Quantum dimer model with a liquid ground-state: topological degeneracy and toy model for a topological quantum-bit

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soon on cond-mat/0410...
Outline

- What is a quantum dimer model?
- What is a $\mathbb{Z}_2$ liquid? Topological degeneracy?
- What is a topological quantum bit?

- Why studying QDM’s on the *kagome* lattice?
  - Very simple: mapping onto (transverse field) Ising models
  - Exact $\mathbb{Z}_2$ liquid ground-state, topological degeneracy, etc.

- How to lift a topological degeneracy?
  - with “scissors” to change the topology
  - with monomers to mix different sectors

- Quantum-bit manipulation
  - Rotation, projection…
What is a quantum dimer model?

- Fully-packed dimer covering
  Several connections exist between QDM & frustrated magnets.
  Example: dimer $\sim$ spin singlet

- Introduce some (simple) quantum dynamics
  \[ H = -t \sum \left[ \left| \begin{array}{c} \text{spin}\ 1 \vspace{5pt} \\ \text{spin}\ 2 \end{array} \right| \times \left| \begin{array}{c} \text{spin}\ 1 \vspace{5pt} \\ \text{spin}\ 2 \end{array} \right| + H.c \right] \]
  \[ + V \sum \left[ \left| \begin{array}{c} \text{spin}\ 1 \vspace{5pt} \\ \text{spin}\ 2 \end{array} \right| \times \left| \begin{array}{c} \text{spin}\ 1 \vspace{5pt} \\ \text{spin}\ 2 \end{array} \right| + \left| \begin{array}{c} \text{spin}\ 1 \vspace{5pt} \\ \text{spin}\ 2 \end{array} \right| \times \left| \begin{array}{c} \text{spin}\ 1 \vspace{5pt} \\ \text{spin}\ 2 \end{array} \right| \right] \]

- Example of phase diagram (T=0)

  \begin{align*}
  \text{Valence-bond crystals (VBC)} & \sim -0.2 \\
  \text{Staggered crystal} & 1
  \end{align*}

  V/J

  Critical point

  No liquid so far…

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What is a $Z_2$ liquid?

- No broken symmetry
  - No long-ranged correlations
  - No local order parameter
  - Short-ranged RVB state: dimer~spin singlet

- Gapped excitations
  
  = Elementary flux (vortex) of a $Z_2$ gauge theory = visons
  

- Deconfined fractional excitations (monomers for instance)

- Topological degeneracy – topological order

Can this be used for something?

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What is a topological quantum bit?

- **Qubit = 2-level system**
  - $|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle$
  - Used to store/process some quantum information

- **Topological qubit:**
  - $|0\rangle$ and $|1\rangle$ are the ground-states of a macro (meso?)scopic system which are degenerate because of the (non-trivial) topology.
  - Example: $\mathbb{Z}_2$ liquid on a cylinder.

- **Advantage:**
  - The two states are *locally* indistinguishable
  - $\Rightarrow$ no local perturbation can introduce decoherence.

- **Problems:**
  - How can it be initialized, manipulated and read?

Are there some examples of such $\mathbb{Z}_2$ liquids?
Examples of QDM with $Z_2$-liquid ground-states

- Triangular lattice
  [Moessner & Sondhi, PRL (2001)]

- Kagome lattice
  [GM, Serban, Pasquier, PRL (2002)]

- 3D non-bipartite lattices (fcc,…)
  [Moessner & Sondhi, PRB (2003)]

- …
  - Heisenberg-like models
    [Sp(N): Read & Sachdev PRL (1991)]
    [Several candidates among 2D frustrated $S=\frac{1}{2}$ models: exact diagonalizations studies in C. Lhuillier’s group]
  - Ising-like models
    [Nayak & Shtengel PRB (2001)]
    [Balents, Fisher & Girvin PRB (2002)]
  - Bose-Hubbard models
    [Senthil & Motrunich PRB (2002); PRL (2002)]
  - Josephson junction arrays
    [Ioffe et al.; Douçot, Feigel’man & Ioffe PRL (2003)]

Let’s look at the simplest example
A solvable QDM with $\mathbb{Z}_2$ liquid ground-state

On a lattice made of corner-sharing triangles (such as kagome), dimer coverings are easily represented with arrows:

$\sigma^x(h)$: Flips the 6 arrows around $h$

$\sigma^x(h)^2 = 1$

$[\sigma^x(h), \sigma^x(h')] = 0 \quad \forall h, h'$

$\sigma^x \leftrightarrow$ Ising pseudo-spin operator

$H = - \sum_{h \in \text{hexagons}} \sigma^x(h)$

Where is the topological degeneracy?
Topological sectors & topological degeneracy

\[ T^x = (-1)^{N_A} \]

Dimer number parity is conserved by any local dimer move \( \Rightarrow 2 \) topological sectors

A QDM can be diagonalized separately in each topological sector.

A dimer liquid has the same ground-state energy in all topological sectors (in the thermodynamic limit).

Bu in a crystal: gap \( \sim O(L) \)

- \( T^x = 1 \)
  - \( E_+ = 0 \)

- \( T^x = -1 \)
  - \( E_- \sim O(L) \)
Ground-state degeneracy

\[ \tau^z_i \Rightarrow H = -\sum_h \sigma^x(h) = -\sum_h \left( \prod_{i=1}^6 \tau^z_i \right) \]

\[ T^z = \prod_{i \in \Delta^*} \tau^z_i \]

= Shifts all the dimers along \( \Delta^* \)

\[ [H, T^z] = 0 \]
\[ [H, T^x] = 0 \]

\[ T^x T^z = -T^z T^x \]

Dimer shift  Dimer number parity

\( \Delta \) \quad \Delta^* \quad T^x = (-1)^{N_\Delta} \]

Is this degeneracy robust? What could lift it?

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From a cylinder to a rectangle

\[ H_\lambda = - \sum_{h \in \text{hexagons}} \sigma^x(h) + 2\lambda N_\Delta \]

\[ = - \sum_{h \in \Delta} \sigma^x(h) - \sum_{h \in \Delta} \sigma^x(h) + \lambda \sum_{i=1}^{L} (1 - \tau_i^x) \]

\[ H_{\text{bulk}} \quad H_{\text{chain}}(\lambda) \]

Dimer number parity \( T^x = (-1)^{N_\Delta} \) is still a conserved but \([H_\lambda, T^z] \neq 0\)

This model is solvable!
Ising chain in transverse field

\[ H_{\text{chain}}(\lambda) = -\sum_{h \in \Delta} \sigma^x(h) + \lambda \sum_{i=1}^{L} (1 - \tau^x_i) \]

\[ = -\sum_{h=1}^{L} \sigma^x(h) - \lambda \sum_{i=0}^{L} \sigma^z(i)\sigma^z(i+1) \]

Topological sector coded by the boundary conditions of the Ising chain

\[ T^x = (-1)^{N_\Delta} = \prod_{i=1}^{L} \tau^x_i \]

\[ \sigma^z(L+1) = \sigma^z(0)\sigma^z(L+1) = \begin{cases} +1 & \text{ferro.} \\ -1 & \text{antiferro.} \end{cases} \]

NB: \( \sigma^z = \) vison creation operator

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Monomers to mix the topological sectors

\[ T^x = (-1)^{N_\Delta} \]

The dimer number parity is **no longer conserved** in presence of mobile monomers

\[ N_{in} = \text{even (0 or 2)} \]

\[ N_{in} = \text{odd (1 or 3)} \]

**Arrow flip**

Solvable model with monomers?
QDM with monomers (I)

\[ H = - \sum_{h} \sigma^x(h) - \mu U \sum_{i \in \Delta^*} \tau^z_i \]

\[ - U \sum_{t \in \text{triangles}} (-1)^{N_{in}(t)} \rightarrow \text{constraint } N_{in} = \text{even} \]

\[ = H_{\text{bulk}} + H_{\text{chain}}(\mu) \]

\[ H_{\text{chain}}(\mu) = -U \sum_{t=0 \cdots L-1} (-1)^{N_{in}(t)} - \mu U \sum_{i=0 \cdots L-1} \tau^z_i \]

\[ = -U \sum_{t=0 \cdots L-1} \bar{\sigma}^x(t) - \mu U \sum_{t=0 \cdots L-1} \bar{\sigma}^z(t) \bar{\sigma}^z(t+1) \]

Ising chain in transverse field
\[ H_{\text{chain}} = -U \sum_{t=0}^{L-1} (-1)^{N_{in}(t)} - \mu U \sum_{i=0}^{L-1} \bar{\sigma}_i^z \]

\[ = -U \sum_{t=0}^{L-1} \bar{\sigma}_i^x(t) - \mu U \sum_{t=0}^{L-1} \bar{\sigma}_i^z(t) \bar{\sigma}_i^z(t+1) \]

\[ T^z = \prod_{i \in \Delta^z} \tau_i^z \] is conserved. Boundary conditions: \( \bar{\sigma}_i^z(L) = T^z \bar{\sigma}_i^z(0) \)

⇒ Periodic/Antiperiodic boundary conditions for the Ising pseudospins.

\[
\langle \bar{\sigma}_i^z \rangle \\
\sim U \mu^L \\
\text{Paramagnet} \\
\sim 2U \mu \\
\text{Ferromagnet} \\
\mu \\
\]

\[
E_{\text{Antiperiodic}} - E_{\text{Periodic}} \\
\]

\[
\uparrow \uparrow \downarrow \downarrow \quad T^z = -1 \\
\uparrow \uparrow \uparrow \uparrow \quad T^z = 1 \\
\]
Quantum bit manipulation (I)

\[ f(\lambda) = E_{T^x=-1} - E_{T^x=+1} \]

\[ g(\mu) = E_{T^z=-1} - E_{T^z=+1} \]

\[ H_{\text{eff}}(\lambda, \mu) = -\frac{1}{2} f(\lambda) T^x - \frac{1}{2} g(\mu) T^z + \cdots \]

⇒ Allow any unitary rotation

Use \( \lambda < 1 \) or \( \mu < 1 \) (perturbative regime):
Quantum bit manipulation (II)

- Gap $\Delta \sim 1/L$ close to the transition.
  - Requires a slow adiabatic (time$\sim L$) evolution to avoid transitions to higher levels close to the critical point.
  - To be compared to $\sim \exp(L)$ if one stays in the perturbative regime [Ioffe et al., Nature 2002].
  - Requires a low temperature $k_B T \ll \Delta$ to avoid thermal excitations across the gap.

- Reading out - projection

$$|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

How to measure $|\alpha|^2$?

Switch on $\lambda$ adiabatically to $\gg 1 \Rightarrow$ minimizes $N_\Delta$

$|N_\Delta \text{ even}\rangle \rightarrow 0 \text{ dimer along } \Delta^*$

$|N_\Delta \text{ odd}\rangle \rightarrow 1 \text{ dimer along } \Delta^*$

- No need to implement physically the non-local operator $T^x = (-1)^{N_\Delta}$
Summary

- Solvable quantum dimer model on kagome realizing a $Z_2$ liquid
- Toy model to investigate perturbations of a topological degeneracy
- Simple illustration of the manipulation of a topological qubit
  Use of the phase transition to optimize the speed
  and the protection against decoherence.