Steep tilings and sequences of interlaced partitions

Jérémie Bouttier
Joint work with Guillaume Chapuy and Sylvie Corteel
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Institut de Physique Théorique, CEA Saclay
Département de mathématiques et applications, ENS Paris

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Introduction

Rhombus tiling

Plane partition

4 3 3 2 1
4 2 2 2
3 2 2 1
2 1 1
Introduction

Rhombus tiling

Plane partition

\[
\begin{align*}
4 & \ 3 & \ 3 & \ 2 & \ 1 \\
4 & \ 2 & \ 2 & \ 2 \\
3 & \ 2 & \ 2 & \ 1 \\
2 & \ 1 & \ 1 \\
\end{align*}
\]

Sequence of interlaced partitions

\[
\begin{align*}
2 & \prec \ 3 & \prec \ 4 & \prec \ 4 & \succ \ 3 & \succ \ 3 & \succ \ 2 & \succ \ 2 & \succ \ 1 \\
1 & \prec \ 2 & \prec \ 2 & \succ \ 2 & \succ \ 1 \\
\end{align*}
\]
An (integer) partition $\lambda$ is a finite non-increasing sequence of integers

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_\ell > 0$$

(By convention we set $\lambda_i = 0$ for $i \geq \ell$.)

We say that $\lambda$ and $\mu$ are (horizontally) interlaced, and denote $\lambda \succ \mu$, iff

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \cdots$$
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Plane partitions correspond to sequences of interlaced partitions:

$$\cdots \lambda^{(-2)} \prec \lambda^{(-1)} \prec \lambda^{(0)} \succ \lambda^{(1)} \succ \lambda^{(2)} \succ \cdots$$

with $\lambda^{(i)} = \emptyset$ for $|i|$ large enough.
Define the **size** of a partition/plane partition as the sum of its entries.
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\[ \sum_{\text{plane partitions}} q^{\text{size}} = \prod_{k=1}^{\infty} \frac{1}{(1 - q^k)^k} \quad \text{[McMahon]} \]
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The sequence of interlaced partitions corresponding to a random plane partition drawn with probability proportional to $q^{\text{size}}$ ($0 \leq q < 1$) forms a Schur process [Okounkov-Reshetikin 2003].

How about tilings made of dominos instead of rhombi?
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For instance the “state” \( \lambda^{(0)} \) of the main diagonal is drawn with probability proportional to

\[ \left( s_{\lambda^{(0)}}(q^{1/2}, q^{3/2}, q^{5/2}, \ldots) \right)^2 . \]
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\[ \left( s_{\lambda(0)}(q^{1/2}, q^{3/2}, q^{5/2}, ...) \right)^2. \]

How about tilings made of dominos instead of rhombi?
Motivations

In general:

- statistical mechanics: rhombus/domino tilings = dimer model on honeycomb/square lattice
- enumerative combinatorics: beautiful enumeration formulas
- probability theory: determinantal correlations, limit shape phenomena, interesting limiting processes related to random matrices
- algebraic geometry: Donaldson-Thomas theory

For our specific work: understand precisely the connection between domino tilings and interlaced partitions, implicitly hinted at in works of Johansson, Borodin, etc.
Motivations

In general:

- statistical mechanics: rhombus/domino tilings = dimer model on honeycomb/square lattice
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For our specific work: understand precisely the connection between domino tilings and interlaced partitions, implicitly hinted at in works of Johansson, Borodin, etc. Have fun with “vertex operators” (a recreation after a reading group on the works from the Kyoto school: solitons, infinite dimensional Lie algebras and all that).
1. Steep tilings

2. Bijection with sequences of interlaced partitions

3. Enumeration via the vertex operator formalism
Outline

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Steep tilings

Steep tilings and interlaced partitions

A domino tiling of the oblique strip $x - 2\ell \leq y \leq x$

Steepness condition: we eventually find only north or east dominos in the NE direction, south or west in the SW direction.
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Steep tilings

A domino tiling of the oblique strip \( x - 2\ell \leq y \leq x \)

Steepness condition: we eventually find only north or east dominos in the NE direction, south or west in the SW direction.
The steepness condition implies that the tiling is eventually periodic in both directions. The two repeated patterns define the asymptotic data \( w \in \{+, -\}^{2\ell} \) of the tiling. For fixed \( w \) there is a unique (up to translation) minimal tiling which is periodic from the start.
Examples

Domino tilings of the Aztec diamond [Elkies et al.]
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Pyramid partitions [Kenyon, Szendrői, Young]
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Outline

1. Steep tilings

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3. Enumeration via the vertex operator formalism
To each steep tiling we may associate a particle configuration by filling each square covered by a N or E domino with a white particle, and each square covered by a S or W domino with a black particle.
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Particles along a diagonal form a “Maya diagram” which codes an integer partition (here 421).
Interlacing of particles

Between two successive even/odd diagonals, the white particles must be adjacent.
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Between two successive even/odd diagonals, the white particles must be adjacent. Conversely, between two successive odd/even diagonals, the black particles must be adjacent.
Interlacing of partitions

Between two successive even/odd diagonals, a finite number of white particles can be moved one site to the left (+) or to the right (−) in the Maya diagram (depending on asymptotic data). This corresponds to adding/removing a horizontal strip to the associated partition.
Interlacing of partitions

Between two successive even/odd diagonals, a finite number of white particles can be moved one site to the left (+) or to the right (−) in the Maya diagram (depending on asymptotic data). This corresponds to adding/removing a horizontal strip to the associated partition. Conversely, between two successive odd/even diagonals, a vertical strip is added/removed.
Interlacing of partitions

For $\lambda, \mu$ two integer partitions, the following are equivalent:

- $\lambda/\mu$ is a horizontal strip,
- $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \cdots$,
- $\lambda'_i - \mu'_i \in \{0, 1\}$ for all $i$.

Notation: $\lambda \triangleright \triangleright \mu$ or $\mu \triangleright \triangleright \lambda$. 

Similarly, the following are equivalent:

- $\lambda/\mu$ is a vertical strip,
- $\lambda'_1 \geq \mu'_1 \geq \lambda'_2 \geq \mu'_2 \geq \lambda'_3 \geq \cdots$,
- $\lambda_i - \mu_i \in \{0, 1\}$ for all $i$.

Notation: $\lambda \triangleright \triangleright \mu'$ or $\mu \triangleright \triangleright \lambda'$. 

Interlacing of partitions

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- $\lambda/\mu$ is a horizontal strip,
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Notation: $\lambda \succ \mu$ or $\mu \prec \lambda$.

Similarly, the following are equivalent:

- $\lambda/\mu$ is a vertical strip,
- $\lambda_1' \geq \mu_1' \geq \lambda_2' \geq \mu_2' \geq \lambda_3' \geq \cdots$,
- $\lambda_i - \mu_i \in \{0, 1\}$ for all $i$.

Notation: $\lambda \succ' \mu$ or $\mu \prec' \lambda$. 
The fundamental bijection

For a fixed word $w \in \{+, -\}^{2\ell}$, there is a one-to-one correspondence between steep tilings of asymptotic data $w$ and sequences of partitions $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(2\ell)})$ satisfying for all $k = 1, \ldots, \ell$:

- $\lambda^{(2k-2)} \prec \lambda^{(2k-1)}$ if $w_{2k-1} = +$, and $\lambda^{(2k-2)} \succ \lambda^{(2k-1)}$ if $w_{2k-1} = -$,
- $\lambda^{(2k-1)} \prec' \lambda^{(2k)}$ if $w_{2k} = +$, and $\lambda^{(2k-1)} \succ' \lambda^{(2k)}$ if $w_{2k} = -$. 

Examples:

Aztec diamond:

$\emptyset = \lambda^{(0)} \prec \lambda^{(1)} \succ \lambda^{(2)} \prec \lambda^{(3)} \succ \cdots \succ \lambda^{(2\ell)} = \emptyset$.

Pyramid partitions:

$\emptyset = \lambda^{(0)} \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \cdots \lambda^{(2\ell)} = \emptyset$.

The size of $\lambda^{(m)}$ is equal to the number of flips on diagonal $m$ in any shortest sequence of flips between the tiling at hand and the minimal tiling. Under natural statistics we obtain a Schur process [Okounkov-Reshetikhin].
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Examples:

- Aztec diamond:
  $\emptyset = \lambda^{(0)} \prec \lambda^{(1)} \succ' \lambda^{(2)} \prec \lambda^{(3)} \succ' \lambda^{(4)} \prec \ldots \succ' \lambda^{(2\ell)} = \emptyset$,

- Pyramid partitions:
  $\emptyset = \lambda^{(0)} \prec \lambda^{(1)} \prec' \lambda^{(2)} \prec \ldots \prec' \lambda^{(\ell)} \succ \ldots \succ' \lambda^{(2\ell)} = \emptyset$. 
The fundamental bijection

For a fixed word \( w \in \{+, -\}^{2\ell} \), there is a one-to-one correspondence between steep tilings of asymptotic data \( w \) and sequences of partitions \((\lambda(0), \lambda(1), \ldots, \lambda(2\ell))\) satisfying for all \( k = 1, \ldots, \ell \):

- \( \lambda(2k-2) < \lambda(2k-1) \) if \( w_{2k-1} = + \), and \( \lambda(2k-2) > \lambda(2k-1) \) if \( w_{2k-1} = - \),
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Examples:

- **Aztec diamond:**
  \[
  \emptyset = \lambda(0) < \lambda(1) <' \lambda(2) < \lambda(3) <' \lambda(4) < \ldots <' \lambda(2\ell) = \emptyset,
  \]

- **Pyramid partitions:**
  \[
  \emptyset = \lambda(0) < \lambda(1) <' \lambda(2) < \ldots <' \lambda(\ell) < \ldots <' \lambda(2\ell) = \emptyset.
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Examples:

- Aztec diamond:
  $\emptyset = \lambda(0) \prec \lambda(1) \succ' \lambda(2) \prec \lambda(3) \succ' \lambda(4) \prec \ldots \succ' \lambda(2\ell) = \emptyset$.

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The size of $\lambda(m)$ is equal to the number of flips on diagonal $m$ in any shortest sequence of flips between the tiling at hand and the minimal tiling.

Under natural statistics we obtain a Schur process [Okounkov-Reshetikhin].
Flips

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Transfer matrices

Enumerating sequences of interlaced partitions is done via transfer matrices, which are here “vertex operators”:

\[
\langle \lambda | \Gamma_+(t) | \mu \rangle = \langle \mu | \Gamma_-(t) | \lambda \rangle = \begin{cases} 
 t|\mu|^{-\lambda} & \text{if } \lambda \prec \mu \\
 0 & \text{otherwise}
\end{cases}
\]

\[
\langle \lambda | \Gamma_+(t) | \mu \rangle = \langle \mu | \Gamma_-(t) | \lambda \rangle = \begin{cases} 
 t|\mu|^{-\lambda} & \text{if } \lambda \prec' \mu \\
 0 & \text{otherwise}
\end{cases}
\]

Example: Aztec diamond:

\[
\langle \emptyset | \Gamma_+(z_1) \Gamma_-(z_2) \Gamma_+(z_3) \Gamma_-(z_4) \cdots | \emptyset \rangle
\]
Bosonic representation

The transfer matrices can be rewritten as

\[
\Gamma_\pm(t) = \exp \sum_{k \geq 1} \frac{t^k}{k} \alpha_\pm k, \quad \Gamma'_\pm(t) = \exp \sum_{k \geq 1} \frac{(-1)^{k-1} t^k}{k} \alpha_\pm k
\]

where \([\alpha_n, \alpha_m] = n\delta_{n+m}\).
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where \([\alpha_n, \alpha_m] = n \delta_{n+m}\). This implies that \(\Gamma\)'s with the same sign commute, and that we have the following nontrivial commutation relations:

\[ \Gamma_+(t) \Gamma_-(u) = \frac{1}{1 - tu} \Gamma_-(u) \Gamma_+(t) \]
\[ \Gamma_+(t) \Gamma'_-(u) = (1 + tu) \Gamma'_-(u) \Gamma_+(t) \]

\(k = 0, 1, 2, \ldots\) \(k = 0, 1\)
Super Schur functions

When $w$ consists only of $+$'s, the partition function with fixed boundary conditions is a so-called super Schur function

$$\langle \mu \vert \Gamma_+ (x_1) \Gamma'_+ (y_1) \Gamma_+ (x_2) \Gamma'_+ (y_2) \cdots \vert \lambda \rangle = S_{\lambda/\mu} (x_1, x_2, \ldots ; y_1, y_2, \ldots).$$

Super Schur functions may be combinatorially defined in terms of super semistandard tableaux or (reverse) plane overpartitions:

$$
\begin{array}{cccc}
1 & 1 & \bar{1} & \bar{2} \\
\bar{1} & 2 & 2 & \bar{2} \\
\bar{1} & \bar{2} \\
2
\end{array}
$$
Pure steep tilings

For general asymptotic data and “pure” (\langle \emptyset | \text{ and } | \emptyset \rangle) boundary conditions the partition function is readily evaluated from the commutation relations.

\[
\langle \emptyset | \Gamma_+ (z_1) \Gamma'_+ (z_2) \Gamma_- (z_3) \Gamma'_- (z_4) | \emptyset \rangle = \]

\[
\langle \emptyset | \Gamma_- (z_3) \Gamma'_- (z_4) \Gamma_+ (z_1) \Gamma'_+ (z_2) | \emptyset \rangle \times \]

\[
\frac{(1 + z_1 z_4)(1 + z_2 z_3)}{(1 - z_1 z_3)(1 - z_2 z_4)}
\]

Equivalently we have a RSK-type bijection between pure steep tilings and suitable fillings of the Young diagram associated with \( w \).
Pure steep tilings

For a general word $w$ and the "$q^{\text{flip}}$" specialization, the partition function of pure steep tilings is given by a hook-length type formula:

$$T_w(q) = \prod_{1 \leq i < j \leq 2\ell} \varphi_{i,j}(q^{j-i}), \quad \varphi_{i,j}(x) = \begin{cases} 1 + x & \text{if } j - i \text{ odd} \\ 1/(1 - x) & \text{if } j - i \text{ even} \end{cases}$$
Aztec diamonds and pyramids

Aztec diamond $w = + - + - + - + - +$ [Elkies et al., Stanley]

$$T_w(q) = (1 + q)^3(1 + q^3)^2(1 + q^5)$$

Pyramid partitions $w = ++++----$. Case $\ell \to \infty$ [Young]:

$$T_w(q) = \prod_{k \geq 1} \frac{(1 + q^{2k-1})^{2k-1}}{(1 - q^{2k})^{2k}}$$
Free boundaries

We may also obtain a closed-form formula for the partition function in the case of free boundary conditions

$$|v\rangle = \sum_{\lambda} v^{\lambda} |\lambda\rangle$$

thanks to the “reflection relations”

$$\Gamma_+(t)|v\rangle = \frac{1}{1 - tv} \Gamma_-(tv^2)|v\rangle$$

$$\Gamma_+^{\prime}(t)|v\rangle = \frac{1}{1 - tv} \Gamma_+^{\prime}(tv^2)|v\rangle$$

$$k = 0, 1, 2, \ldots$$
Free boundaries

Example: \( w = ++++ \ldots \)

\[
\langle u | \Gamma_+ (y_1) \Gamma'_+ (y_2) \Gamma_+ (y_3) \Gamma'_+ (y_4) \cdots | v \rangle = \\
\prod_{k=1}^{\infty} \left( \frac{1}{1 - u^k v^k} \prod_{i=1}^{2 \ell} \frac{1}{1 - u^{k-1} v^k y_i} \right) \\
\prod_{1 \leq i < j \leq 2 \ell} \varphi_{i,j} (u^{2k-2} v^{2k} y_i y_j)
\]
Periodic boundary conditions

When identifying the left and right boundaries we obtain a cylindric steep tiling. The corresponding sequence of interlaced partitions form a periodic Schur process [Borodin].

The partition function may still be written as an infinite product.

Example: $w = + + --$

$$\text{Tr} \left[ \begin{array}{cc} \Gamma_+(z_1) & \Gamma'_+(z_2) \\ \Gamma'_+(z_2) & \Gamma_+(z_3) \\ \Gamma_+(z_3) & \Gamma'_+(z_4) \\ \Gamma'_+(z_4) & q^H \end{array} \right] =$$

$$\frac{(1 + z_1 z_4)(1 + z_2 z_3)}{(1 - z_1 z_3)(1 - z_2 z_4)}$$

$$\prod_{k=1}^{\infty} \frac{(1 + q^{k-1} x_1 x_4)(1 + q^{k-1} x_2 x_3)}{(1 - q^k)(1 - q^{k-1} x_1 x_3)(1 - q^{k-1} x_2 x_4)}$$
Further work

- Correlation functions [joint with C. Boutillier and S. Ramassamy]:
  - straightforward to compute for particles in the pure case, thanks to their free fermionic nature
  - less trivially we deduce an explicit expression for the inverse Kasteleyn matrix, which yields domino correlations
  - more involved in the periodic case [Borodin], how about free boundary case?
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- Random generation and limit shapes [joint with D. Betea and M. Vuletić]
Further work

More general setting
[BBCCR]: Rail Yard Graphs
(interpolate between lozenge and domino tilings)

- connection with octahedron recurrence/cluster algebras?
- deformations? (e.g. Schur $\rightarrow$ McDonald)
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Thanks for your attention!