Classical statistical computation of the Schwinger mechanism

Naoto Tanji
KEK/Saclay
collaboration with F. Gelis

arXiv:1303.4633

h3QCD @ ECT*
18/6/2013

Outline

1. Classical statistical method

   as a Monte Carlo method to calculate 1-loop quantities

2. Schwinger mechanism

   Quantum tunneling phenomena can be described
   by classical statistical field theory

3. Effect of self-interactions

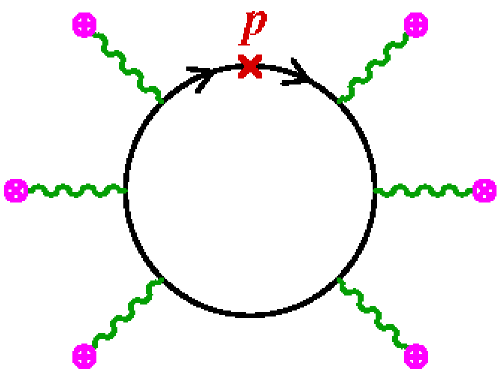
   Renormalization in the classical statistical method

Classical statistical method

- If typical occupancy of a field is large $(f \gg 1)$, its dynamics shows classical behavior.
- Quantum nature of an initial state is incorporated by taking ensemble average over initial conditions.
- Non-equilibrium and non-perturbative.
- Has been used in the studies of the early-universe inflaton dynamics, cold atom systems and the heavy-ion physics as well.

Classical statistical method

- Gives exact results for a field which is quadratic in Lagrangian even if the typical occupancy is small.
- Possible to analyze the quark dynamics in classical gauge fields.



Apply to the Schwinger mechanism

Model

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi + \frac{\lambda}{4}(\phi^*\phi)^2 + J^\mu_{\rm ext}A_\mu$$

$$D_\mu = \partial_\mu - ieA_\mu$$

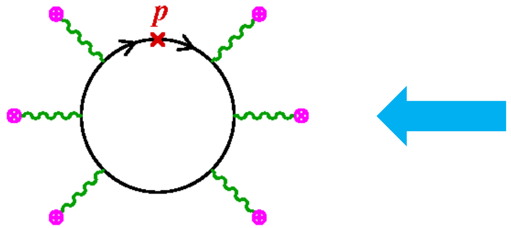
A toy model for QCD (gluodynamics)

- gauge coupling
- self-interaction

- The external source $J^\mu_{\rm ext}$ produce non-perturbatively strong gauge fields, which are treated as classical fields.

$$A \sim 1/e$$

Interaction with the external source (or fields produced by it) must be treated exactly.



Leading Order

No correction by λ
or e not attached to the source

Computation of the two point function

$$G(x,y) = \langle 0_{\rm in}|\phi(x)^\dagger \phi(y)|0_{\rm in}\rangle$$

1. Direct computation

At the LO, the field operator follows the linear equation of motion

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\phi(x) = 0$$

$$\phi(x) = \int \frac{d^3q}{(2\pi)^3 2E_q}\left[\varphi_{\mathbf{q}}(x) a_{\rm in}(\mathbf{q}) + \varphi^*_{\mathbf{q}}(x) b^\dagger_{\rm in}(\mathbf{q})\right]$$

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\varphi_{\mathbf{q}}(x) = 0 \qquad \lim_{x^0\to-\infty} \varphi_{\mathbf{q}}(x) = e^{-iq\cdot x}$$

$$\langle 0_{\rm in}|\phi^\dagger(x)\phi(y)|0_{\rm in}\rangle_{\rm LO} = \int \frac{d^3q}{(2\pi)^3 2E_{\mathbf{q}}}\,\varphi_{\mathbf{q}}(x)\varphi^*_{\mathbf{q}}(y)$$

Computation of the two point function

$$G(x,y) = \langle 0_{\rm in}|\phi(x)^\dagger \phi(y)|0_{\rm in}\rangle$$

1. Direct computation

At the LO, the field operator follows the linear equation of motion

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\phi(x) = 0$$

$$\phi(x) = \int \frac{d^3q}{(2\pi)^3 2E_q}\left[\varphi_{\mathbf{q}}(x) a_{\rm in}(\mathbf{q}) + \varphi^*_{\mathbf{q}}(x) b^\dagger_{\rm in}(\mathbf{q})\right]$$

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\varphi_{\mathbf{q}}(x) = 0 \qquad \lim_{x^0\to-\infty} \varphi_{\mathbf{q}}(x) = e^{-iq\cdot x}$$

$$\langle 0_{\rm in}|\phi^\dagger(x)\phi(y)|0_{\rm in}\rangle_{\rm LO} = \int$$

Numerical cost

$$N_t \times N_{\rm latt}^2$$

expensive in 3+1dim.

Computation of the two point function

2. Monte Carlo computation

Consider the following functional

$$G_{xy}[\varphi_0, \pi_0] \equiv \varphi^*(x)\varphi(y)$$

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\,\varphi = 0 \qquad \varphi(t_0, \mathbf{x}) = \varphi_0(\mathbf{x}),\ \dot{\varphi}(t_0, \mathbf{x}) = \pi_0(\mathbf{x})$$

and a Gaussian average over initial values

$$\langle G_{xy}[\varphi_0, \pi_0]\rangle \equiv \int [D\varphi_0 D\pi_0]\, W[\varphi_0, \pi_0]\; G_{xy}[\varphi_0, \pi_0]$$

We want to calculate the two point function by this Gaussian ensemble.

$$\langle G_{xy}[\varphi_0, \pi_0]\rangle \underset{?}{=} \langle 0_{\rm in}|\phi^\dagger(x)\phi(y)|0_{\rm in}\rangle_{\rm LO}$$

Computation of the two point function

2. Monte Carlo computation

Parameterize the initial values as

$$\varphi_0(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3 2E_q} \left[c_{\mathbf{q}}\,\varphi_{\mathbf{q}}(t_0,\mathbf{x}) + d_{\mathbf{q}}\,\varphi^*_{\mathbf{q}}(t_0,\mathbf{x})\right]$$

$$\pi_0(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3 2E_q} \left[c_{\mathbf{q}}\,\dot{\varphi}_{\mathbf{q}}(t_0,\mathbf{x}) + d_{\mathbf{q}}\,\dot{\varphi}^*_{\mathbf{q}}(t_0,\mathbf{x})\right]$$

with random c-numbers

$$\big\langle c_{\mathbf{q}} c^*_{\mathbf{q}'}\big\rangle = \big\langle d_{\mathbf{q}} d^*_{\mathbf{q}'}\big\rangle = (2\pi)^3 E_q \delta(\mathbf{q}-\mathbf{q}')\ , \quad \text{others} = 0$$

⬇

$$\big\langle G_{xy}[\varphi_0,\pi_0]\big\rangle = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3 2E_q}\Big[\varphi_{\mathbf{q}}(x)\varphi^*_{\mathbf{q}}(y) + \varphi_{\mathbf{q}}(y)\varphi^*_{\mathbf{q}}(x)\Big]$$

$$= \frac{1}{2}\big\langle 0_{\text{in}}\big|\phi^\dagger(x)\phi(y) + \phi(y)\phi^\dagger(x)\big|0_{\text{in}}\big\rangle_{\text{LO}}$$

Computation of the two point function

2. Monte Carlo computation

Parameterize the initial values as

$$\varphi_0(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3 2E_q} \left[c_\mathbf{q}\, \varphi_\mathbf{q}(t_0,\mathbf{x}) + d_\mathbf{q}\, \varphi^*_\mathbf{q}(t_0,\mathbf{x})\right]$$

$$\pi_0(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3 2E_q} \left[c_\mathbf{q}\, \dot\varphi_\mathbf{q}(t_0,\mathbf{x}) + d_\mathbf{q}\, \dot\varphi^*_\mathbf{q}(t_0,\mathbf{x})\right]$$

with ranc

$\langle c_\mathbf{q} c$

The result of the direct QFT calculation is reproduced except the ordering of the operators

$$\left\langle G_{xy}[\varphi_0,\pi_0]\right\rangle = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3 2E_q} \left[\varphi_\mathbf{q}(x)\varphi^*_\mathbf{q}(y) + \varphi_\mathbf{q}(y)\varphi^*_\mathbf{q}(x)\right]$$

$$= \frac{1}{2}\left\langle 0_{\rm in}\middle|\phi^\dagger(x)\phi(y) + \phi(y)\phi^\dagger(x)\middle|0_{\rm in}\right\rangle_{\rm LO}$$

Computation of the two point function

Monte Carlo computation (summary)

1. Solve up to $t = t_0$

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\varphi_{\mathbf{q}}(x) = 0 \qquad \lim_{x^0 \to -\infty} \varphi_{\mathbf{q}}(x) = e^{-iq\cdot x}$$

2. Construct initial conditions at $t = t_0$

$$\varphi_0(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3 2E_q} \left[c_{\mathbf{q}}\, \varphi_{\mathbf{q}}(t_0, \mathbf{x}) + d_{\mathbf{q}}\, \varphi^*_{\mathbf{q}}(t_0, \mathbf{x})\right]$$

$$\pi_0(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3 2E_q} \left[c_{\mathbf{q}}\, \dot\varphi_{\mathbf{q}}(t_0, \mathbf{x}) + d_{\mathbf{q}}\, \dot\varphi^*_{\mathbf{q}}(t_0, \mathbf{x})\right]$$

3. Solve for each ensemble

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\,\varphi = 0 \qquad \varphi(t_0, \mathbf{x}) = \varphi_0(\mathbf{x}),\ \ \dot\varphi(t_0, \mathbf{x}) = \pi_0(\mathbf{x})$$

4. Take the ensemble average

$$\left\langle \mathcal{O}\left[\varphi(t, \mathbf{x}), \dot\varphi(t, \mathbf{x})\right] \right\rangle$$

Computation of the two point function

Monte Carlo computation (summary)

1. Solve up to $t = t_0$

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\varphi_{\mathbf{q}}(x) = 0 \qquad \lim_{x^0 \to}$$

Numerical cost
$N_{\mathrm{ens}} \times N_{\mathrm{latt}}^2$

2. Construct initial conditions at $t = t$

$$\varphi_0(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3 2E_q} \left[c_{\mathbf{q}}\, \varphi_{\mathbf{q}}(t_0, \mathbf{x}) + d_{\mathbf{q}}\, \varphi^*_{\mathbf{q}}(t_0, \mathbf{x})\right]$$

$$\pi_0(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3 2E_q} \left[c_{\mathbf{q}}\, \dot{\varphi}_{\mathbf{q}}(t_0, \mathbf{x}) + d_{\mathbf{q}}\, \dot{\varphi}^*_{\mathbf{q}}(t_0, \mathbf{x})\right]$$

3. Solve for each ensemble

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\,\varphi = 0 \qquad \varphi(t_0, \mathbf{x}) = \varphi_0(\mathbf{x}),\ \ \dot{\varphi}(t_0, \mathbf{x}) = \pi_0(\mathbf{x})$$

4. Take the ensemble average

$$\left\langle \mathcal{O}\left[\varphi(t, \mathbf{x}), \dot{\varphi}(t, \mathbf{x})\right] \right\rangle$$

Computation of the two point function

Monte Carlo computation (summary)

1. Solve up to $t = t_0$

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\varphi_{\mathbf{q}}(x) = 0 \qquad \lim_{x^0 \to}$$

Numerical cost

$N_{\text{ens}} \times N_{\text{latt}}^2$

2. Construct initial conditions at $t = t_0$

$$\varphi_0(\mathbf{x}) = \int \frac{d^3 q}{(2\pi)^3 2E_q} \left[c_{\mathbf{q}}\, \varphi_{\mathbf{q}}(t_0, \mathbf{x}) + d_{\mathbf{q}}\, \varphi^*_{\mathbf{q}}(t_0, \mathbf{x})\right]$$

$$\pi_0(\mathbf{x}) = \int \frac{d^3 q}{(2\pi)^3 2E_q} \left[c_{\mathbf{q}}\, \dot\varphi_{\mathbf{q}}(t_0, \mathbf{x}) + d_{\mathbf{q}}\, \dot\varphi^*_{\mathbf{q}}(t_0, \mathbf{x})\right]$$

$N_{\text{ens}} \times N_t \times N_{\text{latt}}$

3. Solve for each ensemble

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\,\varphi = 0 \qquad \varphi(t_0, \mathbf{x}) = \varphi_0(\mathbf{x}), \;\; \dot\varphi(t_0, \mathbf{x}) = \pi_0(\mathbf{x})$$

4. Take the ensemble average

$$\left\langle \mathcal{O}\left[\varphi(t, \mathbf{x}), \dot\varphi(t, \mathbf{x})\right] \right\rangle$$

Computation of the two point function

Monte Carlo computation (summary)

1. Solve up to $t = t_0$

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\varphi_{\mathbf{q}}(x) = 0$$

Numerical cost

$$N_{\rm ens} \times N_{\rm latt}^2$$

2. Construct initial conditions at $t = t$

$$\varphi_0(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3 2E_q} \left[c_{\mathbf{q}}\, \varphi_{\mathbf{q}}(t_0, \mathbf{x}) + d_{\mathbf{q}}\, \varphi^*_{\mathbf{q}}(t_0, \mathbf{x})\right]$$

$$\pi_0(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3 2E_q} \left[c_{\mathbf{q}} \ldots\right]$$

$$N_{\rm ens} \times N_t \times N_{\rm latt}$$

3. Solve for each ensemble

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\,\varphi = 0 \qquad \varphi(t_0, \mathbf{x}) = \varphi_0(\mathbf{x}),\ \dot\varphi(t_0, \mathbf{x}) = \pi_0(\mathbf{x})$$

4. Take

Total cost

$$N_{\rm ens} \times N_{\rm latt} \times (N_{\rm latt} + N_t)$$

Computation of the two point function

Monte Carlo computation (summary)

1. Solve up to $t = t_0$

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\varphi_{\mathbf{q}}(x) = 0 \qquad \lim_{x^0 \to}$$

Numerical cost

$$N_{\text{ens}} \times N_{\text{latt}}^2$$

2. Construct initial conditions at $t = t$

$$\varphi_0(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3 2E_q} \left[c_{\mathbf{q}}\, \varphi_{\mathbf{q}}(t_0, \mathbf{x}) + d_{\mathbf{q}}\, \varphi^*_{\mathbf{q}}(t_0, \mathbf{x})\right]$$

$$\pi_0(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3 2E_q} \left[c_{\mathbf{q}}\, \dot\varphi_{\mathbf{q}}(t_0, \mathbf{x}) + d_{\mathbf{q}}\, \dot\varphi^*_{\mathbf{q}}(t_0, \mathbf{x})\right]$$

$$N_{\text{ens}} \times N_t \times N_{\text{latt}}$$

3. Solve for each ensemble

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\,\varphi = 0 \qquad \varphi(t_0, \mathbf{x}) = \varphi_0(\mathbf{x}),\ \dot\varphi(t_0, \mathbf{x}) = \pi_0(\mathbf{x})$$

4. Take

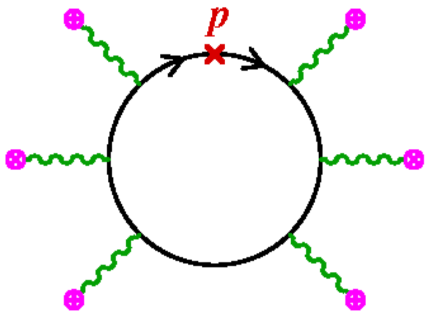
Total cost

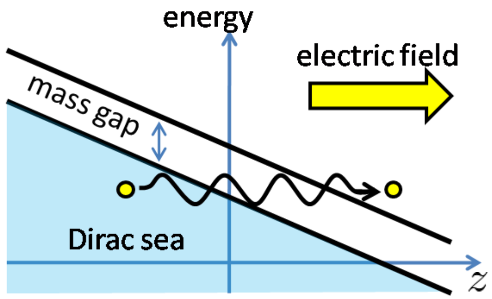
$$N_{\text{ens}} \times N_{\text{latt}} \times (N_{\text{latt}} + N_t) \ll N_t \times N_{\text{latt}}^2$$

if $N_{\text{ens}} \ll N_{\text{latt}}$ and $N_{\text{ens}} \ll N_t$

Schwinger mechanism

Non-perturbative particle pair production in a strong electric field





Schematics of semi-classical tunneling

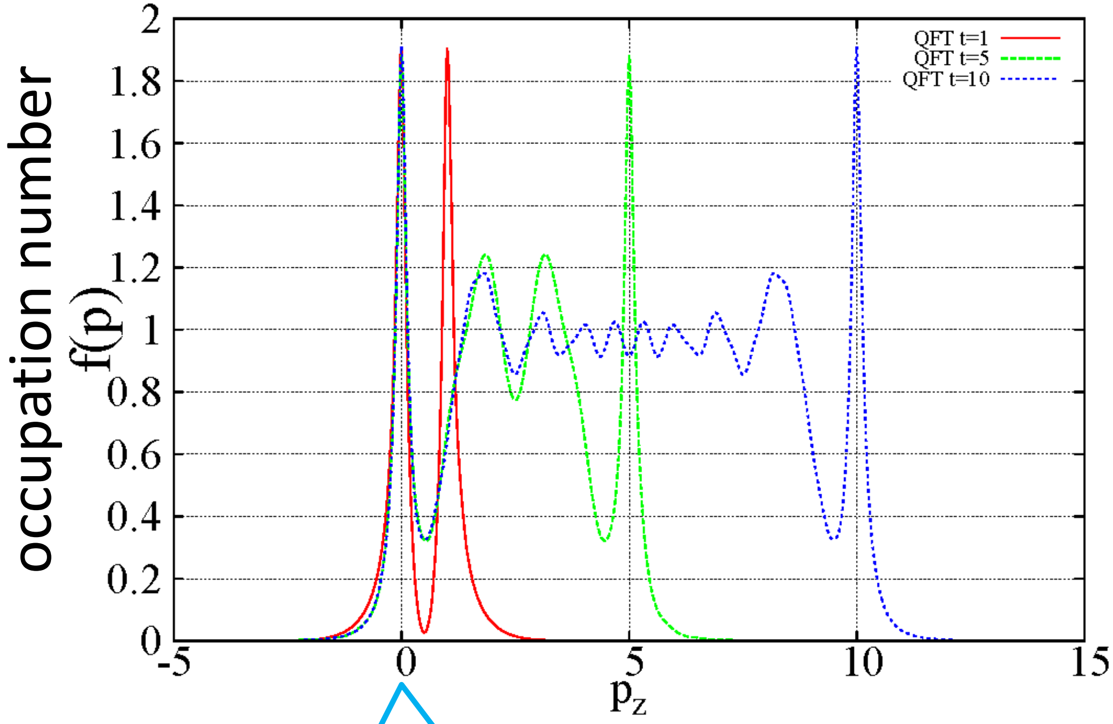
✓ A uniform electric field is turned on at $t = 0$

- The direct QFT calculation is feasible.
- A benchmark against the classical statistical computation

Constant electric field

Time-evolution of the longitudinal momentum distribution
a result by the direct QFT computation

$$eE = 1$$
$$m = 0.1$$

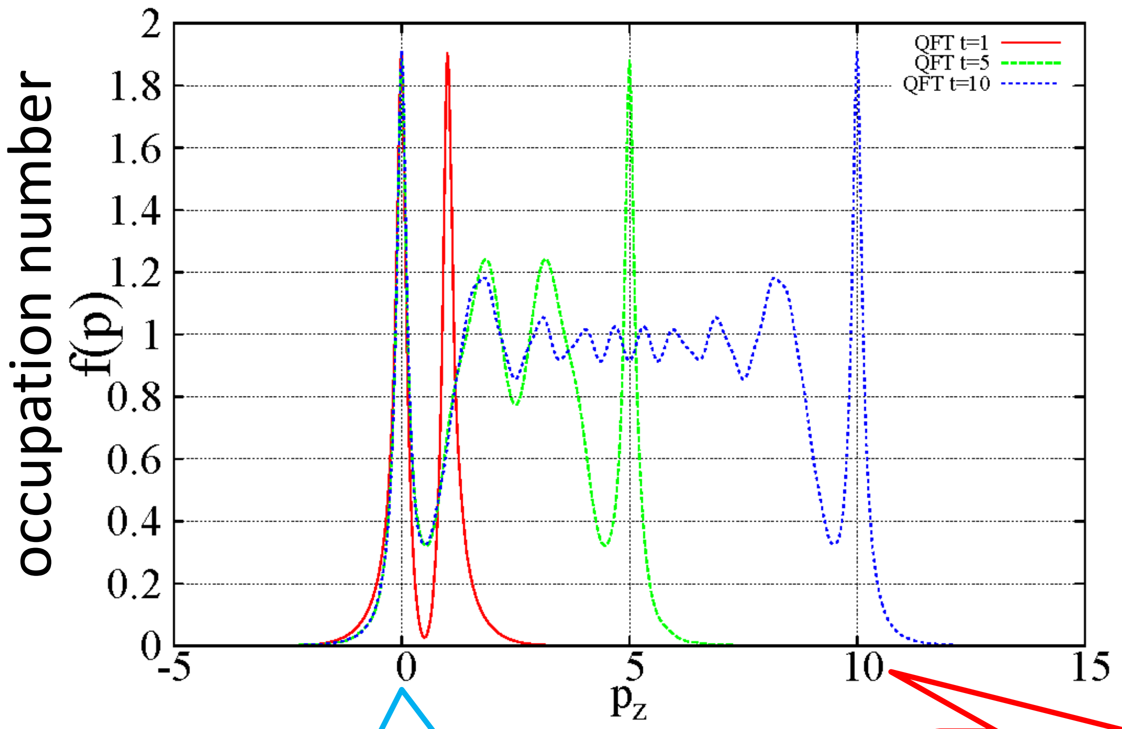


created with approximately
0 longitudinal momentum

Constant electric field

Time-evolution of the longitudinal momentum distribution
a result by the direct QFT computation

$$eE = 1$$
$$m = 0.1$$



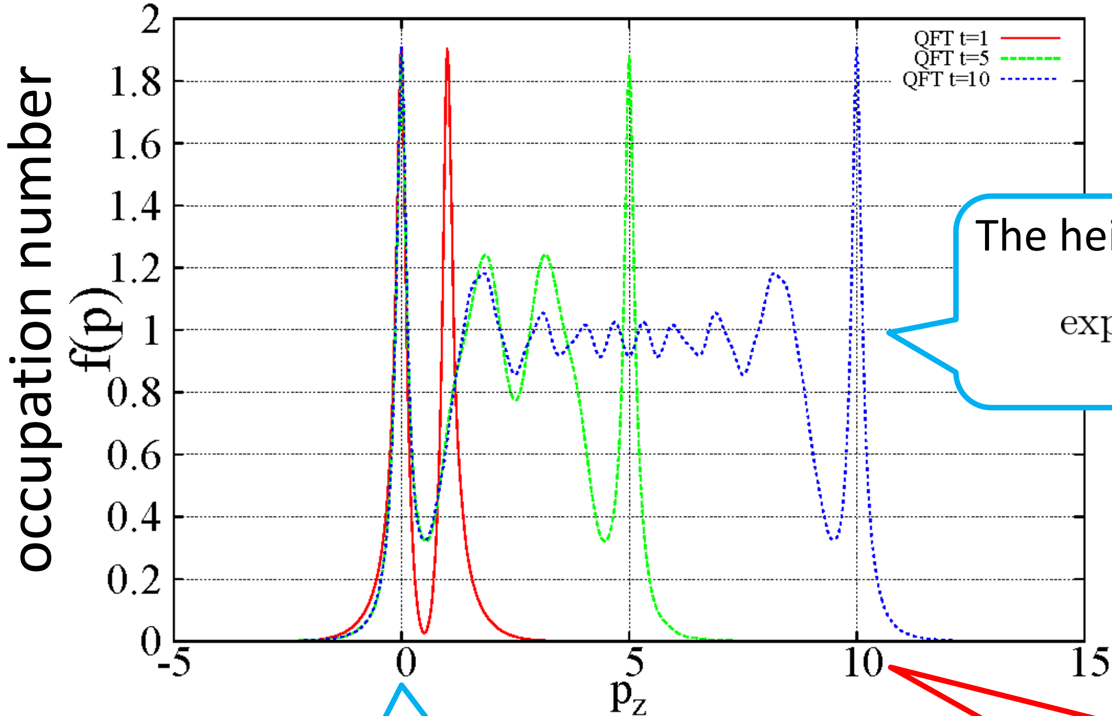
created with approximately 0 longitudinal momentum

accelerated according to classical eq. of motion
$$p_z = eEt$$

Constant electric field

Time-evolution of the longitudinal momentum distribution
a result by the direct QFT computation

$eE = 1$
$m = 0.1$



The height is given by $\exp\left(-\frac{\pi m_\perp^2}{eE}\right)$

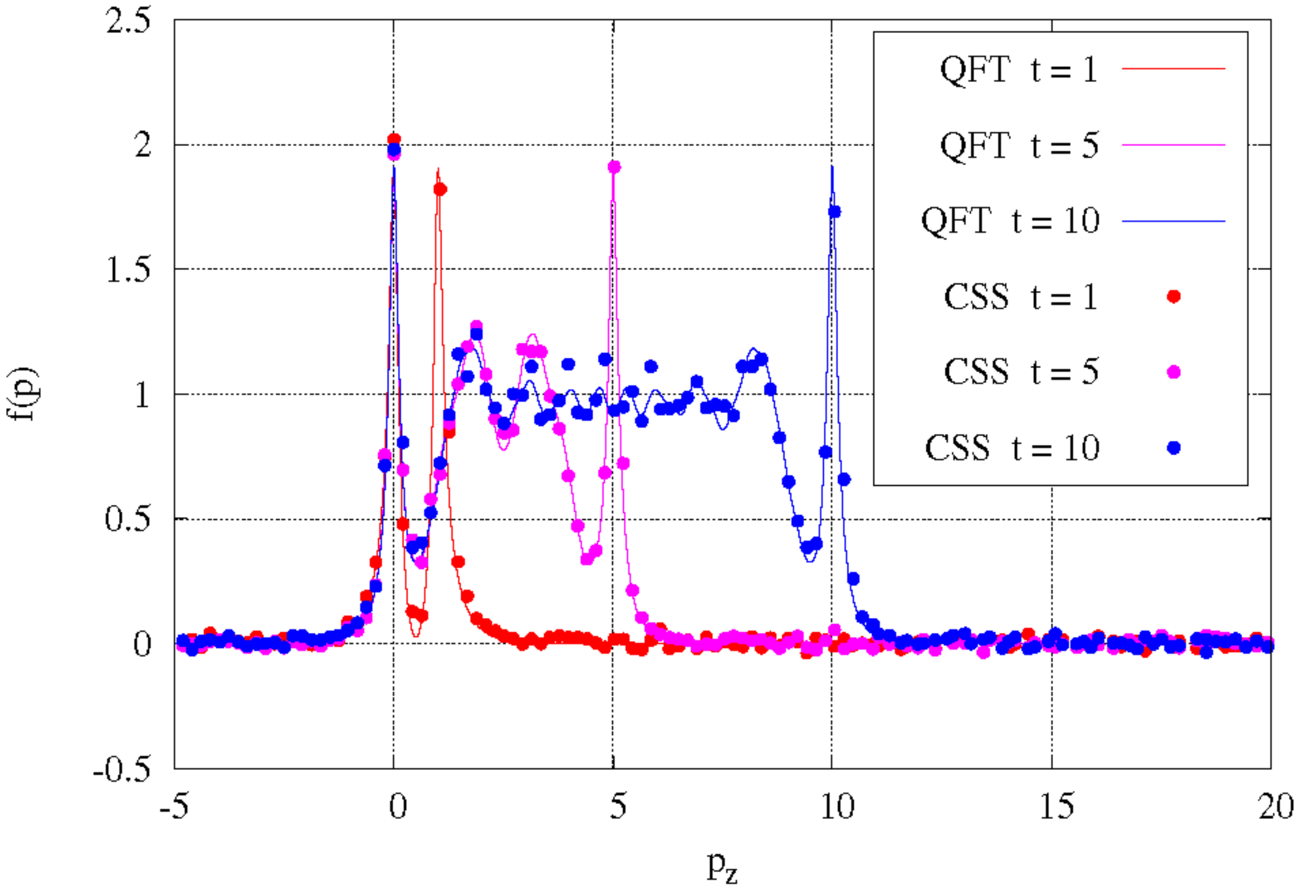
created with approximately 0 longitudinal momentum

accelerated according to classical eq. of motion $p_z = eEt$

Constant electric field

Time-evolution of the longitudinal momentum distribution
the direct QFT computation vs. the Classical Statistical Simulation

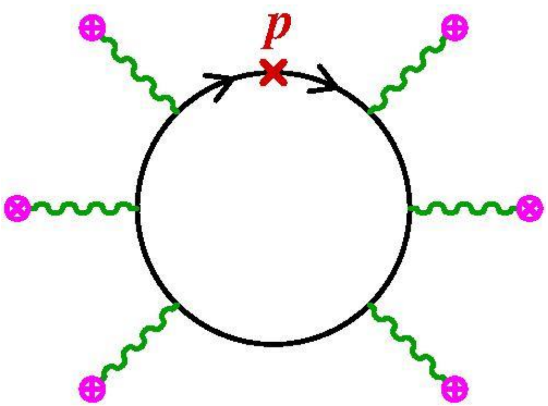
$$eE = 1$$
$$m = 0.1$$
$$N_x = N_y = 32$$
$$N_z = 256$$
$$N_{\text{ens}} = 1024$$



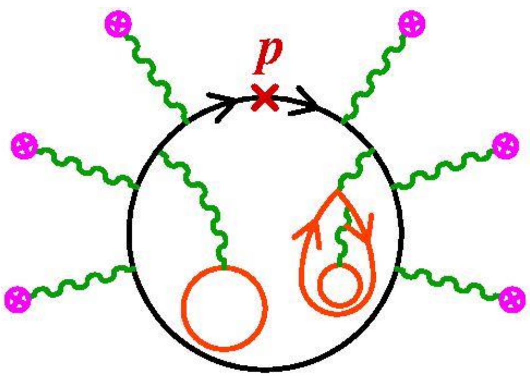
Back reaction

Solve the Maxwell eq. $\partial_\mu F^{\mu\nu} = \langle J^\nu \rangle$ with $\left(D^2 + m^2\right)\phi = 0$

Diagramatically,



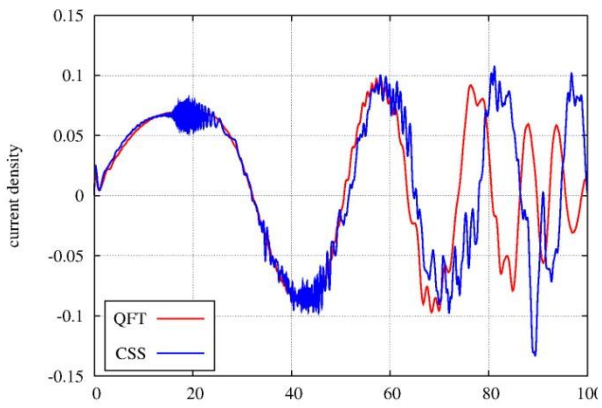
without back reaction



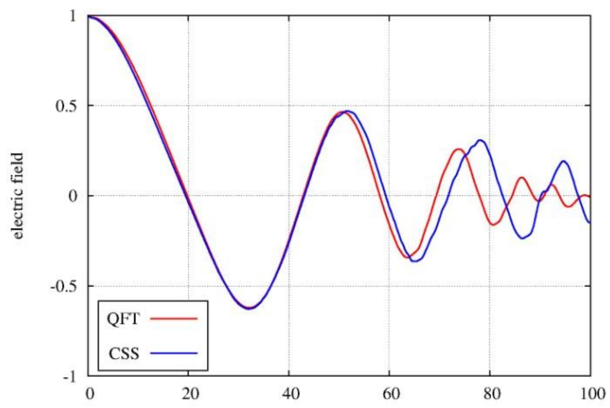
with back reaction

Back reaction

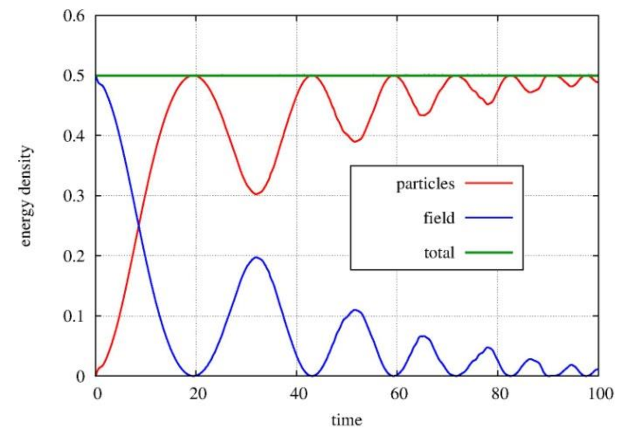
$$eE_0 = 1, e = 1, m = 0.1$$



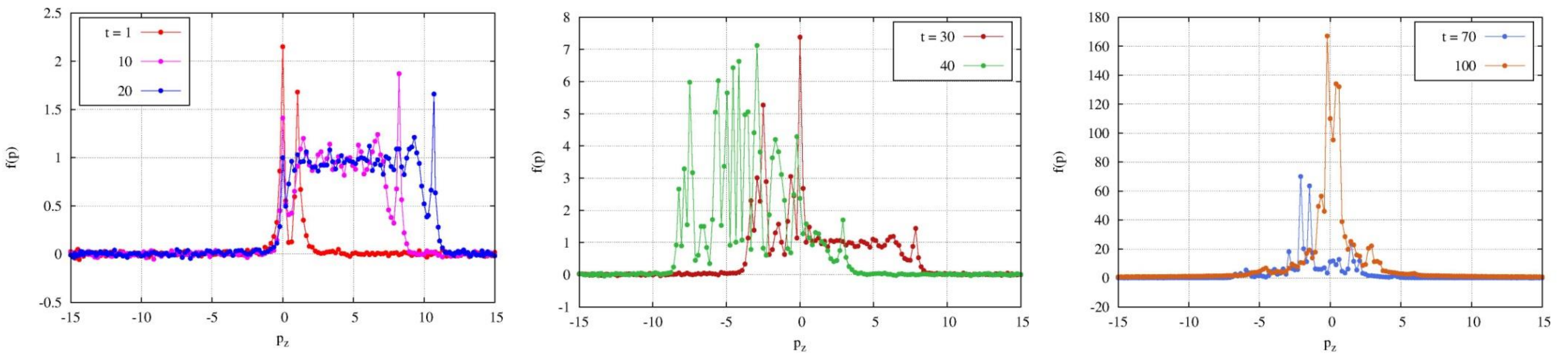
Time-evolution of current



Electric field



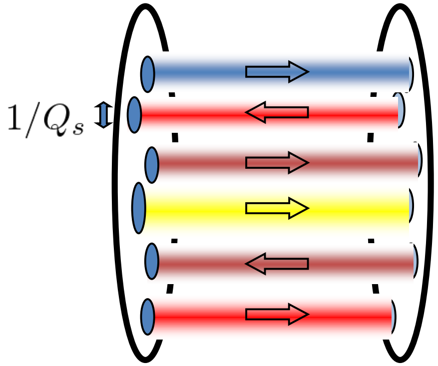
Energy balance



Time-evolution of the distribution function

Merits of the classical statistical method

- Applicable to non-uniform systems with a reasonable numerical cost



Glasma flux tubes

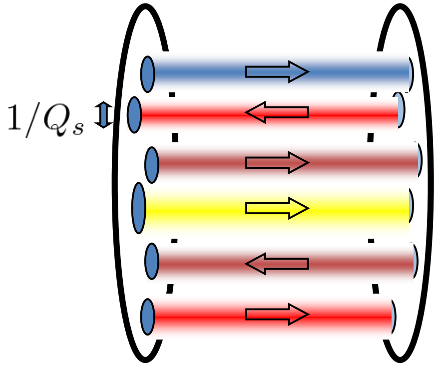
- Taking account of self-interactions is easy

Just add a self-interaction term to a classical equation of motion.

Need to consider renormalization

Merits of the classical statistical method

- Applicable to non-uniform systems with a reasonable numerical cost



Glasma flux tubes

- Taking account of self-interactions is easy

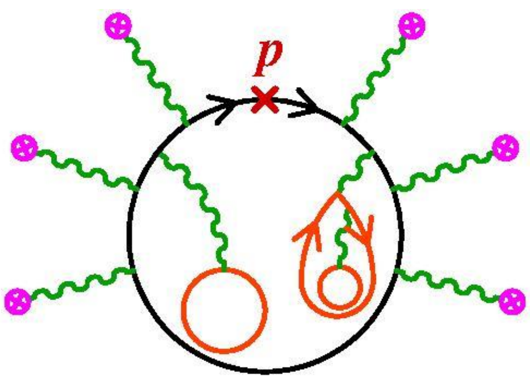
Just add a self-interaction term to a classical equation of motion.

Need to consider renormalization

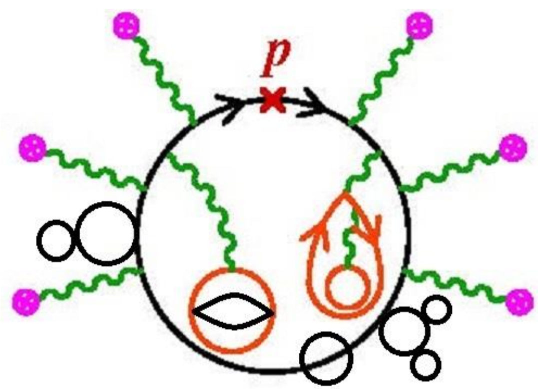
Effect of self-interactions

$$\left(D^2 + m^2\right)\phi + \frac{\lambda}{2}|\phi|^2\phi = 0 \qquad \partial_\mu F^{\mu\nu} = \langle J^\nu \rangle$$

Diagramatically,



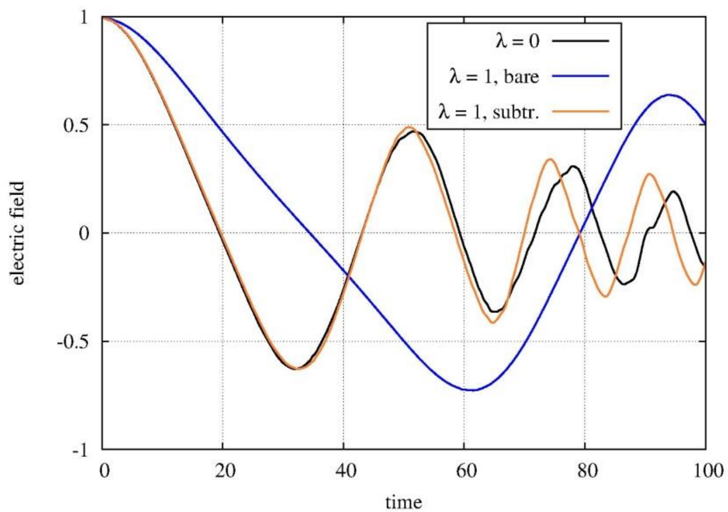
without self-interactions



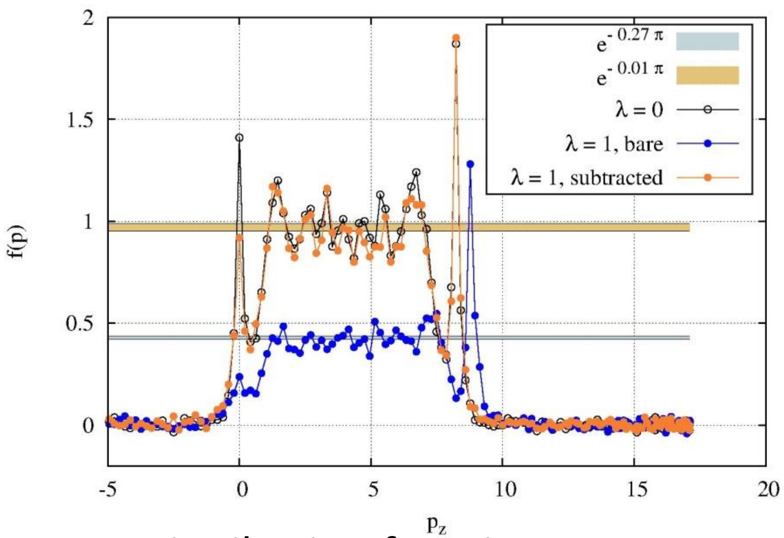
with self-interaction

Effect of self-interactions

$$eE_0 = 1, e = 1, m = 0.1$$

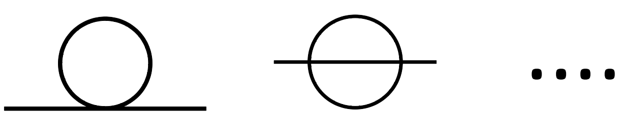


Time-evolution of the electric field



Distribution functions at t=10

Because of self energy, particles get heavier and their production is suppressed.

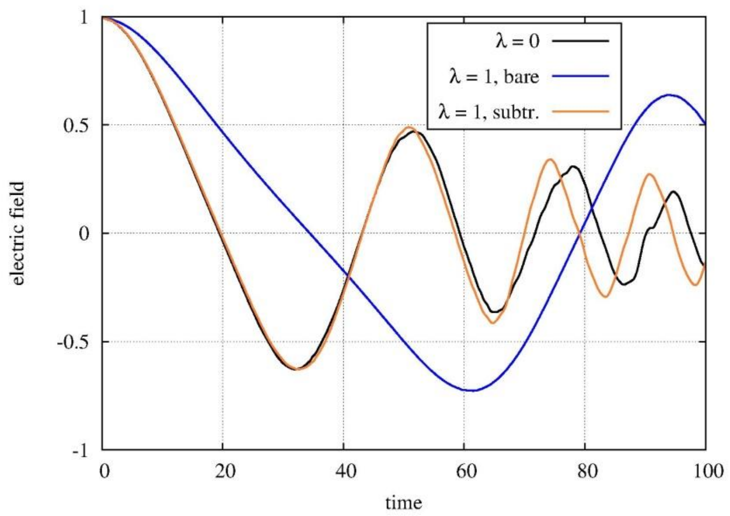


$$\exp\left(-\frac{\pi m_\perp^2}{eE}\right)$$

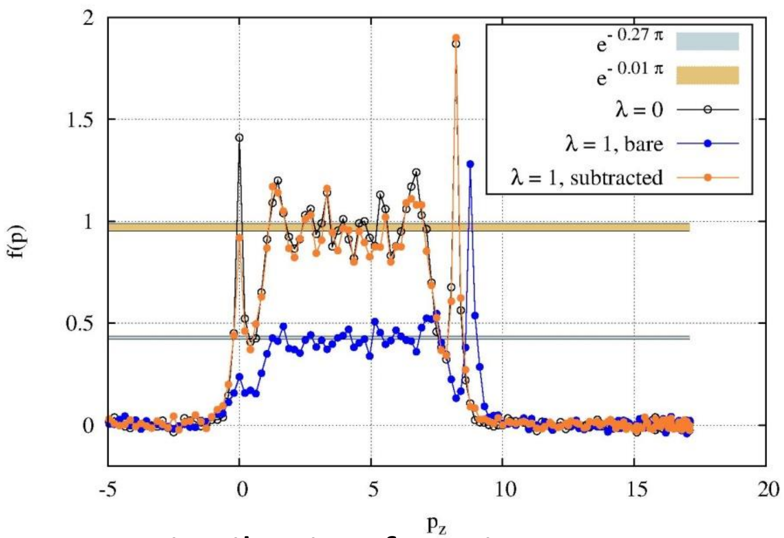
This self energy is finite on the lattice, but depends on the lattice parameters.
Also the results depend on the unphysical lattice parameters.

Effect of self-interactions

$$eE_0 = 1, e = 1, m = 0.1$$



Time-evolution of the electric field



Distribution functions at t=10

This problem is cured by doing renormalization of mass.

$$\left(D^2 + m_{\mathrm{B}}^2\right)\phi + \frac{\lambda}{2}|\phi|^2\phi = 0 \qquad m_{\mathrm{B}}^2 = m^2 + \delta m^2$$

A simple choice of the counter term

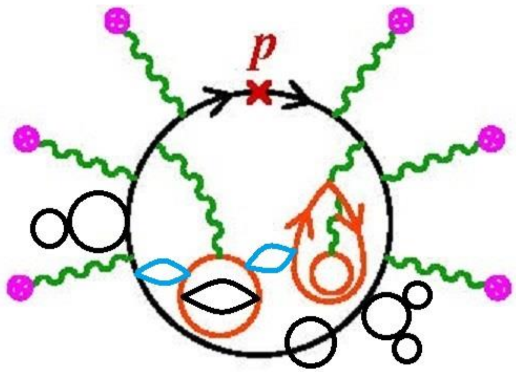
$$\delta m^2 = -\lambda\langle\phi^2(x)\rangle\Big|_{t=0}$$

Early time evolution is not affected by the self interaction.

Summary and outlook

- The classical statistical method gives exact results for a field with a quadratic Lagrangian
- Non-perturbative particle production by quantum tunneling can be described by the classical statistical method.
- A way of mass renormalization in the CSM was presented.

- ✓ Quark production in glasma
  reconsideration of Gelis-Kajantie-Lappi
- ✓ Renormalization condition
- ✓ Dynamical photon



Particle spectrum

$$\frac{dN_1}{d^3\mathbf{p}} = \langle 0_{\rm in}|a^\dagger_{\rm out}(\mathbf{p})a_{\rm out}(\mathbf{p})|0_{\rm in}\rangle$$
$$= \frac{1}{(2\pi)^3 2\omega_{\mathbf{p}}}\int d^4x d^4y\, e^{ip\cdot(x-y)}\,(\Box_x+m^2)(\Box_y+m^2)\,\langle 0_{\rm in}|\phi(x)\phi^*(y)|0_{\rm in}\rangle$$

LSZ reduction formula

$$= \lim_{t\to+\infty}\frac{1}{(2\pi)^3 2\omega_{\mathbf{p}}}\int d^3x d^3y\, e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \times\langle 0_{\rm in}|\left[\dot\phi(t,\mathbf{x})-i\omega_{\mathbf{p}}\phi(t,\mathbf{x})\right]\left[\dot\phi(t,\mathbf{y})+i\omega_{\mathbf{p}}\phi(t,\mathbf{y})\right]|0_{\rm in}\rangle$$

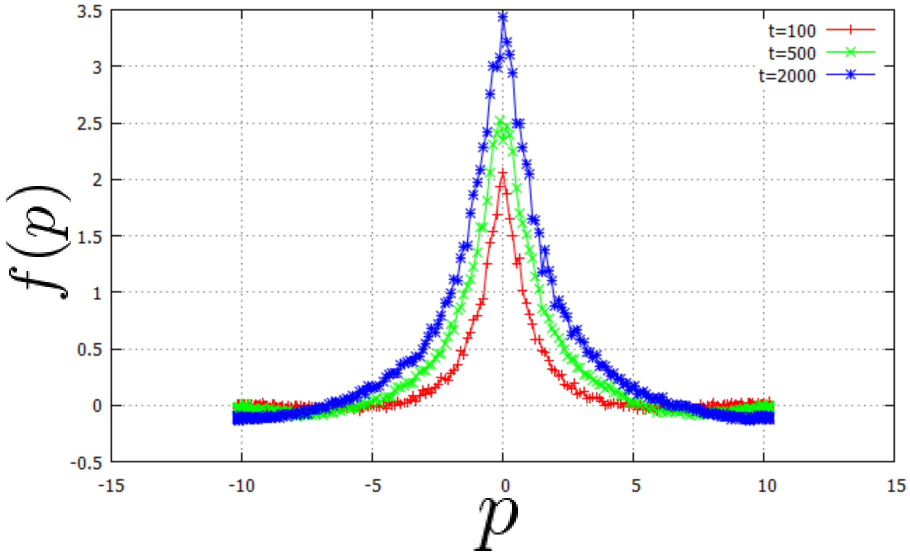
generalize

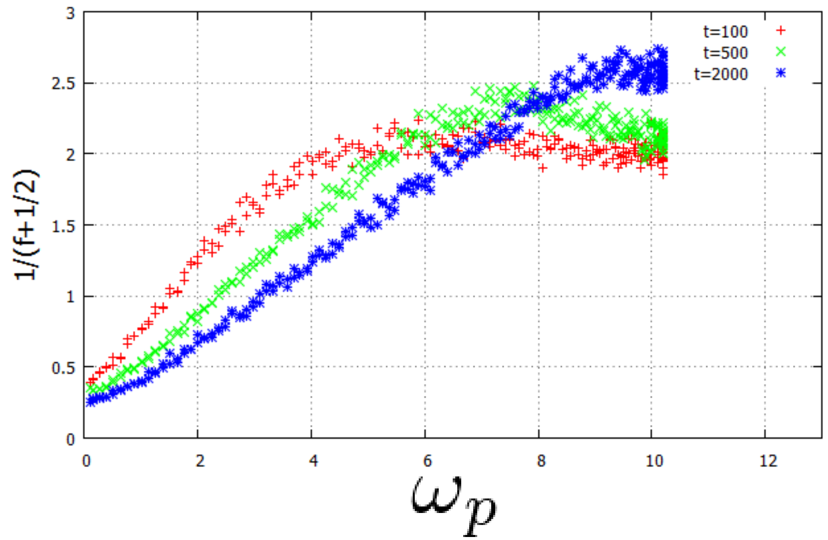
$$f(t,\mathbf{p}) = \frac{1}{2\omega_{\mathbf{p}}V}\int d^3x d^3y\, e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \times\langle 0_{\rm in}|\left[\dot\phi(t,\mathbf{x})-i\omega_{\mathbf{p}}\phi(t,\mathbf{x})\right]\left[\dot\phi(t,\mathbf{y})+i\omega_{\mathbf{p}}\phi(t,\mathbf{y})\right]|0_{\rm in}\rangle$$

“particle number” density at time t

Thermalization?

1+1dim.





“Classical” distribution

$$f(p) = \frac{T}{\omega_p} - \frac{1}{2} \quad \Rightarrow \quad \frac{1}{f(p)+1/2} = \frac{\omega_p}{T}$$

fixed point of the collision term derived from classical statistical field theory

c.f. Mueller-Son, 2004