Comment on "Regional Versus Global Entanglement in Resonating-Valence-Bond States"

In a recent Letter [1], Chandran *et al.* study the entanglement properties of valence bond (VB) states. Their main result is that VB states do not contain (or contain only an insignificant amount of) two-site entanglement, whereas they possess multibody entanglement. Two examples ("RVB gas and liquid") are given to illustrate this claim, which essentially comes from a lower bound derived for spin correlators in VB states. While we do not question that two-site entanglement is generically "small" for isotropic VB states, we show in this Comment that (i) for the "RVB liquid" on the square lattice, the calculations and conclusions of Ref. [1] are incorrect, (ii) a simple analytical calculation gives the exact value of the correlator for the "RVB gas," showing that the bound found in Ref. [1] is tight, and (iii) the lower bound for spin correlators in VB states is equivalent to a celebrated result of Anderson dating from more than 50 years ago.

The $SU(2)$ symmetry of VB states guarantees that any two-spin reduced density matrix is characterized by a single parameter p related to the correlator $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle = -3/4p$ between these spins ("Werner state"). The two spins are entangled if $p > 1/3$. Chandran *et al.* used quantum information concepts such as monogamy of entanglement and quantum telecloning to obtain bounds on p.

(i) The "RVB liquid" is the equal amplitude superposition of all nearest-neighbor (NN) VB coverings of a lattice. Exact results can be obtained for small sizes L of the square $L \times L$ lattice. For $L = 4$, we do not recover the value $p \simeq 0.2004$ of Ref. [1], but find $p = 0.4457579115872$ for periodic boundary conditions (BC) and $p = 0.2281115037$ in the interior of a sample with open BC. However, what really matters is the behavior for large L. We computed by Monte Carlo calculations [2] the NN correlator $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle$ for square lattices up to $L = 128$, using periodic BC. We find $p = 0.3946(3) > 1/3$ in the thermodynamic limit, resulting in an entanglement of formation of $\simeq 0.0215$. Therefore, the "RVB liquid" on the square lattice *does* possess two-site (NN) entanglement, contrary to the claim of Ref. [1].

(ii) The "RVB gas" is the equal amplitude superposition of all *bipartite* VB coverings of a bipartite lattice. This is in fact the projection into the singlet sector of the (magnetically ordered) *Néel state* on this lattice. This observation can be used to calculate p exactly. The total spins S_A and S_B on sublattices A and B are maximal, couple antiferromagnetically, and form a singlet (total spin $S = 0$). For a system of $2N$ spins, $S_A = S_B = N/2$. One then easily obtains that $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle = -1/4 - 1/(2N)$ if i and j belong to different sublattices. The equivalent exact result $p = 1/3 + 2/(3N)$ shows that the telecloning bound $p \leq 1/3 + 2/(3N)$ is tight. Two-site entanglement is therefore present in any finite "RVB gas" and vanishes only in the thermodynamic limit.

(iii) The telecloning bound on p in Ref. [1] reproduces an inequality of Anderson [3], who derived a lower bound for the energy of antiferromagnetic spin models. Take a spin at site i, separated by any distance from a number z of symmetry-equivalent spins j: $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle$ is identical for all z spins at sites j. In this case, the telecloning bound is $p \leq 1/3 + 2/(3z)$ or equivalently $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \geq -1/4 - 1/(2z)$, the result derived by Anderson. His result (of variational nature) on correlators is very general: it holds also for states other than singlets, is independent of any Hamiltonian and can be refined further (see, e.g., Ref. [4]).

In conclusion, the bound obtained with quantum information techniques [1] has been familiar in the condensed matter context for a long time. Nevertheless, it is interesting to see that it can be derived in a totally different framework. For the two examples chosen in Ref. [1], typical condensed matter methods allowed us to provide in one case an exact solution, and to show that the results of Ref. [1] are incorrect in the other one.

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