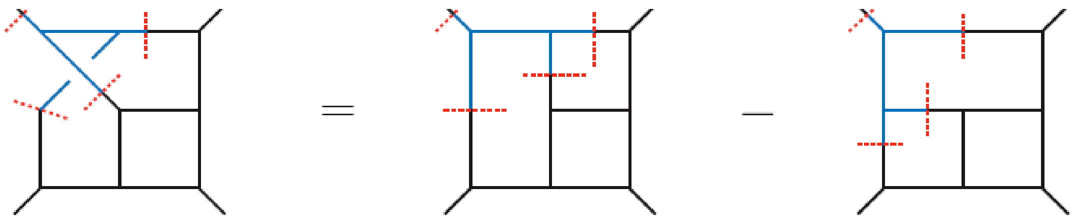
Generally, planar is simpler than non-planar. Can we obtain non-planar from planar? The answer is yes!



New tree level relations for kinematic numerators: They satisfy identities similar to color Jacobi identities.



Numerator relations



From planar results we can immediately obtain most non-planar contributions.

Applies to all gauge theories including QCD. Interlocking set of equations.

Method of Maximal Cuts: Confirmation of Results

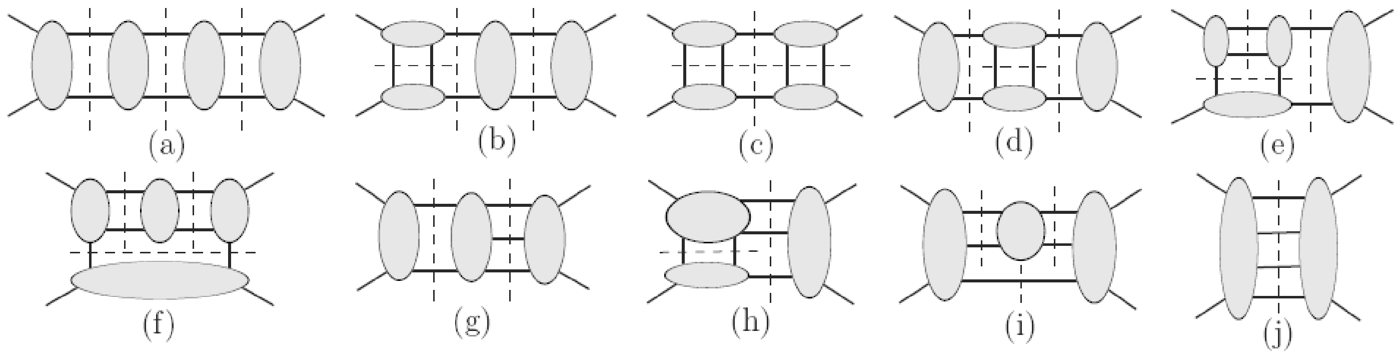
Some technicalities:

- *D*= 4 kinematics used in maximal cuts – need *D* dimensional cuts. Pieces may otherwise get dropped.
- Singlet cuts should be independently verified.

Once we have an ansatz from maximal cuts, we confirm using more standard generalized unitarity

ZB, Dixon, Kosower

At three loops, following cuts guarantee nothing is lost:



$N = 1$, $D = 10$ sYM equivalent to $N = 4$, $D = 4$

KLT Relations

A remarkable relation between gauge and gravity amplitudes exist at tree level which we will exploit.

At *tree level* Kawai, Lewellen and Tye have derived a relationship between closed and open string amplitudes.
In field theory limit, relationship is between gravity and gauge theory

$$M_4^{\rm tree}(1,2,3,4) = s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,$$

$$M_5^{\rm tree}(1,2,3,4,5) = s_{12}s_{34} A_5^{\rm tree}(1,2,3,4,5) A_5^{\rm tree}(2,1,4,3,5) + s_{13}s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5)$$

Gravity amplitude

Color stripped gauge theory amplitude

where we have stripped all coupling constants

$$\mathcal{A}_4^{\rm tree} = g^2 \sum\nolimits_{\rm non-cyclic} {\rm Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) A_4^{\rm tree}(1,2,3,4)$$

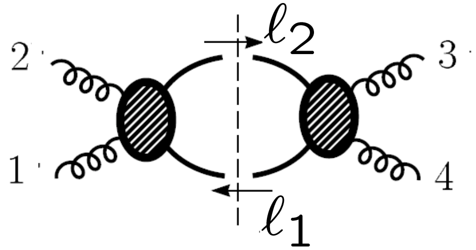
Full gauge theory amplitude

Holds for any external states.
See review: gr-qc/0206071

Progress in gauge theory can be imported into gravity theories

$N = 8$ Supergravity from $N = 4$ Super-Yang-Mills

Using unitarity and KLT we express cuts of $N = 8$ supergravity amplitudes in terms of $N = 4$ amplitudes.



$$\sum_{N=8 \text{ states}} M_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times M_4^{\text{tree}}(-\ell_2, 3, 4, \ell_1)$$

$$= s^2 \sum_{N=4 \text{ states}} \Big(A_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times A_4^{\text{tree}}(\ell_2, 3, 4, -\ell_1) \Big) \times \sum_{N=4 \text{ states}} \Big(A_4^{\text{tree}}(\ell_2, 1, 2, -\ell_1) \times A_4^{\text{tree}}(\ell_1, 3, 4, -\ell_2) \Big)$$

Key formula for $N = 4$ Yang-Mills two-particle cuts:

$$\sum_{N=4 \text{ states}} A_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times A_4^{\text{tree}}(-\ell_2, 3, 4, \ell_1) = -\frac{st\, A_4^{\text{tree}}(1,2,3,4)}{(\ell_1 - k_1)^2 (\ell_2 - k_3)^2}$$

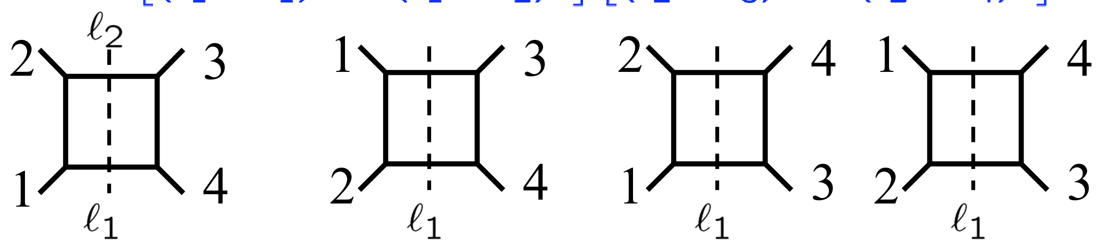
Key formula for $N = 8$ supergravity two-particle cuts:

Note recursive structure!

$$\sum_{N=8 \text{ states}} M_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times M_4^{\text{tree}}(-\ell_2, \ell_3, -\ell_4, \ell_1)$$

$$= istu\, M_4^{\text{tree}}(1,2,3,4) \left[\frac{1}{(\ell_1 - k_1)^2} + \frac{1}{(\ell_1 - k_2)^2}\right] \left[\frac{1}{(\ell_2 - k_3)^2} + \frac{1}{(\ell_2 - k_4)^2}\right]$$

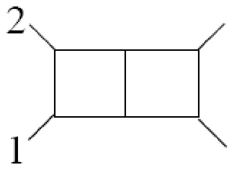
Generates all contributions with *s*-channel cuts.

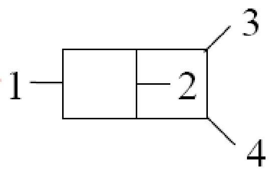


Two-Loop $N = 8$ Amplitude

Z.B., Dixon, Dunbar, Perelstein and Rozowsky

From two- and three-particle cuts we get the $N = 8$ amplitude:

$(s\,K)^2$ 

$+$ $(s\,K)^2$ 

$+$ perms

where $K = st A_4^{\text{tree}}$ ← **Yang-Mills tree**

$$(sK)^2 = stu M_4^{tree}$$

↗ **gravity tree**

First divergence is in $D = 7$

$$\mathcal{M}_4^{\text{2-loop},\ D=7-2\epsilon}|_{\text{pole}} = \frac{1}{2\epsilon\ (4\pi)^7} \frac{\pi}{3} (s^2 + t^2 + u^2)\ stu M_4^{\text{tree}}$$

Note: theory diverges at one loop in $D = 8$

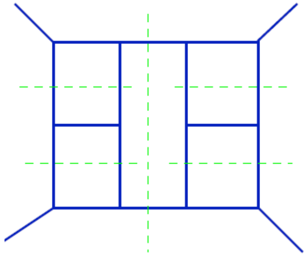
Counterterms are derivatives acting on R^4

$$t_8 t_8 R^4 \equiv t_8^{\mu_1 \mu_2 \cdots \mu_8}\, t_8^{\nu_1 \nu_2 \cdots \nu_8}\, R_{\mu_1 \mu_2 \nu_1 \nu_2}\, R_{\mu_3 \mu_4 \nu_3 \nu_4}\, R_{\mu_5 \mu_6 \nu_5 \nu_6}\, R_{\mu_7 \mu_8 \nu_7 \nu_8}$$

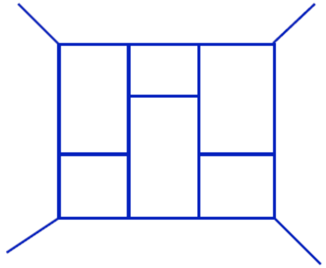
For D=5, 6 the amplitude is finite contrary to traditional superspace power counting. First indication of better behavior.

Iterated Two-Particle Cuts to All Loop Orders

ZB, Dixon, Dunbar, Perelstein, Rozowsky (1998)



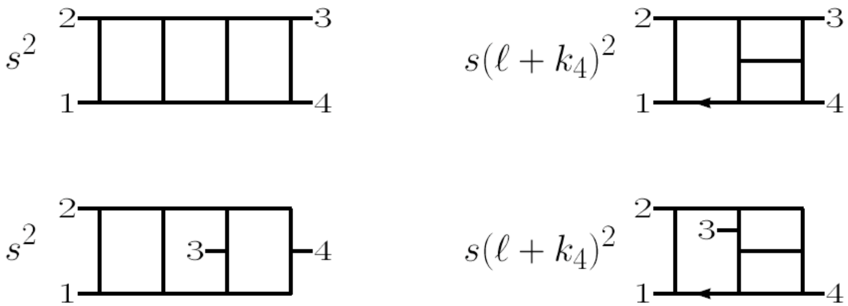
constructible from iterated 2 particle cuts

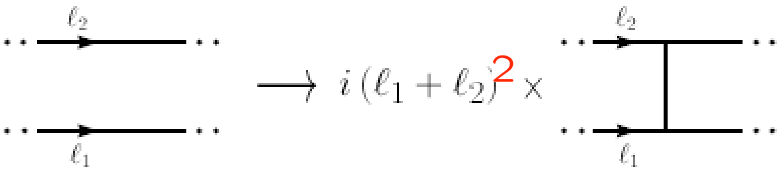


not constructible from iterated 2 particle cuts

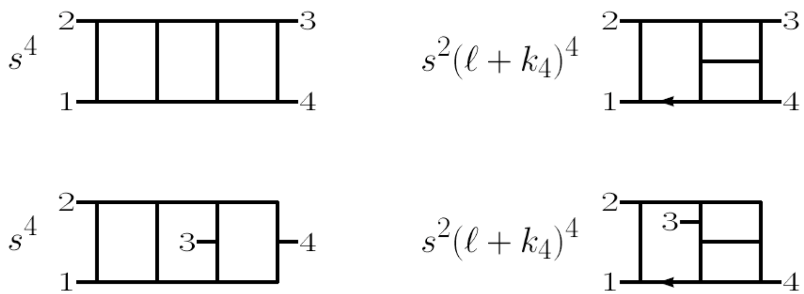
Rung rule for iterated two-particle cuts

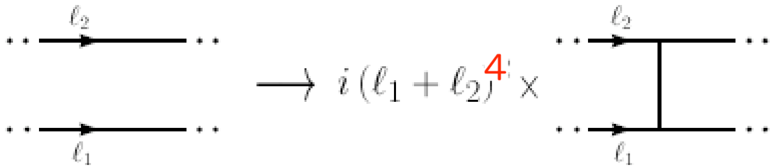
N = 4 super-Yang-Mills





N = 8 supergravity



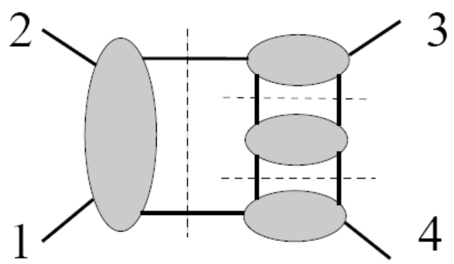


Power Counting To All Loop Orders

Z.B., Dixon, Dunbar, Perelstein and Rozowsky

From '98 paper:

- **Assumed rung-rule contributions give the generic UV behavior.**
- **Assumed no cancellations with other uncalculated terms.**



- **No evidence was found that more than 12 powers of loop momenta come out of the integrals.**
- **This is precisely the number of loop momenta extracted from the integrals at two loops.**

Elementary power counting for 12 loop momenta coming out of the integral gives finiteness condition:

$$D < \frac{10}{L} + 2 \qquad (L > 1)$$

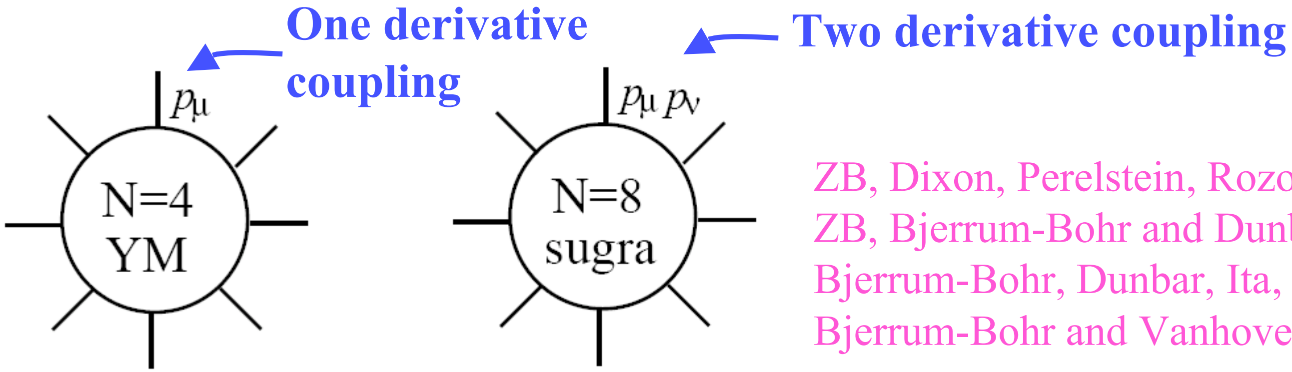
In $D = 4$ finite for $L < 5$.
L is number of loops.

$D^4 R^4$ **counterterm expected in $D = 4$, for $L = 5$**

Cancellations at One Loop

Key hint of additional cancellation comes from one loop.

Surprising cancellations not explained by *any* known susy mechanism are found beyond four points



ZB, Dixon, Perelstein, Rozowsky (1998);
ZB, Bjerrum-Bohr and Dunbar (2006);
Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager (2006)
Bjerrum-Bohr and Vanhove (2008)

Two derivative coupling means $N = 8$ supergravity should have a worse diagram-by-diagram power counting relative to $N = 4$ super-Yang-Mills theory.

However, this is not really how it works!

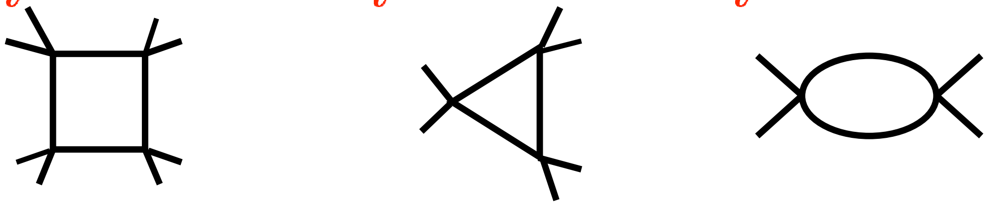
No-Triangle Hypothesis

See Arkani-Hamed's talk

ZB, Bjerrum-Bohr and Dunbar (2006)
Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager (2006)

One-loop $D = 4$ theorem: Any one loop amplitude is a linear combination of scalar box, triangle and bubble integrals with rational coefficients:

$$A_n^{\text{1-loop}} = \sum_i d_i I_4^{(i)} + \sum_i c_i I_3^{(i)} + \sum_i b_i I_2^{(i)}$$



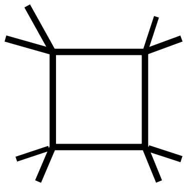
$$\int \frac{d^4 p}{(p^2)^4} \qquad \int \frac{d^4 p}{(p^2)^3} \qquad \int \frac{d^4 p}{(p^2)^2}$$

- **In $N = 4$ Yang-Mills *only box* integrals appear. No triangle integrals and no bubble integrals.**
- **The "no-triangle hypothesis" is the statement that same holds in $N = 8$ supergravity. Recent proof for external gravitons by Bjerrum-Bohr and Vanhove.**

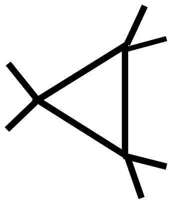
No-Triangle Hypothesis Comments

- NTH *not* a statement of improved UV behavior.
  - — Can have excellent UV properties, yet violate NTH.
  - — NTH can be satisfied, yet have bad UV scaling at higher loops.
- Really just a technical statement on the type of analytic functions that can appear at one loop.
- Used only to demonstrate cancellations beyond those of 1998 paper, otherwise wrong analytic structure.

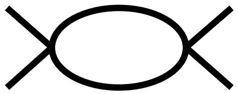
ZB, Dixon, Roiban



$$\int \frac{d^4 p}{(p^2)^4}$$



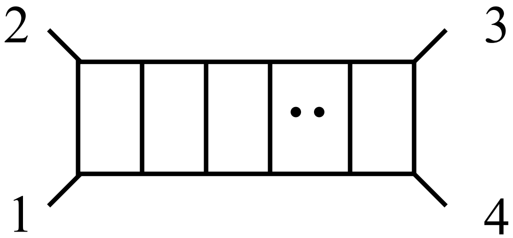
$$\int \frac{d^4 p}{(p^2)^3}$$



$$\int \frac{d^4 p}{(p^2)^2}$$

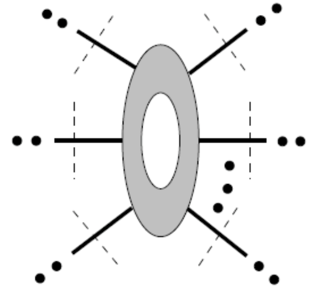
L-Loop Observation

ZB, Dixon, Roiban



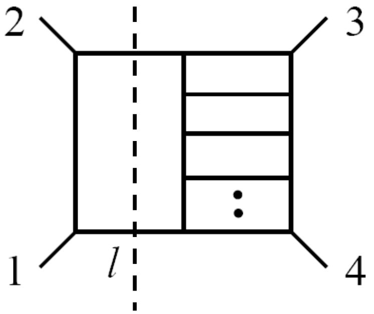
$$[(k_1 + k_2)^2]^{2(L-2)}$$

numerator factor



From 2 particle cut:

1 in $\mathcal{N} = 4$ YM

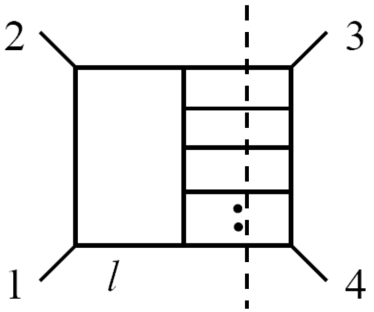


$$[(l + k_4)^2]^{2(L-2)}$$

numerator factor

Using generalized unitarity and no-triangle hypothesis *all* one-loop subamplitudes should have power counting of $\mathcal{N} = 4$ Yang-Mills

From L-particle cut:

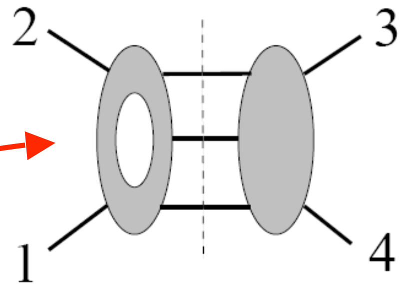


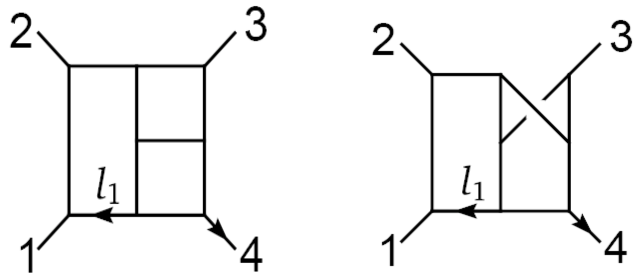
Above numerator violates no-triangle hypothesis. Too many powers of loop momentum.

There must be additional cancellation with other contributions!

$\mathcal{N} = 8$ All Orders Cancellations

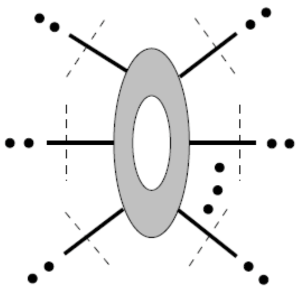
5-point 1-loop known explicitly





$$[(l_1 + k_4)^2]^2$$

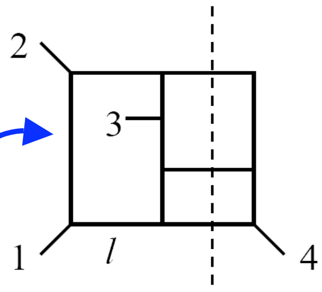
must have cancellations between planar and non-planar



Using generalized unitarity and no-triangle hypothesis *any* one-loop subamplitude should have power counting of $\mathcal{N} = 4$ Yang-Mills

But contributions with bad overall power counting yet no violation of no-triangle hypothesis might be possible.

One-loop hexagon OK



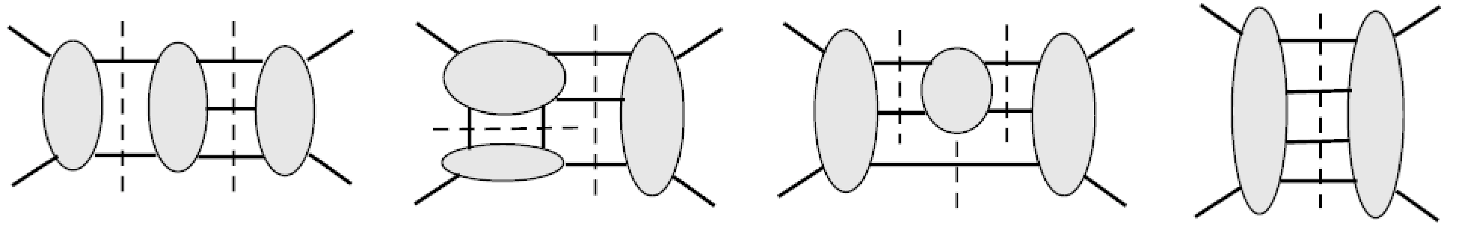
$$(l + k_4)^4$$

Total contribution is worse than for $\mathcal{N} = 4$ Yang-Mills.

Full Three-Loop Calculation

ZB, Carrasco, Dixon,
Johansson, Kosower, Roiban

Besides iterated two-particle cuts need following cuts:



reduces everything to product of tree amplitudes

For first cut have:

$$\sum_{N=8 \text{ states}} M_4^{\text{tree}}(1,2,l_3,l_1) \times M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) \times M_5^{\text{tree}}(3,4,-q_1,-q_2,-q_3)$$

Use KLT

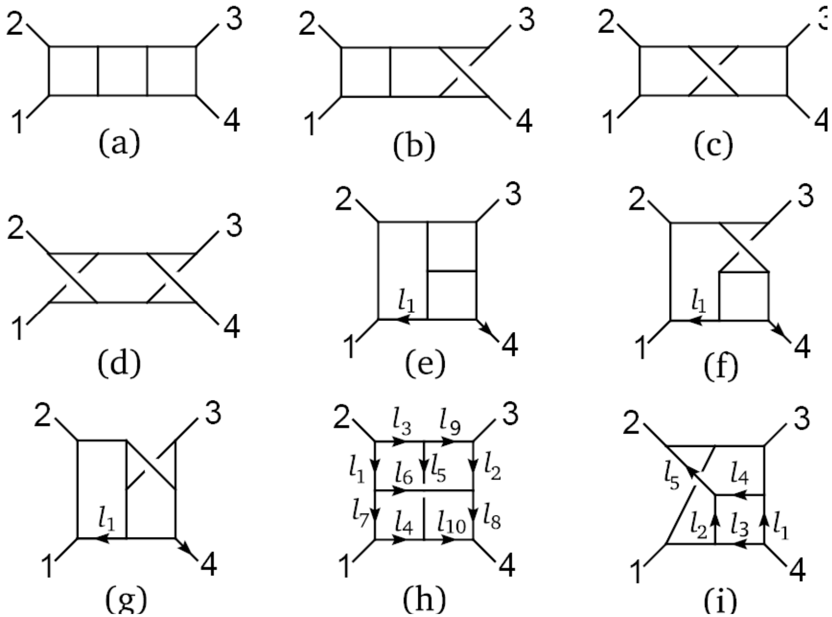
$$M_4^{\text{tree}}(1,2,l_3,l_1) = -is_{12} A_4^{\text{tree}}(1,2,l_3,l_1) A_4^{\text{tree}}(2,1,l_3,l_1)$$

$$M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) = i\, s_{l_1 q_1} s_{l_3 q_3} A_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1)\, A_5^{\text{tree}}(-l_1,q_1,q_3,-l_3,q_2) + \{l_1 \leftrightarrow l_3\}\,,$$

supergravity ↑ super-Yang-Mills ↑

$N = 8$ supergravity cuts are sums of products of $N = 4$ super-Yang-Mills cuts

Complete three loop result



ZB, Carrasco, Dixon, Johansson, Kosower, Roiban; hep-th/0702112

All obtainable from rung rule, except (h), (i) which are new.

$$l_{i,j}^2 = (l_i + l_j)^2$$

Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills	$\mathcal{N}=8$ Supergravity
(a)–(d)	s^2	$[s^2]^2$
(e)–(g)	$s(l_1+k_4)^2$	$[s\,(l_1+k_4)^2]^2$
(h)	$sl_{1,2}^2 + tl_{3,4}^2 - sl_5^2 - tl_6^2 - st$	$(sl_{1,2}^2 + tl_{3,4}^2 - st)^2 - s^2(2(l_{1,2}^2 - t) + l_5^2)l_5^2 - t^2(2(l_{3,4}^2 - s) + l_6^2)l_6^2$ $- s^2(2l_1^2l_8^2 + 2l_2^2l_7^2 + l_1^2l_7^2 + l_2^2l_8^2) - t^2(2l_3^2l_{10}^2 + 2l_4^2l_9^2 + l_3^2l_9^2 + l_4^2l_{10}^2) + 2stl_5^2l_6^2$
(i)	$sl_{1,2}^2 - tl_{3,4}^2 - \frac{1}{3}(s-t)l_5^2$	$(sl_{1,2}^2 - tl_{3,4}^2)^2 - (s^2l_{1,2}^2 + t^2l_{3,4}^2 + \frac{1}{3}stu)l_5^2$

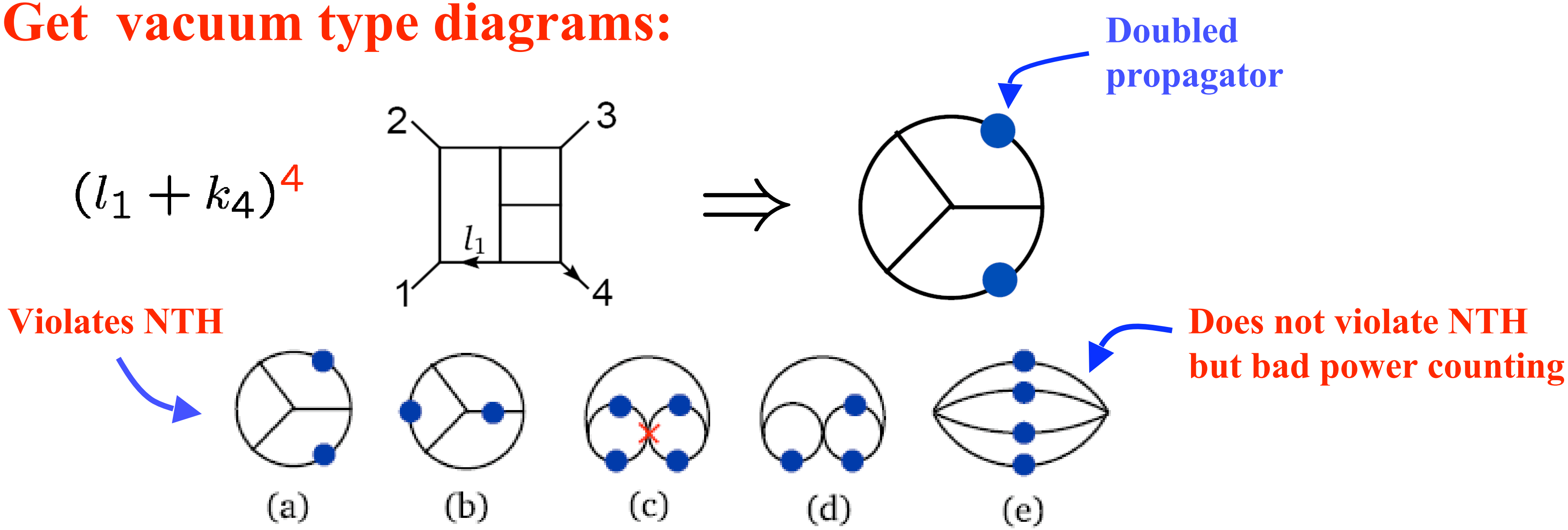
$$M_4^{(3)} = \left(\frac{\kappa}{2}\right)^8 stu M_4^{\rm tree} \sum_{S_3} \Big[I^{(a)} + I^{(b)} + \tfrac{1}{2} I^{(c)} + \tfrac{1}{4} I^{(d)} + 2I^{(e)} + 2I^{(f)} + 4I^{(g)} + \tfrac{1}{2} I^{(h)} + 2I^{(i)} \Big]$$

Cancellation of Leading Behavior

To check leading UV behavior we can expand in external momenta keeping only leading term.

Get vacuum type diagrams:

$$(l_1 + k_4)^4$$



After combining contributions:

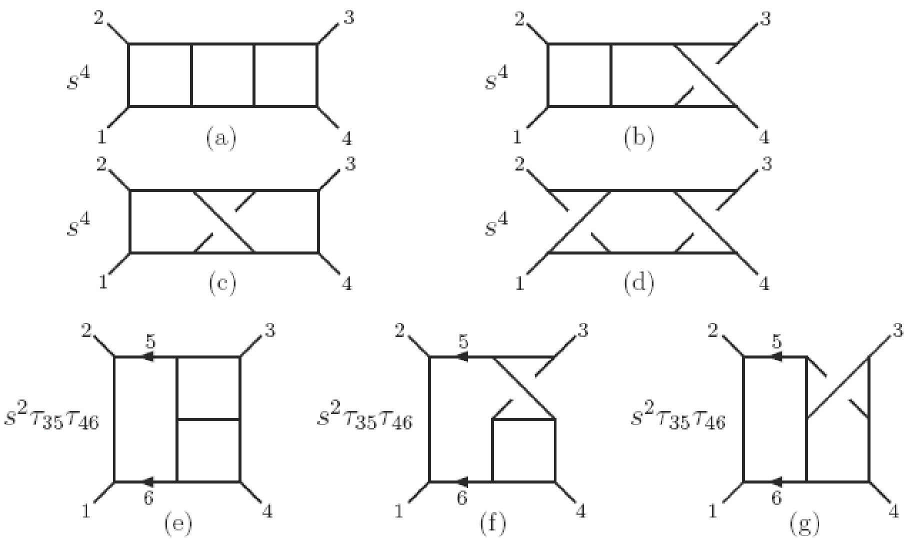
The leading UV behavior cancels!!

Manifest UV Behavior

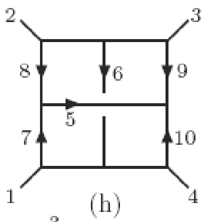
ZB, Carrasco, Dixon, Johansson, Roiban (to appear)

Using maximal cuts method we obtained a better integral representation of amplitude:

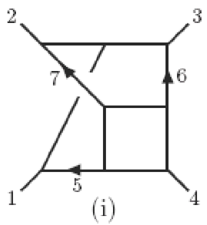
$$\tau_{ij} = 2k_i \cdot k_j$$



$$(s(\tau_{26}+\tau_{36})+t(\tau_{15}+\tau_{25})+st)^2$$
$$+(s^2(\tau_{26}+\tau_{36})-t^2(\tau_{15}+\tau_{25}))(\tau_{17}+\tau_{28}+\tau_{39}+\tau_{4,10})$$
$$+s^2(\tau_{17}\tau_{28}+\tau_{39}\tau_{4,10})+t^2(\tau_{28}\tau_{39}+\tau_{17}\tau_{4,10})$$
$$+u^2(\tau_{17}\tau_{39}+\tau_{28}\tau_{4,10})$$



$$(s\tau_{45}-t\tau_{46})^2-\tau_{27}(s^2\tau_{45}+t^2\tau_{46})$$
$$-\tau_{15}(s^2\tau_{47}+u^2\tau_{46})-\tau_{36}(t^2\tau_{47}+u^2\tau_{45})$$
$$+l_5^2 s^2 t + l_6^2 s t^2 - \tfrac{1}{3} l_7^2 stu$$



$\mathcal{N} = 8$ supergravity manifestly has same power counting as $\mathcal{N} = 4$ super-Yang-Mills!

By integrating this we have demonstrated $D = 6$ divergence.

Finiteness Conditions

Through $L = 3$ loops the correct finiteness condition is ($L > 1$):

"superfinite" in $D = 4$

$$D < \frac{6}{L} + 4$$

same as $N = 4$ super-Yang-Mills bound is saturated

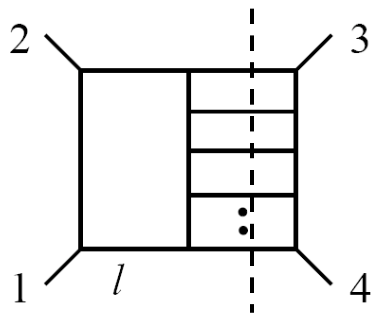
***not* the weaker result from iterated two-particle cuts:**

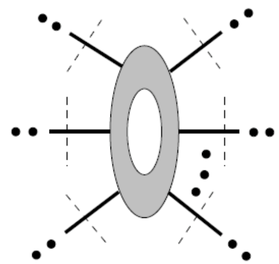
finite in $D = 4$ for $L = 3,4$

$$D < \frac{10}{L} + 2$$

(old prediction)

Beyond $L = 3$, as already explained, from special cuts we have strong evidence that the cancellations continue.





All one-loop subdiagrams should have same UV power-counting as $N = 4$ super-Yang-Mills theory.

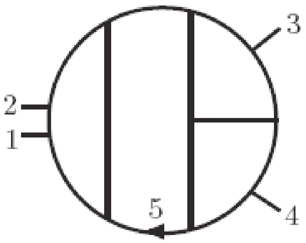
No known susy argument explains these cancellations

$\mathcal{N}$=8 Four-Loop Calculation in Progress

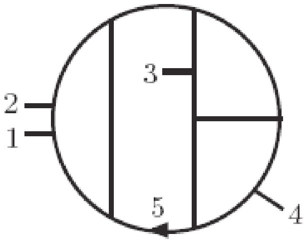
ZB, Carrasco, Dixon, Johansson, Roiban

Some $\mathcal{N}$ =4 YM contributions:

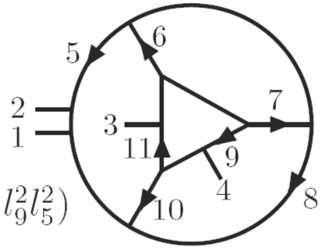
$s_{12}^2 s_{45}$



$s_{12}^2 s_{45}$



$$s_{12}^2 s_{98} - s_{12} s_{35} s_{67} + \tfrac{1}{3} l_9^2 s_{12}(s_{35} - s_{12}) + s_{12}(l_5^2 s_{4,10} - l_5^2 l_{11}^2 - \tfrac{1}{3} l_9^2 l_5^2)$$



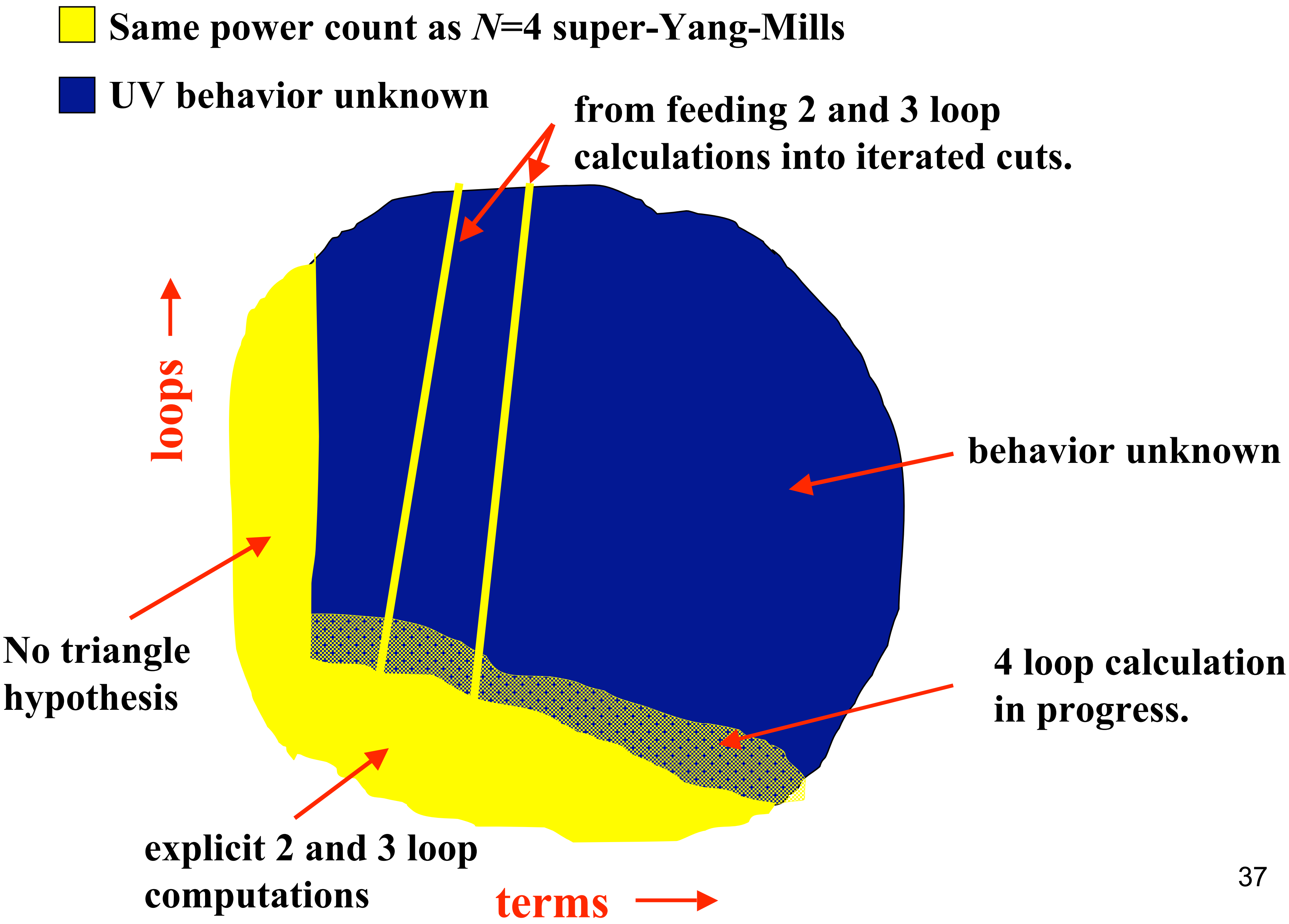
50 distinct planar and non-planar diagrammatic topologies

$\mathcal{N}$ = 4 super-Yang-Mills case is complete.
$\mathcal{N}$ = 8 supergravity still in progress.

Four-loop will teach us a lot: **See Stelle's talk**

1. **Direct challenge to potential superspace explanations.**
2. **Study of cancellations will lead to better understanding.**
3. **Need 16 not 14 powers of loop momenta to come out of integrals to get power counting of $\mathcal{N}$ = 4 sYM**

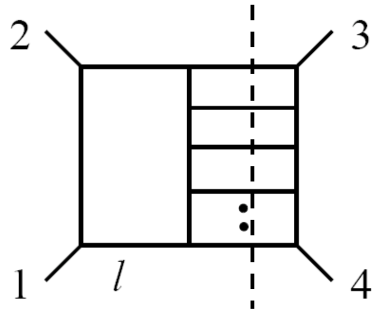
Schematic Illustration of Status



Origin of Cancellations?

There does not appear to be a supersymmetry explanation for observed cancellations, especially as the loop order increases.

If it is *not* supersymmetry what might it be?



Tree Cancellations in Pure Gravity

Unitarity method implies all loop cancellations come from tree amplitudes. Can we find tree cancellations?

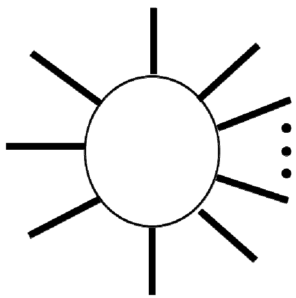
You don't need to look far: proof of BCFW tree-level on-shell recursion relations in gravity relies on the existence such cancellations!

Susy not required

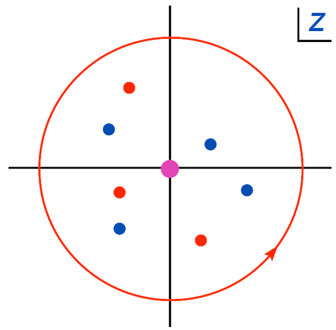
Britto, Cachazo, Feng and Witten;
Bedford, Brandhuber, Spence and Travaglini
Cachazo and Svrcek; Benincasa, Boucher-Veronneau and Cachazo
ZB, Carrasco, Forde, Ita, Johansson; Arkani-Hamed and Kaplan

Consider the shifted tree amplitude:

$$k_1^\mu \to k_1^\mu + \frac{z}{2}\langle k_1^-|\gamma^\mu|k_2^-\rangle, \qquad k_2^\mu \to k_2^\mu - \frac{z}{2}\langle k_1^-|\gamma^\mu|k_2^-\rangle,$$



How does $M(z)$ behave as $z \to \infty$?

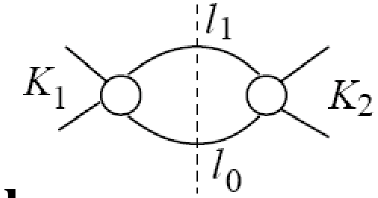


Proof of BCFW recursion requires $M(z) \to 0$

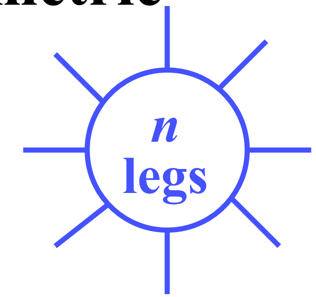
Loop Cancellations in Pure Gravity

ZB, Carrasco, Forde, Ita, Johansson

Powerful new one-loop integration method due to Forde makes it much easier to track the cancellations. Allows us to directly link one-loop cancellations to tree-level cancellations.



Observation: Most of the one-loop cancellations observed in $N = 8$ supergravity leading to "no-triangle hypothesis" are already present even in non-supersymmetric gravity. Susy cancellations are on top of these.



$$(l^\mu)^{2n} \rightarrow (l^\mu)^{n+4} \times (l^\mu)^{-8}$$

Maximum powers of loop momenta

Cancellation generic to Einstein gravity

Cancellation from $N = 8$ susy

Key Proposal: This continues to higher loops, so that most of the observed $N = 8$ multi-loop cancellations are *not* due to susy but in fact are generic to gravity theories! If N=8 is finite suspect also $N = 5, 6$ to be finite.

Summary

- Unitarity method gives us means to calculate at high loop orders – maximal cuts very helpful.
- Gravity ~ (gauge theory) x (gauge theory) at tree level.
- Unitarity method gives us means of exploiting KLT relations at loop level. Map gravity to gauge theory.
- $N = 8$ supergravity has cancellations with no known supersymmetry explanation.
  - No-triangle hypothesis implies cancellations strong enough for finiteness to *all* loop orders, in a limited class of terms.
  - At four points three loops, *established* that cancellations are complete and $N = 8$ supergravity has the same power counting as $N = 4$ Yang-Mills.
  - Key cancellations appear to be generic in gravity.
- Four-loop $N = 8$ – if superfiniteness holds it will directly challenge potential superspace explanation.

Summary

***N* = 8 supergravity may well be the first example of a unitary point-like perturbatively UV finite theory of gravity.**

Demonstrating this remains a challenge.

Extra transparancies

Where are the $N = 8$ Divergences?

Depends on who you ask and when you ask.

Howe and Lindstrom (1981)
Green, Schwarz and Brink (1982)
Howe and Stelle (1989)
Marcus and Sagnotti (1985)

3 loops: Conventional superspace power counting.

5 loops: Partial analysis of unitarity cuts. ZB, Dixon, Dunbar, Perelstein, and Rozowsky (1998)

If harmonic superspace with $N = 6$ susy manifest exists

Howe and Stelle (2003)

6 loops: *If* harmonic superspace with $N = 7$ susy manifest exists

Howe and Stelle (2003)

7 loops: If a superspace with $N = 8$ susy manifest were to exist.

Grisaru and Siegel (1982)

8 loops: Explicit identification of potential susy invariant counterterm with full non-linear susy. Kallosh; Howe and Lindstrom (1981)

9 loops: *Assume* Berkovits' superstring non-renormalization theorems can be naively carried over to $N = 8$ supergravity. Also need to extrapolate. Green, Vanhove, Russo (2006)

Superspace gets here with additional speculations. Stelle (2006)

Note: none of these are based on demonstrating a divergence. They are based on arguing susy protection runs out after some point.