Scattering Amplitudes and Strings on AdS

Luis Fernando Alday

Utrecht University

Wonders of gauge theory and supergravity
Paris - June 2008

Motivations

We will be interested in gluon scattering amplitudes of planar $\mathcal{N}=4$ super Yang-Mills.

Motivation: It can give non trivial information about more realistic theories but is more tractable. In the last years, many tools become available.

- Perturbative computations are easier. In particular higher loop (MHV) amplitudes appear to have a remarkable iterative structure, leading to a proposal for all loops n-point amplitudes.
- The strong coupling regime can be studied, by means of the gauge/string duality, through a weakly coupled string sigma model.

Aim of these lectures

Prescription for computing scattering amplitudes of planar $\mathcal{N}=4$ super Yang-Mills at strong coupling by using the AdS/CFT correspondence.

1. Introduction
   - Gauge theory results
   - AdS/CFT duality
2. String theory set up
3. Four point amplitude at strong coupling
   - Important ingredients
4. Scattering amplitudes vs. WL and testing BDS
5. Conclusions and outlook

Gauge theory results

Bern, Dixon, Smirnov, (Anastasiou, Carrasco, Johansson, Kosower,...)

- Study gluon amplitudes of planar MSYM, in the color decomposed form:

$$A_n^{L,Full} \sim \sum_\rho Tr(T^{a_{\rho(1)}} \ldots T^{a_{\rho(n)}}) A_n^{(L)}(\rho(1), \ldots, \rho(2))$$

- Leading N_c color ordered $n-$points amplitude at L loops: $A_n^{(L)}$
- The amplitudes are divergent so we need to introduce a regulator.
- Dimensional regularization $D = 4 - 2\epsilon \rightarrow A_n^{(L)}(\epsilon) = 1/\epsilon^{2L} + \ldots$
- Focus on MHV amplitudes and scale out the tree amplitude $M_n^{(L)}(\epsilon) = A_n^{(L)}/A_n^{(0)}$.
- Up to few loops, $M_n^{(L)}(\epsilon)$ can be written in terms of lower order amplitudes!

Motivated by explicit computations, the infrared behavior and collinear factorization of multi-loop amplitudes...

MHV amplitudes: all loops proposal!

$$\mathcal{M}_n \equiv 1 + \sum_{L=1} \alpha^L M_n^{(L)}(\epsilon) = exp\left[\sum_{\ell=1}^{\infty} \alpha^\ell \left(f^\ell(\epsilon) M_n^{(1)}(\ell\epsilon) + C^{(\ell)} + \mathcal{O}(\epsilon)\right)\right]$$

$$\alpha \sim \frac{\lambda \mu^{2\epsilon}}{8\pi^2}, \qquad f^\ell(\epsilon) = f_0^\ell + \epsilon f_1^\ell + \epsilon^2 f_2^\ell$$

We will perform explicit computations for $n = 4$.

4 point amplitude

$$\mathcal{A} = \mathcal{A}_{tree} \left(\mathcal{A}_{div,s}\right)^2 \left(\mathcal{A}_{div,t}\right)^2 \exp\left\{\frac{f(\lambda)}{8}(\ln s/t)^2 + const\right\}$$

$$\mathcal{A}_{div,s} = \exp\left\{-\frac{1}{8\epsilon^2} f^{(-2)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right) - \frac{1}{4\epsilon} g^{(-1)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right)\right\}$$

- Divergent piece plus kinematical part of finite term are characterized by two functions.
- $\left(\lambda\frac{d}{d\lambda}\right)^2 f^{(-2)}(\lambda) = f(\lambda)$: Cusp anomalous dimension, controls leading divergence.
- $\lambda\frac{d}{d\lambda} g^{(-1)}(\lambda) = g(\lambda)$: Subleading divergence.

AdS/CFT duality

AdS/CFT duality (Maldacena)

Four dimensional maximally SUSY Yang-Mills $\Leftrightarrow$ Type IIB string theory on $AdS_5 \times S^5$.

$$\sqrt{\lambda} \equiv \sqrt{g_{YM}^2 N} = \frac{R^2}{\alpha'} \qquad \frac{1}{N} \approx g_s$$

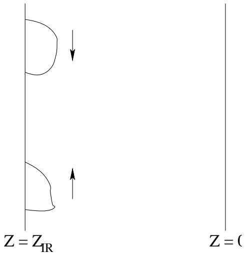
We will study scattering amplitudes at strong coupling by using the AdS/CFT duality.

- Set up the computation: Use a $D - brane$ as IR cut-off.
- Actual computations: Dimensional regularization.

String theory set up

$$ds^2 = R^2 \frac{dx_{3+1}^2 + dz^2}{z^2}$$

- Place a D-brane extended along x_{3+1} and located at some large z_{IR}.



- The asymptotic states are open strings ending on the D-brane.
- Consider the scattering of these open strings.
- The proper momentum of these strings, $k_{pr} = k \frac{z_{IR}}{R}$ is very large, so we are interested in the regime of fixed angle and very high momentum.

This regime was considered in flat space (Gross and Mende)

Key feature

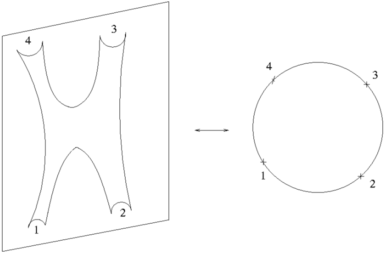
The amplitude is dominated by a saddle point of the classical action.

$$\Downarrow$$

We need to consider a classical string on AdS

- Important difference: k doesn't need to be too large.

World-sheet with the topology of a disk with vertex operator insertions (corresponding to external states)



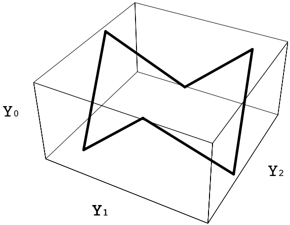
- Near each vertex operator, the momentum of the external state fixes the form of the solution.
- In the boundary of the world-sheet $z = z_{IR}$

- "T-duality": $ds^2 = w^2(z)dx_\mu dx^\mu \rightarrow \partial_\alpha y^\mu = i w^2(z)\epsilon_{\alpha\beta}\partial_\beta x^\mu$
- Boundary conditions: x^μ carries momentum $k^\mu \rightarrow y^\mu$ has winding $\Delta y^\mu = 2\pi k^\mu$.
- After a change of coordinates $r = R^2/z$ we end up again with AdS_5

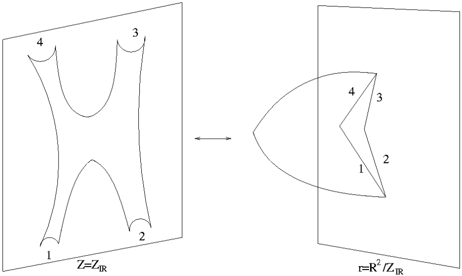
$$ds^2 = R^2 \frac{dy_{3+1}^2 + dr^2}{r^2}$$

World-sheet whose boundary is located at $r = R^2/z_{IR}$ and is a particular line constructed as follows...

- For each particle with momentum k^μ draw a segment joining two points separated by $\Delta y^\mu = 2\pi k^\mu$



- Concatenate the segments according to the ordering of the insertions on the disk (particular color ordering)
- Momentum conservation: Closed diagram.
- Massless gluons → light-like edges.



- As $z_{IR} \to \infty$ the boundary of the world-sheet moves to $r = 0$.
- Exactly same computation as when computing the expectation value of a Wilson-Loop given by a sequence of light-like segments!

Prescription

- $\mathcal{A}_n$: Leading exponential behavior of the $n-$point scattering amplitude.
- $A_{min}(k_1^\mu, k_2^\mu, ..., k_n^\mu)$: Area of a minimal surface that ends on a sequence of light-like segments on the boundary.

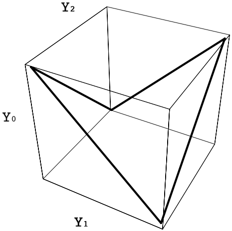
$$\mathcal{A}_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}$$

- Prefactors are subleading in $1/\sqrt{\lambda}$, and we don't compute them.
- In particular our computation is blind to helicity (and hence works also for non MHV)

Four point amplitude at strong coupling

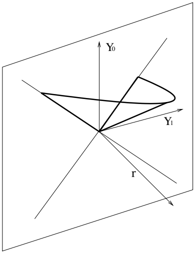
Consider $k_1 + k_3 \rightarrow k_2 + k_4$

$$s = -(k_1 + k_2)^2, \qquad t = -(k_1 + k_4)^2$$



Need to find the minimal surface ending on such sequence of light-like segments.

- Warm up: Try to find the solution near one of the cusps.



The surface can be embedded in AdS_3

$$ds^2 = \frac{-dy_0^2 + dy_1^2 + dr^2}{r^2}$$

$$r = \sqrt{2}\sqrt{y_0^2 - y_1^2}$$

- $y_0 > y_1$, nice cusp (space-like momentum transfer)
- $y_1 > y_0$ not nice cusp (time-like momentum transfer)

- S_{NG}: Poincare coordinates (r, y_0, y_1, y_2) and parametrize the surface by its projection to (y_1, y_2) plane.
- Action for two fields $r(y_1, y_2), y_0(y_1, y_2)$. *E.g.* if $s = t$ the fields live on a square parametrized by y_1, y_2.

$$S_{NG} = \frac{R^2}{2\pi} \int dy_1 dy_2 \frac{\sqrt{1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}}{r^2}$$

- By scale invariance, edges of the square at $y_1, y_2 = \pm 1$

Boundary conditions

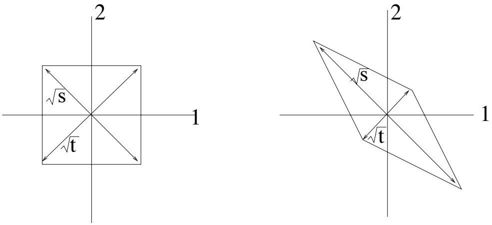
$$r(\pm 1, y_2) = r(y_1, \pm 1) = 0, \quad y_0(\pm 1, y_2) = \pm y_2, \quad y_0(y_1, \pm 1) = \pm y_1$$

- We know the solution near the cusps. We can make some guess

$$y_0(y_1, y_2) = y_1 y_2, \quad r(y_1, y_2) = \sqrt{(1 - y_1^2)(1 - y_2^2)}$$

- Easily seen to satisfy all the conditions and actually solves the eoms!
- However, $s = t$ is somehow a boring case...

- We would like to capture the kinematical dependence of the amplitude. We need to consider $s \neq t$.
- The square will be deformed to a rhombus



(a) (b)

Embedding coordinates

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1$$

$$Y^\mu = \frac{y^\mu}{r}, \quad \mu = 0, ..., 3$$

$$Y_{-1} + Y_4 = \frac{1}{r}, \quad Y_{-1} - Y_4 = \frac{r^2 + y_\mu y^\mu}{r}$$

Embedding coordinates surface

$$Y_0 Y_{-1} = Y_1 Y_2 \qquad Y_3 = Y_4 = 0$$

- We can perform $SO(2,4)$ transformations and get new solutions. This is a "dual" conformal symmetry.
- *e.g.* a boost in the $0-4$ direction gives a new solution with $s \neq t$.

Conformal gauge action

$$iS = -\frac{R^2}{2\pi}\int du_1 du_2 \frac{1}{2}\frac{\partial r \partial r + \partial y_\mu \partial y^\mu}{r^2}$$

Solution for the rhombus

$$r = \frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2},$$

$$y_0 = r\sqrt{1+b^2}\sinh u_1 \sinh u_2$$

$$y_1 = r \sinh u_1 \cosh u_2, \qquad y_2 = r \cosh u_1 \sinh u_2$$

- The parameters a and b encode the kinematical information.

$$-s(2\pi)^2 = \frac{8a^2}{(1-b)^2}, \qquad -t(2\pi)^2 = \frac{8a^2}{(1+b)^2}$$

Let's compute the area...

- Small problem: The area diverges!
- Dimensional reduction scheme: Theory in $D = 4 - 2\epsilon$ dimensions but with 16 supercharges.
- For integer D this is exactly the low energy theory living on $Dp-$branes ($p = D - 1$)

Gravity dual

$$ds^2 = h^{-1/2} dx_D^2 + h^{1/2}\left(dr^2 + r^2 d\Omega_{9-D}^2\right), \qquad h = \frac{c_D \lambda_D}{r^{8-D}}$$

$$\lambda_D = \frac{\lambda \mu^{2\epsilon}}{(4\pi e^{-\gamma})^{\epsilon}} \qquad c_D = 2^{4\epsilon}\pi^{3\epsilon}\Gamma(2+\epsilon)$$

T-dual coordinates

$$ds^2 = \sqrt{\lambda_D c_D}\left(\frac{dy_D^2 + dr^2}{r^{2+\epsilon}}\right) \rightarrow S_\epsilon = \frac{\sqrt{\lambda_D c_D}}{2\pi}\int \frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon}$$

- Presence of ϵ will make the integrals convergent.
- The eoms will depend on ϵ but if we plug the original solution into the new action, the answer is accurate enough.
- plugging everything into the action...

$$iS = -\frac{\sqrt{\lambda_D c_D}}{2\pi a^\epsilon}\left(\frac{\pi\Gamma\left[-\frac{\epsilon}{2}\right]^2}{\Gamma\left[\frac{1-\epsilon}{2}\right]}\ {}_2F_1\left(\frac{1}{2}, -\frac{\epsilon}{2}, \frac{1-\epsilon}{2}; b^2\right) + 1/2\right) + \mathcal{O}(\epsilon)$$

- Just expand in powers of ϵ...

Final answer

$$\mathcal{A} = e^{iS} = \exp\left[iS_{div} + \frac{\sqrt{\lambda}}{8\pi}\left(\log\frac{s}{t}\right)^2 + \tilde{C} \right]$$

$$S_{div} = 2S_{div,s} + 2S_{div,t}$$

$$S_{div,s} = -\frac{1}{\epsilon^2}\frac{1}{2\pi}\sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}} - \frac{1}{\epsilon}\frac{1}{4\pi}(1-\log 2)\sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}}$$

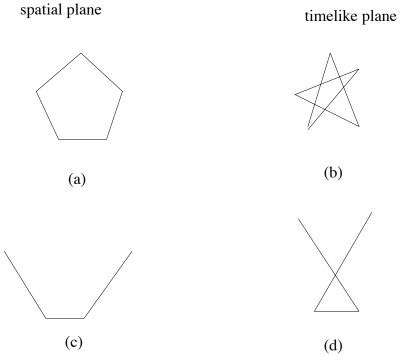
- Should be compared to the field theory answer

$$\mathcal{A} \sim (\mathcal{A}_{div,s})^2 (\mathcal{A}_{div,t})^2 \exp\left\{ \frac{f(\lambda)}{8}(\ln s/t)^2 + const \right\}$$

$$\mathcal{A}_{div,s} = \exp\left\{ -\frac{1}{8\epsilon^2} f^{(-2)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right) - \frac{1}{4\epsilon} g^{(-1)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right) \right\}$$

Can we have n odd and only nice cusps?

- Yes, if we use $R^{2,2}$ signature.
- Contour → curve on (y_1, y_2) and (t_1, t_2) plane.
- LIght-like segments → same length in the two planes.
- Space like momentum transfer → angles in the space-like plane are bigger.



- The IR structure at strong coupling can be computed (Buchbinder)

$$\mathcal{A}_n = \sum_{i=1}^{n} \left(f \frac{1}{\epsilon^2} \sqrt{\frac{\mu^{2\epsilon}}{(x_{i-1,i+1}^2)^\epsilon}} + g \frac{1}{\epsilon} \sqrt{\frac{\mu^{2\epsilon}}{(x_{i-1,i+1}^2)^\epsilon}} \right) + Fin(x_i)$$

- We can also use a radial cut-off

$$\mathcal{A}_n = \sum_{i=1}^{n} \left(f \log^2 \left(\frac{r_c^2}{x_{i-1,i+1}^2} \right) + g \log \left(\frac{r_c^2}{x_{i-1,i+1}^2} \right) \right) + Fin(x_i)$$

The two constants (up to constant pieces) agree!

- This computation shows a relation between Wilson loops and scattering amplitudes.
- This relation holds also at weak coupling!
  - Four legs at one loop (Drummond, Korchemsky, Sokatchev)
  - n legs at one loop (Brandhuber, Heslop, Travaglini)
  - Up to six legs at two loops (Drummond, Henn, Korchemsky, Sokatchev)

Will hear more at Korchemsky's talk!

- $SO(2,4)$ transformations fixed somehow the kinematical dependence of the finite piece.
- This dual conformal symmetry constrains the form of the amplitude

Dual Ward identity

$$\mathcal{O}_K \mathcal{A} = 0 \quad \rightarrow \quad \mathcal{O}_k Fin = -\mathcal{O}_k Div$$

For $n = 4, 5$ the solution is unique and agrees with BDS! for $n = 6$ there is some freedom.

- Also perturbatively and even dual super-conformal symmetry on NMHV (Sokatchev's talk).
- Dual super-conformal symmetry present for all values of the coupling! (Berkovits's talk)

What about the BDS ansatz?

- The strong coupling computation agrees with BDS for $n = 4$ and $n = 5$.
- But symmetries "protect" BDS from corrections, we need to consider $n > 5$.
- BDS can be disproved by considering a configuration with a large number of gluons!

$$\frac{16\pi^2}{\Gamma(1/4)^4} = \frac{3^2-1}{3^2}\frac{5^2}{5^2-1}\frac{7^2-1}{7^2}\cdots$$ (Magnus and Oberhettinger (1949))

- Later on, indeed: BDS fails for six gluons at two loops! (Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich)

Wilson loops vs. Scattering amplitudes

- This computation shows a relation between Wilson loops and scattering amplitudes.
- This relation holds also at weak coupling! Drummond, Korchemsky, Sokatchev

Write BDS on a slightly different way

$$\log \mathcal{M}_n = Div_n + \frac{f(\lambda)}{4} a_1(k_1, k_2, ..., k_n) + h(\lambda) + nk(\lambda)$$

Scattering amplitudes vs. WL (Brandhuber, Heslop, Travaglini)

$$< W_{k_i} >= 1 + \lambda \left(Div + w_1(k_1, ..., k_n) + c + n\tilde{c} \right)$$

$$\Downarrow$$

$$w_1(k_1, ..., k_n) = a_1(k_1, ..., k_n)$$

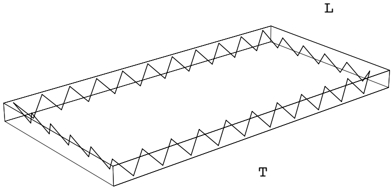
- BDS $\Rightarrow \quad a_{strong} = f^{strong} a_1(k_1, ..., k_n)$
- WL vs. Amplitudes at strong coupling $\Rightarrow a_{strong} = w_{strong}$
- WL vs. Amplitudes at weak coupling $\Rightarrow a_1 = w_1$

$$\Downarrow$$

$$w_{strong} = f^{strong} w_1(k_1, ..., k_n)$$

- For $n = 4$ and $n = 5$ that is the case! but fixed by symmetries.
- We need to take $n > 5$, what about $n = \infty$?

We choose a zig-zag configuration that approximates the rectangular Wilson loop.



- $\log < W_{rect}^{weak} >= \frac{\lambda}{8\pi} \frac{T}{L}$
- $\log < W_{rect}^{strong} >= \sqrt{\lambda} \frac{4\pi^2\sqrt{2}}{\Gamma(\frac{1}{4})^4} \frac{T}{L}$

- The strong coupling result is not what we would expect from the BDS ansatz, hence something needs to be revised...

At which order in perturbation theory and for how many gluons will BDS fail?

- BDS $\Rightarrow \quad a_\ell = f^{(\ell)} a_1(k_1, ..., k_n)$
- WL vs. Amplitudes at all orders $\quad \Rightarrow a_\ell = w_\ell$

$$\Downarrow$$

$$w_2 = f^{(2)} w_1(k_1, ..., k_n)$$

- An explicit computation for the rectangular Wilson loop, shows that either the BDS conjecture or the relation between WL and amplitudes (or both!) fail at two loops for a large number of gluons.

- A month ago (Dec. 31) it was shown that this is indeed the case for $n = 6$!!

What things need to be done?

- Try to make explicit computations for $n > 4$, *e.g.* $n = 6$ is a good one.
- We haven't assume/use at all the machinery of integrability.
- Subleading corrections in $1/\sqrt{\lambda}$? Information about helicity of the particles, etc.
- Gross and Mende computed higher genus amplitudes (in flat space) using similar ideas, can we do the same?
- Can we repeat the computation in other backgrounds?
- Deeper relation among Wilson loops and scattering amplitudes?
- Some powerful alternative to BDS?