SURPRISES IN THE $AdS_5 \times S^5$ SUPERSTRING

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$\Bigg\}$ $AdS_5 \times S^5$ at small r_{AdS}

NB + J. Maldacena, to appear

T-duality and dual superconf. in

Type IIB supergravity backgrounds with maximal supersymmetry:

1) Flat background: N=2 d=10 super-Poincaré inv.

$$\{q_\alpha, q_\beta\} = \gamma^a_{\alpha\beta} P_a \,, \quad \{\hat{q}_{\hat\alpha}, \hat{q}_{\hat\beta}\} = \gamma^a_{\hat\alpha\hat\beta} P_a \,, \quad \{q_\alpha, \hat{q}_{\hat\alpha}\} = 0 \qquad \begin{matrix} \alpha, \hat\alpha = 1 \ldots 16 \\ a = 0 \ldots 9 \end{matrix}$$

Superstring worldsheet Lagrangian is super-Poincaré invariant up to a total derivative.

Worldsheet action is quadratic using pure spinor formalism

$\Rightarrow$ Easy to covariantly quantize

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2) $AdS_5 \times S^5$ background: PSU(2,2|4) inv.

$$\{q_\alpha, q_\beta\} = \gamma^a_{\alpha\beta} P_a \,, \quad \{\hat{q}_{\hat\alpha}, \hat{q}_{\hat\beta}\} = \gamma^a_{\hat\alpha\hat\beta} P_a \,, \quad \{q_\alpha, \hat{q}_{\hat\beta}\} = \frac{1}{r} \eta_{\alpha\hat\gamma} (\gamma^{ab})^{\hat\gamma}_{\hat\beta} M_{ab}$$

r: AdS_5 radius, $\quad \eta_{\alpha\hat\gamma} = (\gamma^{01234})_{\alpha\hat\gamma}$

Superstring worldsheet Lagrangian is PSU(2,2|4) inv.

Worldsheet action is not quadratic

$\Rightarrow$ Covariant quantization is easy only when $r \to \infty$ or $r \to 0$.

3 surprises:

1) Measure factor for scattering amplitudes in $AdS_5 \times S^5$ is simpler than in flat space.

2) $AdS_5 \times S^5$ sigma model can be expressed in worldsheet $N=(2,2)$ superspace.

3) $AdS_5 \times S^5$ sigma model is invariant under combination of bosonic and fermionic T-duality. Explains "dual superconformal invariance" of $\mathcal{N}=4$ $d=4$ super-Yang-Mills amplitudes.

I. Basics of pure spinor formalism

Worldsheet variables (Type IIB) : x^m $\underbrace{\theta^\alpha, p_\alpha \quad \hat\theta^\alpha, \hat p_\alpha}_{\text{fermions}}$ $\underbrace{\lambda^\alpha, \omega_\alpha \quad \hat\lambda^\alpha, \hat\omega_\alpha}_{\text{bosons}}$

$$\boxed{\begin{array}{c}\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0 \\ \hat\lambda^\alpha \gamma^m_{\alpha\beta} \hat\lambda^\beta = 0\end{array}} \Rightarrow$$

λ^α has 11 indep. (complex) degrees of freedom

$$\lambda^\alpha \in \mathbb{C} \times \frac{SO(10)}{U(5)}$$

"pure spinor in $d=10$"

Worldsheet action in flat background :

$$S = \frac{1}{\alpha'} \int d^2z \Big[\tfrac{1}{2} \partial x^m \bar\partial x_m + p_\alpha \bar\partial \theta^\alpha + \hat p_\alpha \partial \hat\theta^\alpha + \omega_\alpha \bar\partial \lambda^\alpha + \hat\omega_\alpha \partial \hat\lambda^\alpha \Big]$$

ω_α has gauge inv. $\delta \omega_\alpha = \Lambda^m (\gamma_m \lambda)_\alpha$

$\Rightarrow$ ω_α has 11 gauge inv. components

$T = \frac{1}{2} \partial x^m \partial x_m + p_\alpha \partial \theta^\alpha + \omega_\alpha \partial \lambda^\alpha$ has zero central charge $(+10 - 32 + 22 = 0)$

Physical states are determined from cohomology of nilpotent "BRST" operator

$$Q = \int dz \, \lambda^\alpha d_\alpha \quad , \quad \bar{Q} = \int d\bar{z} \, \hat{\lambda}^\alpha \hat{d}_\alpha$$

$$d_\alpha = p_\alpha + (\gamma_m \theta)_\alpha \partial x^m + \frac{1}{8} (\gamma_m \theta)(\theta \gamma^m \partial \theta)$$

is spacetime supersymmetric and satisfies OPE's

$$d_\alpha(y) \, d_\beta(z) \to \gamma^m_{\alpha\beta} \frac{\partial x_m + \theta \gamma_m \partial \theta}{y-z} \equiv \gamma^m_{\alpha\beta} \frac{\Pi_m(z)}{y-z}$$

$$d_\alpha(y) \, V(x(z), \theta(z)) \to \frac{1}{y-z} D_\alpha V(x(z), \theta(z))$$

$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - (\gamma_m \theta)_\alpha \frac{\partial}{\partial x^m}$ is supersymmetric derivative

$$\lambda \gamma^m \lambda = 0 \Rightarrow Q^2 = 0$$

In Green-Schwarz formalism, $d_\alpha = 0$ is the fermionic Dirac constraint which has 8 first-class and 8 second-class constraints.

S_{pure} can be written in terms of GS action as

$$S_{pure} = \frac{1}{\alpha'}\int d^2z \left[\frac{1}{2}\partial x^m \bar\partial x_m + p_\alpha \bar\partial\theta^\alpha + \hat p_\alpha \partial\hat\theta^\alpha + w_\alpha \bar\partial\lambda^\alpha + \hat w_\alpha \partial\hat\lambda^\alpha\right]$$

$$= S_{GS} + \frac{1}{\alpha'}\int d^2z \left[d_\alpha \bar\partial\theta^\alpha + \hat d_\alpha \partial\hat\theta^\alpha + w_\alpha \bar\partial\lambda^\alpha + \hat w_\alpha \partial\hat\lambda^\alpha\right]$$

$$S_{GS} = \frac{1}{\alpha'}\int d^2z \left[\frac{1}{2}\Pi^m \bar\Pi_m + \epsilon^{ij}\left[\partial_i x^m (\theta\gamma_m \partial_j\theta - \hat\theta\gamma_m\partial_j\hat\theta) + (\theta\gamma^m\partial_i\theta)(\hat\theta\gamma_m\partial_j\hat\theta)\right]\right]$$

Lagrangian changes by total derivative under spacetime supersymmetry.

S_{GS} is invariant under K-symmetry but cannot be covariantly quantized in a simple manner.

S_{pure} is invariant under BRST symmetry generated by $Q = \int dz\, \lambda^\alpha d_\alpha$ and $\bar Q = \int d\bar z\, \hat\lambda^\alpha \hat d_\alpha$ and is easy to covariantly quantize.

For open superstring, massless super-YM vertex operator is $V = \lambda^\alpha A_\alpha(x,\theta)$

$$QV = 0 \Rightarrow \lambda^\alpha \lambda^\beta D_\alpha A_\beta = 0 \Rightarrow \gamma_{mnpqr}^{\alpha\beta} D_\alpha A_\beta = 0$$

$$\delta V = Q\Omega \Rightarrow \lambda^\alpha \delta A_\alpha = \lambda^\alpha D_\alpha \Omega \Rightarrow \delta A_\alpha = D_\alpha \Omega$$

$\Rightarrow A_\alpha(x,\theta)$ describes on-shell $d=10$ super-YM

$$A_\alpha(x,\theta) = a_m(x)(\gamma^m\theta)_\alpha + \psi^\beta(x)(\gamma^m\theta)_\beta(\gamma_m\theta)_\alpha + \ldots$$

where $\ldots$ involves derivatives of $a_m(x)$, $\psi^\beta(x)$

For Type IIB superstring, massless supergravity vertex operator is $V = \lambda^\alpha \hat\lambda^\beta A_{\alpha\beta}(x,\theta,\hat\theta)$

$QV = \hat{Q}V = 0 \Rightarrow A_{\alpha\beta}(x,\theta,\hat\theta)$ describes on-shell $d=10$ Type IIB sugra fields

$$A_{\alpha\beta}(x,\theta,\hat\theta) = (g_{mn}(x) + b_{mn}(x) + \eta_{mn}\varphi(x))(\gamma^m\theta)_\alpha(\gamma^n\hat\theta)_\beta$$
$$+ (\psi_m^\gamma + \gamma_m^{\gamma\delta}\chi_\delta)(\gamma^m\theta)_\alpha(\gamma^n\hat\theta)_\gamma(\gamma_n\hat\theta)_\beta + \text{complex conjugate}$$
$$+ F^{\gamma\rho}(x)(\gamma^m\theta)_\gamma(\gamma_m\theta)_\alpha(\gamma^n\hat\theta)_\rho(\gamma_n\hat\theta)_\beta + \ldots$$

Superstring tree amplitudes are computed as in bosonic string theory:

$$\mathcal{A} = \left\langle V_1(z_1)\, V_2(z_2)\, V_3(z_3) \int d^2 z_4\, U_4(z_4) \dots \int d^2 z_N\, U_N(z_N) \right\rangle$$

Integrated vertex op. satisfies $Q\bar{Q}\,U = \partial\bar{\partial} V$.

Open massless: $U = \underset{\substack{\uparrow \\ \text{gauge fields}}}{A_M(Y)}\,\partial Y^M + \underset{\substack{\uparrow \\ \text{field-strengths}}}{W^\alpha(Y)}\, d_\alpha + F^{mn}(Y)\, N_{mn}$

$Y^M = (x^m, \theta^\alpha)$ $\quad \boxed{N_{mn} = \frac{1}{2} w \gamma_{mn} \lambda}$

Closed massless $=$ | Open massless $|^2$

$$U = (G_{MN} + B_{MN})\,\partial Y^M \bar{\partial} Y^N + \underset{\substack{\uparrow \\ \text{R-R field strength}}}{F^{\alpha\hat{\beta}}}\, d_\alpha \hat{d}_{\hat{\beta}} + \underset{\substack{\uparrow \\ \text{curvature}}}{R^{mnpq}}\, N_{mn} \hat{N}_{pq} + \dots$$

Zero mode measure factor:

$(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$ is unique state in cohom. of Q at ghost-number $+3$ (analog of $c\,\partial c\,\partial^2 c$)

$\Rightarrow \langle (\lambda^3\theta^5)(\hat{\lambda}^3\hat{\theta}^5) \rangle = 1$ is measure factor

$\delta_{susy}\, (\lambda^3\theta^5)(\hat{\lambda}^3\hat{\theta}^5)$ is BRST-trivial

$\Rightarrow$ scattering amplitude is spacetime supersymmetric

II. $AdS_5 \times S^5$ Sigma Model

Pure spinor version of $AdS_5 \times S^5$ sigma model is BRST-invariant generalization of GS action

$$S_{GS} = r^2 \int d^2z \, \left(G_{MN}(Y) + B_{MN}(Y) \right) \partial Y^M \bar\partial Y^N$$

r: AdS_5 radius, $Y^M = (x^m, \theta^\alpha, \hat\theta^{\hat\alpha})$

Convenient to use Metsaev-Tseytlin coset

$$g \in \frac{PSU(2,2|4)}{SO(4,1)\times SO(5)} \qquad = \underset{AdS_5}{\frac{SO(4,2)}{SO(4,1)}} + \underset{S^5}{\frac{SO(6)}{SO(5)}} + 32 \text{ fermions}$$

$g \simeq g\Omega$ for $\Omega \in SO(4,1)\times SO(5)$ local

Under global $\Sigma \in PSU(2,2|4)$, $g \to \Sigma g$

Define left-invariant currents

$$J^I = (g^{-1}\partial g)^I \,, \qquad \bar J^I = (g^{-1}\bar\partial g)^I$$

$$I = (m, [mn], \tilde m, [\tilde m \tilde n], \alpha, \hat\alpha)$$

$m = 0$ to 4, $\tilde m = 5$ to 9, $\alpha = 1$ to 16, $\hat\alpha = 1$ to 16

J^I's are <u>invariant</u> under global $PSU(2,2|4)$ and action can be expressed in terms of J^I's.

$$S_{GS} = r^2 \int d^2z \left[J^m \bar{J}_m + J^{\tilde{m}} \bar{J}_{\tilde{m}} + \frac{1}{4} \eta_{\alpha\hat{\beta}} \left(J^{\hat{\beta}} \bar{J}^{\alpha} - \bar{J}^{\hat{\beta}} J^{\alpha} \right) \right]$$

$$\eta_{\alpha\hat{\beta}} = (\gamma^{01234})_{\alpha\hat{\beta}}$$

S_{GS} is invariant under K-symmetry but cannot be covariantly quantized.

S_{pure} adds d_α, λ^α variables such that S_{GS} is BRST-inv:

$$S_{pure} = S_{GS} + r^2 \int d^2z \left[d_\alpha \bar{J}^{\alpha} + \hat{d}_{\hat{\alpha}} J^{\hat{\alpha}} + \eta^{\alpha\hat{\beta}} d_\alpha \hat{d}_{\hat{\beta}} + w_\alpha \bar{\nabla} \lambda^\alpha + \hat{w}_{\hat{\alpha}} \nabla \hat{\lambda}^{\hat{\alpha}} + N^{mn} \bar{N}_{mn} - N^{\tilde{m}\tilde{n}} \bar{N}_{\tilde{m}\tilde{n}} \right]$$

$$\bar{\nabla} \lambda^\alpha = \bar{\partial} \lambda^\alpha + \bar{J}^{[mn]} (\gamma_{mn} \lambda)^\alpha + \bar{J}^{[\tilde{m}\tilde{n}]} (\gamma_{\tilde{m}\tilde{n}} \lambda)^\alpha$$

$$S^{AdS}_{pure} = S^{flat}_{pure} + \int d^2z \, U_{R\text{-}R\ flux} + \text{back-reaction}$$

Since d_α and $\hat{d}_{\hat{\alpha}}$ are auxiliary, can integrate them out to obtain

$$S_{pure} = r^2 \int d^2z \left[J^m \bar{J}_m + J^{\tilde{m}} \bar{J}_{\tilde{m}} - \frac{1}{4} \eta_{\alpha\hat{\beta}} \left(\bar{J}^{\hat{\beta}} J^{\alpha} + 3 J^{\hat{\beta}} \bar{J}^{\alpha} \right) + w_\alpha \bar{\nabla} \lambda^\alpha + \hat{w}_{\hat{\alpha}} \nabla \hat{\lambda}^{\hat{\alpha}} + N^{mn} \bar{N}_{mn} - N^{\tilde{m}\tilde{n}} \bar{N}_{\tilde{m}\tilde{n}} \right]$$

S_{pure} is invariant under BRST transformations generated by $Q = \int dz\, \lambda^\alpha d_\alpha = \int dz\, \eta_{\alpha\hat\beta} \lambda^\alpha J^{\hat\beta}$

and $\bar Q = \int d\bar z\, \hat\lambda^{\hat\alpha} \hat d_{\hat\alpha} = \int d\bar z\, \eta_{\alpha\hat\beta} \hat\lambda^{\hat\beta} \bar J^\alpha$.

If $T^I = [T^m, T^{(mn)}, T^{\tilde m}, T^{(\tilde m\tilde n)}, T^\alpha, T^{\hat\alpha}]$ are the $PSU(2,2|4)$ Lie-algebra generators, BRST transformations act on g as

$$Q g = g\,(\eta_{\alpha\hat\beta} \lambda^\alpha T^{\hat\beta})$$

$$\bar Q g = g\,(\eta_{\alpha\hat\beta} \hat\lambda^{\hat\beta} T^\alpha).$$

Invariance of S_{pure} under $\delta g = g\Omega$ where $\Omega \in SO(4,1)\times SO(5)$ or $\Omega = \eta_{\alpha\hat\beta}(\lambda^\alpha T^{\hat\beta} + \hat\lambda^{\hat\beta} T^\alpha)$ is very restrictive.

Can prove to all orders in $\frac{1}{r}$ that action is conformally invariant and that there exist quantum non-local conserved charges.

Can perform perturbative computations in $\frac{1}{r}$ with manifest $PSU(2,2|4)$ invariance. (Mikhailov + Schäfer-Nameki)

Physical states in $AdS_5 \times S^5$ background are in $Q+\bar{Q}$ cohomology at ghost-number $(1,1)$.

Supergravity states: $V = \lambda^\alpha \hat{\lambda}^{\hat{\beta}} A_{\alpha\hat{\beta}}(x,\theta,\hat{\theta})$

$$QV = \bar{Q}V = 0 \Rightarrow \gamma_{mnpqr}^{\alpha\gamma} \nabla_\alpha A_{\gamma\hat{\beta}} = \gamma_{mnpqr}^{\hat{\alpha}\hat{\beta}} \nabla_{\hat{\alpha}} A_{\gamma\hat{\beta}} = 0$$

$\nabla_\alpha = E_\alpha^M \nabla_M$ is covariant spinor derivative in $AdS_5 \times S^5$

Examples: $\int d^2z \, \mathcal{L}_{pure}$ is integrated vertex op. for dilaton at zero momentum

$$\Rightarrow Q\bar{Q}\mathcal{L}_{pure} = \partial\bar{\partial} V_{dilaton}$$

In flat background, $V_{dilaton} = (\lambda\gamma^m\theta)(\hat{\lambda}\gamma_m\hat{\theta})$

In $AdS_5 \times S^5$ background, $V_{dilaton} = \eta_{\alpha\hat{\beta}} \lambda^\alpha \hat{\lambda}^{\hat{\beta}}$.

Other supergravity states obtained from $V_{dilaton}$ by applying supersymmetry transformations.

E.g. State with J R-charge in 56 direction of $SO(6)$

$$V_J^{\pm} = (\eta_{\alpha\hat{\beta}} \lambda^\alpha \hat{\lambda}^{\hat{\beta}}) \, x_5^{\pm J} \left(e^{i\tilde{x}_{56}}\right)^J + \ldots$$

where $\ldots$ depends on $(\theta, \hat{\theta})$ and is determined by BRST invariance.

To compute tree amplitudes, need zero-mode measure factor which is $PSU(2,2|4)$-invariant.

In flat background, $\langle (\lambda^3\theta^5)(\hat\lambda^3\hat\theta^5)\rangle = 1$.

Sugra vertex op. is $V = \lambda\theta\,\hat\lambda\hat\theta\,(g_{mn} + \theta\theta\partial_p g_{mn} + \ldots)$

$\Rightarrow$ 3-point tree amplitude is

$$\langle V_1 V_2 V_3\rangle = \delta^{10}(k_1+k_2+k_3)\,(g_1 k_2 g_2 k_3 g_3 + \ldots)$$

In $AdS_5\times S^5$ background, zero-mode measure factor is $\langle (\eta_{\alpha\hat\beta}\lambda^\alpha\hat\lambda^{\hat\beta})^3\rangle = 1$.

Vertex op. $V_J^{\pm} = (\eta_{\alpha\hat\beta}\lambda^\alpha\hat\lambda^{\hat\beta})\, x_5^{\pm J}\,(e^{i\tilde{x}_{56}})^J + \ldots$

$\Rightarrow$ 3-point tree amplitude is

$$\langle V^+_{J_1} V^+_{J_2} V^+_{J_3}\rangle = \delta(J_1+J_2+J_3).$$

For $\frac{1}{2}$-BPS states, pure spinor ghosts appear to play a trivial role.
Suggests there should be an alternative description of $AdS_5\times S^5$ superstring.

Pure spinor formalism closely resembles an $N=2$ topological string where BRST current and B ghost are fermionic twisted $N=2$ generators.

In flat background, complicated B ghost implies $N=2$ worldsheet susy is very non-linear.

But in $AdS_5 \times S^5$ background, $N=2$ worldsheet susy can act <u>linearly</u>.

$$g(x,\theta,\hat\theta) \in \frac{PSU(2,2|4)}{SO(4,1)\times SO(5)} \quad \text{implies}$$

$$g(x,\theta,\hat\theta) = G(\theta,\hat\theta)\, H(x)\, \tilde H(\tilde x)$$

$$G(\theta,\hat\theta) = e^{\theta^\alpha T_\alpha + \hat\theta^{\hat\alpha} T_{\hat\alpha}} \in \frac{PSU(2,2|4)}{SU(2,2)\times SU(4)} = \begin{pmatrix} 1 & \theta_J^A \\ \bar\theta_A^J & 1 \end{pmatrix}$$

$$H(x) = e^{x^m T_m} \in \frac{SU(2,2)}{SO(4,1)} = H^A_{A'}(x)$$

$$\tilde H(\tilde x) = e^{\tilde x^{\tilde m} T_{\tilde m}} \in \frac{SU(4)}{SO(5)} = \tilde H^J_{J'}(\tilde x)$$

$A=1$ to 4 is $SU(2,2)$ index, $J=1$ to 4 is $SU(4)$ index

$A'=1$ to 4 is $SO(4,1)$ spinor index, $J'=1$ to 4 is $SO(5)$ spinor index

Under $N=(2,2)$ worldsheet susy generated by $(g^\pm, \bar{g}^\pm)$,

$$g^+\theta^A_J = Z^A_J\,,\quad g^-\theta^A_J = 0\,,\quad \bar{g}^+\theta^A_J = \bar{Y}^A_J\,,\quad \bar{g}^-\theta^A_J = 0$$

$$g^+\bar{\theta}^J_A = 0\,,\quad g^-\bar{\theta}^J_A = Y^J_A\,,\quad \bar{g}^+\bar{\theta}^J_A = 0\,,\quad \bar{g}^-\bar{\theta}^J_A = \bar{Z}^J_A$$

where $Z^A_J = H^A_{A'}(x)\,\tilde{H}^{-1}{}^{J'}_{J}(\tilde{x})\,\lambda^{A'}_{J'}$

$$\bar{Z}^J_A = H^{-1}{}^{A'}_{A}(x)\,\tilde{H}^J_{J'}(\tilde{x})\,\bar{\lambda}^{J'}_{A'}$$

$$Y^J_A = H^{-1}{}^{A'}_{A}(x)\,\tilde{H}^J_{J'}(\tilde{x})\,\omega^{J'}_{A'}$$

$$\bar{Y}^A_J = H^A_{A'}(x)\,\tilde{H}^{-1}{}^{J'}_{J}(\tilde{x})\,\bar{\omega}^{A'}_{J'}$$

Although $(\lambda^{A'}_{J'}, \bar{\lambda}^{J'}_{A'})$ have 22 indep. components because of constraint $\lambda\gamma^m\lambda = \lambda\gamma^{\tilde{m}}\lambda = \bar{\lambda}\gamma^m\bar{\lambda} = \bar{\lambda}\gamma^{\tilde{m}}\bar{\lambda} = 0$, $(Z^A_J, \bar{Z}^J_A)$ are (generically) unconstrained and have 32 independent components.

Map from $(x, \tilde{x}, \lambda, \bar{\lambda}) \to (Z, \bar{Z})$ can be used to write S_{pure} in terms of $(\theta, \bar{\theta}, Z, \bar{Z}, Y, \bar{Y})$ variables.

Up to a BRST-trivial term,

$$S_{pure} = r^2 \int d^2 z \, \Big[(G^{-1}\partial G)^A_J (G^{-1}\bar\partial G)^J_A$$

$$+ Y^J_A \bar\nabla Z^A_J + \bar Y^A_J \nabla \bar Z^J_A + (ZY)^A_B (\bar Y \bar Z)^B_A - (YZ)^J_K (\bar Z \bar Y)^K_J \Big]$$

$$= r^2 \int d^2 z \int d^2\kappa^+ \int d^2 \kappa^- \; \mathrm{Tr} \left[\log \left(\delta^J_K + \bar{\Theta}^J_A \, \Theta^A_K \right) \right]$$

$$\Theta^A_K (\kappa^+, \bar\kappa^+) = \theta^A_K + \kappa^+ Z^A_K + \bar\kappa^+ \bar Y^A_K + \kappa^+ \bar\kappa^+ f^A_K$$

auxiliary

$$\bar\Theta^K_A (\kappa^-, \bar\kappa^-) = \bar\theta^K_A + \kappa^- Y^K_A + \bar\kappa^- \bar Z^K_A + \kappa^- \bar\kappa^- \bar f^K_A$$

If one defines the BRST operators as

$Q' = \int dz \, g^+$ and $\bar Q' = \int d\bar z \, \bar g^-$,

S_{pure} is a topological A-model.

But map from $(x, \tilde x, \lambda, \hat\lambda) \to (z, \bar z)$ does not map the pure spinor BRST operators $Q = \int dz \, \lambda^\alpha d_\alpha$ and $\bar Q = \int d\bar z \, \hat\lambda^{\hat\alpha} \hat d_{\hat\alpha}$ into Q' and $\bar Q'$.

So original pure spinor action (valid when $r \to \infty$) is not a topological A-model.

III. Fermionic T-duality

Suppose GS (or pure spinor) action

$$S = \int d^2z \, \left(G_{MN}(Y) + B_{MN}(Y)\right) \partial Y^M \bar{\partial} Y^N + \ldots$$

is invariant under $\theta^1 \to \theta^1 + \rho$ ← constant

$Y^M = (x^m, \theta^\mu)$ "abelian supersymmetry"

If $B_{11} \neq 0$, can use Buscher procedure to replace $\partial\theta^1 \to \nabla\theta^1 = \partial\theta^1 + A$ and T-dualize by integrating out $(A, \bar{A})$.

$$S = \int d^2z \left[B_{11} \nabla\theta^1 \bar{\nabla}\theta^1 + L_{1M} \nabla\theta^1 \bar{\partial} Y^M + L_{M1} \partial Y^M \bar{\nabla}\theta^1 + L_{MN} \partial Y^M \bar{\partial} Y^N + \xi (\partial\bar{A} - \bar{\partial} A) \right]$$

$L_{MN} = G_{MN} + B_{MN}$

$$\to S = \int d^2z \left[B'_{11} \partial\xi \bar{\partial}\xi + L'_{1M} \partial\xi \bar{\partial} Y^M + L'_{M1} \partial Y^M \bar{\partial}\xi + L'_{MN} \partial Y^M \bar{\partial} Y^N \right]$$

After performing fermionic T-duality,

$$B'_{11} = -(B_{11})^{-1}, \quad L'_{1M} = -(B_{11})^{-1} L_{1M},$$

$$L'_{M1} = -(B_{11})^{-1} L_{M1}, \quad L'_{MN} = L_{MN} \pm \frac{1}{4}(B_{11})^{-1} L_{M1} L_{1N}$$

and measure factor from $(A, \bar{A})$ integration

$$\Rightarrow \quad \varphi' = \varphi + \frac{1}{2}\log B_{11} \leftarrow$$ opposite sign from bosonic T-duality

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In pure spinor action, also have couplings

$$S_{pure} = S_{GS} + \int d^2 z \left[E_M^{\alpha} d_{\alpha} \bar{\partial} Y^M + E_M^{\hat{\alpha}} \partial Y^M \hat{d}_{\hat{\alpha}} + P^{\alpha\hat{\beta}} d_{\alpha} \hat{d}_{\hat{\beta}} + \ldots \right]$$

After fermionic T-duality,

$$P'^{\alpha\hat{\beta}} = P^{\alpha\hat{\beta}} - (B_{11})^{-1} E_1^{\alpha} E_1^{\hat{\beta}}, \quad E_1'^{\alpha} = -(B_{11})^{-1} E_1^{\alpha}, \quad E_1'^{\hat{\alpha}} = -(B_{11})^{-1} E_1^{\hat{\alpha}}$$

$$E_M'^{\alpha} = E_M^{\alpha} - \frac{1}{2}(B_{11})^{-1} L_{1M} E_1^{\alpha}, \quad E_M'^{\hat{\alpha}} = E_M^{\hat{\alpha}} - \frac{1}{2}(B_{11})^{-1} L_{M1} E_1^{\hat{\alpha}}$$

Same relative sign of $E_1'^{\alpha}$ and $E_1'^{\hat{\alpha}}$ implies

T-duality <u>does</u> <u>not</u> change D-brane boundary cond'ns.

... duality transformations of component fields, use that

1) $E_1^{\alpha}|_{\theta=0}$ and $E_1^{\hat\alpha}|_{\theta=0}$ are parameters of abelian susy transf. $\delta_{susy} = \varepsilon_1^{\alpha} q_{\alpha} + \varepsilon_1^{\hat\alpha} q_{\hat\alpha}$

2) $P^{\alpha\hat\beta}|_{\theta=0} = e^{-\varphi} f^{\alpha\hat\beta}$ $\quad f^{\alpha\hat\beta}$ is R-R field-strength

3) $H_{\alpha\beta c} = \gamma_{\alpha\beta c}$ and $H_{\hat\alpha\hat\beta c} = -\gamma_{\hat\alpha\hat\beta c}$ $\quad H_{ABC} = E_A{}^M E_B{}^N E_C{}^P \partial_{[M}$

$$\Rightarrow \partial^c B_{11} = E_1^{\alpha} \gamma^c_{\alpha\beta} E_1^{\beta} = - E_1^{\hat\alpha} \gamma^c_{\hat\alpha\hat\beta} E_1^{\hat\beta}$$

So superfield T-duality transformations

$$\Rightarrow \quad \varphi' = \varphi + \tfrac{1}{2}\log C$$

$$e^{-\varphi'} f'^{\alpha\hat\beta} = e^{-\varphi} f^{\alpha\hat\beta} - \varepsilon^{\alpha}\varepsilon^{\hat\beta} C^{-1}$$

$$\varepsilon'^{\alpha} = -C^{-1}\varepsilon^{\alpha}, \quad \varepsilon'^{\hat\alpha} = -C^{-1}\varepsilon^{\hat\alpha}$$

$(\varepsilon^{\alpha}, \varepsilon^{\hat\alpha})$ are supersymmetry parameters

and $C = B_{11}|_{\theta=0}$.

In $AdS_5 \times S^5$ background, sigma model has 8 abelian supersymmetries q^a_j $a=1$ $j=1$

since $\{q^a_j, q^b_k\} = 0$.

After T-dualizing for 8 θ^j_a variables,

$$e^{-\varphi'} f'^{\alpha\hat{\beta}} = e^{-\varphi} f^{\alpha\hat{\beta}} - \varepsilon^{\alpha j}_a \varepsilon^{\beta k}_b C^{ab}_{jk}$$

$$= e^{-\varphi} (\gamma_{01234})^{\alpha\hat{\beta}}$$

$$- e^{-\varphi} [\gamma_{01234} - \gamma_4]^{\alpha\hat{\beta}} = e^{-\varphi} \gamma_4^{\alpha}$$

$$\varphi' = \varphi + 4 \log |r|.$$

This transformation can be undone by now T-dualizing with respect to bosonic $d=4$ coordinates (x^0, x^1, x^2, x^3)

$$\Rightarrow \quad \varphi'' = \varphi' - 4 \log |r| = \varphi$$

$$e^{-\varphi''} f''^{\alpha\hat{\beta}} = e^{-\varphi} (\gamma_{01234})^{\alpha\hat{\beta}} = e^{-\varphi} f^{\alpha\hat{\beta}}$$

So $AdS_5 \times S^5$ sigma model is invariant under combined T-duality of $\theta_a^{\ j}$ and $x^{a\dot{a}}$ variables. Can be easily verified using Metsaev-Tseytlin or pure spinor version of sigma model action.

T-duality exchanges superconf. inv. and "dual superconf. inv"

$p^{a\dot{a}}$	$\to$ trivial	$p^{a\dot{a}}$
$K^{a\dot{a}}$	$\to$ non-local	$K^{a\dot{a}}$
$M^{ab}, M^{\dot{a}\dot{b}}$	$\longrightarrow$	$M^{ab}, M^{\dot{a}\dot{b}}$
$R^j_{\ k}$	$\longrightarrow$	$R^j_{\ k}$
D	$\longrightarrow$	$-D$
$q^a_{\ j}$	$\to$ trivial	$q^j_{\ a}$
$s^j_{\ a}$	$\to$ non-local	$s^a_{\ j}$
$\bar{q}^j_{\ \dot{a}}$	$\longrightarrow$	$\bar{s}^j_{\ \dot{a}}$
$\bar{s}^{\dot{a}}_{\ j}$	$\longrightarrow$	$\bar{q}^{\dot{a}}_{\ j}$

SUMMARY

Manifestly $PSU(2,2|4)$-inv. description of $AdS_5 \times S^5$ superstring led to new surprising features:

1) Simple measure factor $\langle (\lambda^\alpha \hat{\lambda}^{\hat{\beta}} \eta_{\alpha\hat{\beta}})^3 \rangle = 1$.

   Compute non-trivial scattering amplitudes?

2) Sigma model in $N=(2,2)$ worldsheet superspace

   YM Feynman diagrams = topological amplitudes?

3) Invariance under combined bosonic + fermionic T-duality

   Prove YM amplitude = Wilson-line conjecture?