Turbulent thermalization of the Quark Gluon Plasma

Soeren Schlichting
In collaboration with
J. Berges, K. Boguslavski, D. Sexty, R. Venugopalan

h3QCD ECT* Trento
18/06/13
Heavy-ion collisions

- fluid-like behavior from very early time on
- very special transport properties, such as small $\eta/s$

**How is local isotropization/thermalization achieved?**

**Progress in a first-principle understanding from two limiting cases**

**Holographic thermalization:**

a) strong coupling?  
Heller, Janik, Witaszczyk; Chesler, Yaffe ...

Sizeable anisotropy at transition to hydrodynamic regime

**Turbulent thermalization:**

b) weak coupling but highly occupied?  
CGC: McLerran, Venugopalan ...

Energy density of gluons with typical momentum $Q_s$ (at time $\sim 1/Q_s$)

$$\epsilon \sim \frac{Q_s^4}{\alpha_s}$$  
i.e. ‘occupation numbers’  
$$n(p \lesssim Q_s) \sim \frac{1}{\alpha_s}$$

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**Introduction**

Turbulent Thermalization  
Heavy Ion collisions at asymptotic energies

Soeren Schlichting | Univ. Heidelberg  
06/18/13
Non-equilibrium dynamics

Initial state: Far from equilibrium
Non-equilibrium dynamics
Final state: Thermal equilibrium

How is thermal equilibrium achieved?
Solve *Initial Value Problem* in QCD

- **Initial conditions:**
  
  based on *color glass condensate* (CGC) description of heavy ion collisions \((n(p)\sim 1/\alpha)\)

- **Non-equilibrium dynamics:**
  
  - Classical-statistical lattice simulations (SU(2) for numerical studies) \((n(p)>>1)\)
  
  - Kinetic theory \((n(p)<\sim 1/\alpha)\) (analytic discussion)
Non-equilibrium phenomena may be shared by a large class of strongly correlated many-body systems

I) Thermalization in scalar field theory – Cosmology
(Micha, Tkachev PRD 70 (2004) 043538)

II) Thermalization in Yang-Mills theory in Minkowski space

III) Thermalization in heavy-ion collisions at ultra-relativistic energies, weak coupling and large nuclei
(Berges, Boguslavski, SS, Venugopalana arXiv:1303.5650 [hep-ph])
The thermalization process is described by a *quasi-stationary evolution* with *scaling exponents*  

- Dynamic: $\alpha = -4/5$  
- Spectral: $\kappa = -3/2$  
- $\beta = -1/5$

*(Micha, Tkachev PRD 70 (2004) 043538)*
Wave turbulence


- Stationary solution with universal non-thermal scaling exponents

- Quasi-stationary solution with universal non-thermal spectral exponents

- Self-similar evolution with universal dynamical scaling exponents

Uriel Frisch, Turbulence. The Legacy of A. N. Kolmogorov. (CUP, 1995)
Evolution in kinetic theory

- Search for **self-consistent scaling solutions**
  
  \[ f(p, t) = t^\alpha \ f_S(t^\beta p) \quad \partial_t f(p, t) = C[f](p, t) \]

- Fixed point equation + Scaling relation

  \[ \alpha f_S(p) + \beta \partial_p f_S(p) = C[f_S](p) \quad \alpha - 1 = \mu(\alpha, \beta) \]

(Cosmology: Micha, Tkachev PRD 70 (2004) 043538)
Evolution in kinetic theory

- The self-similar evolution in time is characterized by *universal scaling exponents* fixed by

\[ \alpha - 1 = \mu(\alpha, \beta) + \text{conservation laws} \]

*Dynamic scaling relation*

<table>
<thead>
<tr>
<th>Interaction</th>
<th>Spectral Shape (Exponent (\kappa))</th>
<th>(\Lambda) evolution (Exponent (\beta))</th>
<th>Occupancy evolution (Exponent (\alpha))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2(-\rightarrow)1+soft</td>
<td>3/2</td>
<td>-1/5</td>
<td>-4/5</td>
</tr>
<tr>
<td>2(-\rightarrow)2</td>
<td>4/3</td>
<td>-1/7</td>
<td>-4/7</td>
</tr>
<tr>
<td>2(-\rightarrow)3</td>
<td>??</td>
<td>-1/7</td>
<td>-4/7</td>
</tr>
<tr>
<td>(gauge theory)</td>
<td></td>
<td></td>
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</tbody>
</table>

- **Scalar theory**: turbulent cascade is driven by 2\(-\rightarrow\)(1+soft) interaction and leads to a *transient condensate* formation

*(Cosmology: Micha, Tkachev PRD 70 (2004) 043538)*
Consider homogenous and *isotropic* systems which are initially *highly occupied* and initially characterized by a single momentum scale $Q$.

How does thermalization proceed? What are the relevant processes? Turbulence? Bose Einstein Condensation?

Occupation number

- Non-perturbative calculation – occupation number is a **gauge dependent** quantity

- Chose **temporal axial + Coulomb type gauge** to fix the residual gauge freedom – interpretation of physical degrees of freedom

- Define occupation number from **equal time correlation functions** in temporal axial + Coulomb type gauge

Turbulent Spectra

- Spectrum at **early times** consistent with $\kappa=3/2$ as in the scalar theory – Possible sign of condensation

- Spectrum at **late times** always closer to $\kappa=4/3$ - Inelastic processes much more efficient in gauge theory as compared to scalars

- The appearance of the exponent $\kappa=3/2$ also depends on initial conditions and **disappears for smaller initial over-occupation**

Dynamic Scaling

- Evolution at late times shows a **self-similar** behavior with dynamic scaling exponents
  \[ \alpha = -4/7 \]
  \[ \beta = -1/7 \]

- Consistent with both **elastic and inelastic** scattering processes

Turbulent thermalization

Thermalization of over-occupied systems proceeds as a *turbulent cascade* with a self-similar evolution associated to the presence of a *non-thermal fixed point*.

Schematically:

- **nonthermal fixed points:** $n(t,p) \sim t^n \eta_0(t^\theta p)$
- **self-similar evolution, turbulence, ...**
- **overpopulation**
- **initial conditions** $n(t=0,p)$
- **quantum effects**
- **relaxation:** $n(t,p) - n_T(p) \sim e^{-\gamma t}$
- **thermal equilibrium** $n_{BE}$

Does this picture hold for expanding systems as encountered in ultra-relativistic heavy-ion collisions?
There is a natural **competition** between **interactions** and the **longitudinal expansion** which renders the system **anisotropic** on large time scales.

**Longitudinal Expansion:**
- Red-shift of longitudinal momenta $p_z$ → increase of anisotropy
- Dilution of the system

**Interactions:**
- Isotropize the system
Different scenarios of how thermalization proceeds have been proposed in the literature:

- **Baier et al. (BMSS)**, PLB 502 (2001) 51-58

- **Kurkela, Moore (KM)**, JHEP 1111 (2011) 120

- **Blaizot et al. (BGLMV)**, Nucl. Phys. A 873 (2012) 68-80

Classical-statistical lattice simulations can be used to determine which solution is realized!
Consider a large class of different initial conditions.

- The anisotropy of the system increases due to the longitudinal expansion.

- The system remains strongly interacting throughout the entire evolution.

- At late times, the evolution becomes insensitive to the details of the initial conditions.

(Berges, Boguslavski, SS, Venugopalan arXiv:1303.5650 [hep-ph])
The typical *longitudinal momentum* of hard excitations exhibits a *universal scaling* behavior

\[ \Lambda_L^2/Q^2 \sim (Qt)^{-2\gamma} \]

\[ 2\gamma = 0.67 \pm 0.07 \]

The typical *transverse momentum* of hard excitations remains approximately *constant*

\[ \Lambda_T^2/Q^2 \sim (Qt)^{-2\beta} \]

\[ 2\beta \approx 0 \]
Expanding systems - Spectrum

**Transverse spectrum**

(Coulomb type gauge)

*“Thermal-like”* \( T/p_T \) spectrum with decreasing amplitude

\[ n_{\text{Hard}}(t) = f(p_T = Q, p_z = 0, t) \]

**Longitudinal spectrum**

(Coulomb type gauge)

- Decrease of the characteristic longitudinal momentum

- \( Qt = 750 \text{ to } 4000 \)
The spectrum of hard excitations shows a **self-similar evolution** characteristic of **wave turbulence** \( f(p_\perp, p_z, t) = t^\alpha f_S(t^\beta p_\perp, t^\gamma p_z) \)

**stationary fixed point distribution**
Non-thermal fixed point

**Boltzmann equation** with generic collision term for longitudinal expansion:

\[
\left[ \partial_t - \frac{p_z}{t} \partial_{p_z} \right] f(p_T, p_z, t) = C[p_T, p_z, t; f]
\]

**Self-similar evolution:**

\[
f(p_T, p_z, t) = t^\alpha f_S(t^\beta p_T, t^\gamma p_z), \quad C[p_T, p_z, t; f] = t^\mu C[t^\beta p_T, t^\gamma p_z; f_S]
\]

→ **a) fixed point equation for stationary distribution:**

\[
\alpha f_S(p_T, p_z) + \beta p_T \partial_{p_T} f_S(p_T, p_z + (\gamma - 1) p_z \partial_{p_z} f_S(p_T, p_z) = C[p_T, p_z; f_S]
\]

→ **b) scaling condition:**

\[
\alpha - 1 = \mu(\alpha, \beta, \gamma)
\]
Non-thermal fixed point

Interpret scaling condition with energy/number conserving* Fokker-Planck-type dynamics for the collisional broadening of longitudinal momentum distribution:

\[ C^{(elast)}[p_T, p_z; f] = \hat{q} \partial_{p_z}^2 f(p_T, p_z, t) \]

with momentum diffusion parameter:

\[ \hat{q} \sim \alpha^2 \int \frac{d^2 p_T}{(2\pi)^2} \int \frac{dp_z}{2\pi} f^2(p_T, p_z, t) \]

→ 1) \[ \mu = 3\alpha - 2\beta + \gamma \]
\[ \frac{\alpha - 1}{\mu(\alpha, \beta, \gamma)} \rightarrow 2\alpha - 2\beta + \gamma + 1 = 0 \]

2) number conservation
\[ \alpha - 2\beta - \gamma + 1 = 0 \]

3) energy conservation
\[ \alpha - 3\beta - \gamma + 1 = 0 \]

\[ \alpha = -2/3, \quad \beta = 0, \quad \gamma = 1/3 \]

remarkable agreement with lattice data!

*cf. early stages of Baier, Mueller, Schiff, Son, PLB 502 (2001) 51 (‘BMSS’)
The attractor solution

- Universal scaling behavior for different initial conditions
- Qualitative agreement with “bottom-up” thermalization scenario
Conclusions & Outlook

Classical-statistical lattice simulations demonstrate the existence of a non-thermal fixed point also for the expanding plasma at high energies.

- self-similar evolution with universal scaling exponents/functions characteristic for wave turbulence (novel universality class because of expansion)

- zero longitudinal pressure fixed point with ‘thermal‘ $1/p_T$ spectrum

- self-similar evolution leads to BMSS attractor in the classical regime

- after an initial transient regime dominated by plasma instabilities the evolution becomes independent of initial condition

What happens in the quantum regime? Does the weak coupling behavior survive at realistic values of the coupling constant?
## Backup - Definitions

### Gauge invariant definition of $\Lambda_L^2$:  
\[
\Lambda_L^2(\tau) = \frac{\langle \mathcal{H}_x^x(\tau) \rangle + \langle \mathcal{H}_y^y(\tau) \rangle - \langle \mathcal{H}_n^n(\tau) \rangle}{\epsilon(\tau)}
\]

\[
\mathcal{H}_{\mu}^\mu(\tau) = \frac{1}{V_L L_\eta} \int d^2 \bar{x}_\perp d\eta \ D^{ab}(x) F^{\alpha\mu}_b(x) D^{\beta\nu}_c(x) F_{\beta\mu}^c(x)
\]

### Perturbative expression:
\[
\Lambda_L^2(\tau) \approx \frac{\int d^2 \bar{p}_T \ dp_z \ p_z^2 \ \omega_p \ f(\bar{p}_T, p_z, \tau)}{\int d^2 \bar{p}_T \ dp_z \ \omega_p \ f(\bar{p}_T, p_z, \tau)}
\]

### Fock Schwinger + Coul. Gauge:  
\[
A_\tau = 0 + \tau^{-2} \partial_\eta A_\eta(x) + \sum_i \partial_i A_i(x)|_\tau = 0
\]

### Particle Number:  
\[
f(\bar{p}_T, p_z, \tau) = \frac{\tau^2}{N_g V_L L_\eta} \sum_{a=1}^{N_\tau-1} \sum_{\lambda=1,2} \left\langle \left| g^{\mu\nu} \left[ (\xi^{(\lambda)}_\mu \bar{p}_T \nu)(\tau))^* \stackrel{\leftrightarrow}{\partial_\tau} A_\nu^a(\tau, \bar{p}_T, \nu) \right] \right|^2 \right\rangle
\]
Mode functions in Fock Schwinger + Coul. Gauge:

\[
\xi^{(1)}_{\mu \vec{p}^{\pm}}(x) = \frac{\sqrt{\pi}}{2p_{\perp}} \frac{e^{\pm \pi \nu /2}}{p_{\perp}} \left( \begin{array}{c} -p_y \\ p_x \\ 0 \end{array} \right) H^{(2/1)}_{i\nu}(p_{\perp} \tau) \\
\xi^{(2)}_{\mu \vec{p}^{\pm}}(x) = \left( \begin{array}{c} \nu p_x / (p_{\perp}^2 \tau_{0}^2) R_{\perp}^{\vec{p}^{\pm}}(\tau) \\ \nu p_y / (p_{\perp}^2 \tau_{0}^2) R_{\perp}^{\vec{p}^{\pm}}(\tau) \\ -R_{\eta}^{\vec{p}^{\pm}}(\tau) \end{array} \right)
\]

\[
R_{\perp}^{\vec{p}^{\pm}}(\tau) = -\frac{p_{\perp} \tau_{0}}{\nu^2 + p_{\perp}^2 \tau_{0}^2} c_{2/1}^\pm H^{(2/1)}_{i\nu}(p_{\perp} \tau_{0}) + \int_{\tau_{0}}^{\tau} d\tau' \tau'^{-1} c_{2/1}^\pm H^{(2/1)}_{i\nu}(p_{\perp} \tau') ,
\]

\[
R_{\eta}^{\vec{p}^{\pm}}(\tau) = -\frac{p_{\perp} \tau_{0}}{\nu^2 + p_{\perp}^2 \tau_{0}^2} c_{2/1}^\pm H^{(2/1)}_{i\nu}(p_{\perp} \tau_{0}) + \int_{\tau_{0}}^{\tau} d\tau' \tau'^{-2} c_{2/1}^\pm H^{(2/1)}_{i\nu}(p_{\perp} \tau') .
\]
Backup – Bose Condensation