Strongly Interacting Fermions in Curved Space

- a "story" about geometry and chiral symmetry -

Antonino Flachi
CENTRA
IST - UTL - LISBON

ECT*
h_3QCD (high energy, high density and hot QCD)
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outline

Interacting Fermion Field Theories in curved space

heat-kernel & zeta-regularization techniques

2 examples

flat space with non-trivial boundary conditions

black holes

interacting fermions

examples

(Polyakov-)Nambu-Jona Lasinio model
Gross-Neveu model
Quark-Meson model

central in different contexts

particle/nuclear physics
condensed matter

share global symmetry of QCD
display the phenomena of chiral symmetry breaking
[in curved space]
Large-N allow (to some extent) analytic treatment

flat space

chiral symmetry is dynamically broken below T_c
dynamical mass generation [appearance of a condensate]

low density: spatially constant condensate
higher density: spatial variations

geography & morphology of the phase diagram

chiral density wave approach

Ginzburg-Landau

2D-GN

[Fukushima and Sasaki (2013)]

Curved space

manifolds:

$\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ $\mathbb{S}^2 \times \mathbb{S}^1$ $\mathbb{H}^2 \times \mathbb{S}^1$ $\mathbb{S}^3$

homogeneous shift of the [flat space] condensate

boundary-less
constant curvature
weakly curved
maximally symmetric

expanding cavities
boundary effects
strong curvature
singularities
AdS/CFT

Interacting Fermions

$$S = \int d^D x \sqrt{g} \left\{ \bar{\psi} i \gamma^\mu \nabla_\mu \psi + \frac{\lambda}{2N} \left(\bar{\psi}\psi\right)^2 + \cdots \right\}$$

$N = N_f \times N_c$ - large

λ is the coupling constant

$g_{\mu\nu}$ is the metric tensor

ψ is a $D \times N$ component Dirac Spinor

Geometries

- Ultrastatic geometries

  AF & T. Tanaka, JHEP 1102 (2011) 026
  AF, arXiv: 1304:6880

- Singular geometries [conifolds]

  AF, JHEP 1201 (2012) 023

- Boundaries

  AF, Phys. Rev. D86 (2012) 104047
  AF, Phys. Rev. Lett. 110 (2013) 060401

- Black Holes

  AF & T.Tanaka, Phys. Rev. D84 (2011) 061503
  AF, arXiv: 1305:5348

Ultra-static

D-dimensional ultrastatic manifold

$$ds^2 = dt^2 - g_{ij} dx^i dx^j$$

g_{ij} : metric of the spatial section

Bosonization [Hubbard-Stratonovich] $\sigma \sim \langle \bar{\psi}\psi \rangle$
Large-N

$$S_{eff} \;=\; -\int d^D x \sqrt{g} \left(\frac{\sigma^2}{2\lambda} \right) + \ln \mathrm{Det} \left(i\gamma^\mu \nabla_\mu - \sigma + \mu\gamma^0 \right)$$

Effective Action

$$S_{eff} = -\int d^D x \sqrt{g} \left(\frac{\sigma^2}{2\lambda} \right) + \delta\Gamma$$

$$\delta\Gamma = \frac{1}{2} \sum_{\epsilon} \sum_{n=-\infty}^{\infty} \ln \mathrm{Det}\, D^{(n)}$$

$$D^{(n)} \equiv -\Delta + \omega_n^2 + \frac{1}{4} R + \sigma^2 + \epsilon \left| \partial\sigma \right|$$

$$\omega_n = \frac{2\pi}{\beta} \left(n + \frac{1}{2} \right), \; \Delta = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j \right), \; \epsilon = \pm 1$$

Zeta Regularization

Define a generalised ζ-function [Mellin Transform of the heat-Trace]

$$\zeta(s) = \frac{1}{\Gamma(s)} \sum_{n,\epsilon} \int_0^\infty dt\, t^{s-1} \mathrm{Tr}\, e^{-tD^{(n)}}$$

then:

$$\delta\Gamma = \frac{1}{2} \int d^d x \sqrt{g} \left(\zeta(0) \ln \ell^2 + \zeta'(0) \right)$$

Using

$$\mathrm{Tr}\, e^{-tD^{(n)}} \doteqdot \frac{1}{(4\pi t)^{d/2}} \sum_{j=0}^{\infty} \vartheta_j\, t^j$$

returns:

$$S_{GL} = \frac{\alpha_2}{2}\sigma^2 + \frac{\alpha_4}{4}\left[\sigma^4 + (\nabla\sigma)^2\right] + \\ + \frac{\alpha_6}{6}\left[\sigma^6 + 5\,(\nabla\sigma)^2\,\sigma^2 + \frac{1}{2}\,(\Delta\sigma)^2\right] + \cdots$$

Resummation

partial resummation

$$\mathrm{Tr}\, e^{-tD^{(n)}} \;\breve{\doteq}\; \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-t\mathcal{Q}} \sum_k \mathcal{C}_\epsilon^{(k)}\, t^k$$

$$\mathcal{Q} = \omega_n^2 + R/12 + \sigma^2 - \mu^2 - 2i\mu\omega_n + \epsilon\,|\partial\sigma|$$

$$\begin{aligned} \mathcal{C}_\epsilon^{(0)} &= 1\,, \quad \mathcal{C}_\epsilon^{(1)} = 0\,, \\ \mathcal{C}_\epsilon^{(2)} &= \mathcal{R} + \frac{1}{6}\Delta\left(\sigma^2 + \epsilon\,|\partial\sigma|\right) \end{aligned}$$

$$\mathcal{R} = \frac{1}{180} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} - \frac{1}{120}\Delta R$$

resummation of powers of the condensate and of the scalar curvature

Zeta function

$$\zeta(s) = \frac{1}{\Gamma(s)} \sum_{k,\epsilon} \int_0^\infty dt \, \frac{t^{s-1+k}}{(4\pi t)^{\frac{d}{2}}} \mathcal{C}_\epsilon^{(k)} e^{-t\mathcal{X}_\epsilon} \mathcal{F}_{\beta,\mu}(t)$$

geometry

thermodynamics

zeta function assumes a factorized form

$$\mathcal{X}_\epsilon = \left(R/12 + \sigma^2 + \epsilon\,|\partial\sigma|\right)$$

$$\mathcal{F}_{\beta,\mu}(t) = \sum_{n=-\infty}^{\infty} e^{-t\left(\omega_n^2 - 2i\mu\omega_n - \mu^2\right)}$$

- convergent for $d/2 - k - 2 > s \in \mathbb{C}$
- analytical continuation to $s = 0$
- regular for small μ

$\zeta(0)$ and $\zeta'(0)$

$$
\begin{aligned}
\zeta(0) &= \frac{\beta}{(4\pi)^{D/2}} \sum_{\epsilon} \sum_{k=0}^{[D/2]} \gamma_k(D) \mathcal{C}_\epsilon^{(k)} \mathcal{X}_\epsilon^{D/2-k} \\
\zeta'(0) &= \frac{\beta}{(4\pi)^{D/2}} \sum_{\epsilon} \sum_{k=0}^{\infty} \Big(a_k(D) \mathcal{C}_\epsilon^{(k)} \mathcal{X}_\epsilon^{D/2-k} + \gamma_k(D) \mathcal{C}_\epsilon^{(k)} \mathcal{X}_\epsilon^{D/2-k} \ln \mathcal{X}_\epsilon \\
&+ 2^{D/2+1-k} \mathcal{C}_\epsilon^{(k)} \left(\mathcal{X}_\epsilon\right)^{D/4-k/2} \sum_{n=1}^{\infty} (-1)^n \frac{\cosh(\beta\mu n)}{(n\beta)^{D/2-k}} K_{k-D/2}\left(n\beta\sqrt{\mathcal{X}_\epsilon}\right) \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_k(D) &= \lim_{s\to 0} \frac{\Gamma(s+k-D/2)}{\Gamma(s)} , \\
a_k(D) &= \lim_{s\to 0} \frac{\Gamma(s+k-D/2)}{\Gamma(s)} \left(\psi^{(0)}(s+k-D/2) - \psi^{(0)}(s)\right)
\end{aligned}
$$

other cases

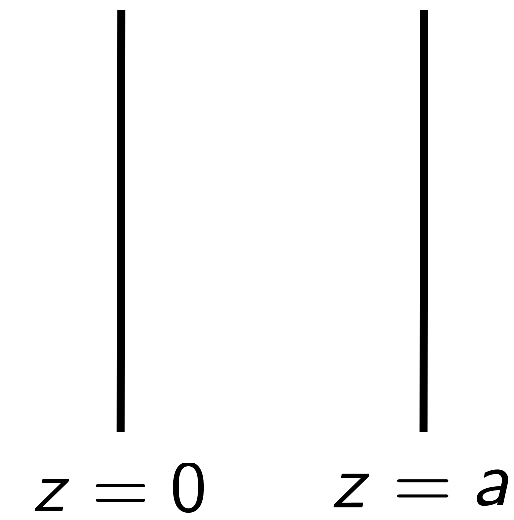
- phase dependent boundary conditions
- geometries with singularities
- different forms of resummation (CPTh)
- flat space with boundaries
- black holes

boundaries

boundaries

- free fermions [Dirac equation]
- impose boundary conditions at $z = 0, a$

$$(1 + \imath\gamma_z)\psi\Big|_{z=0,a} = 0$$



quantization

$$\Phi(k_z) := m \sin(k_z a) + k_z \cos(k_z a) = 0$$

vacuum energy

$$\mathcal{E} = -\lim_{s \to 0} \frac{\ell^{-2s}}{2\pi} \frac{\Gamma(s - 3/2)}{\Gamma(s - 1/2)} \sum_{k_z \in \{\Phi(k_z)=0\}} \left(k_z^2 + m^2\right)^{3/2-s}$$

boundaries

m=0 $\mathcal{E} = -7\pi^2 a^{-3}/2880$

m≠0
$$\mathcal{E} = -\frac{1}{a^3\pi^2}\int_0^\infty du u^{1/2}(u+\xi)(u+2\xi)^{1/2} \times \times \ln\left(1+\frac{u}{u+2\xi}e^{-2(u+\xi)}\right) + \cdots$$

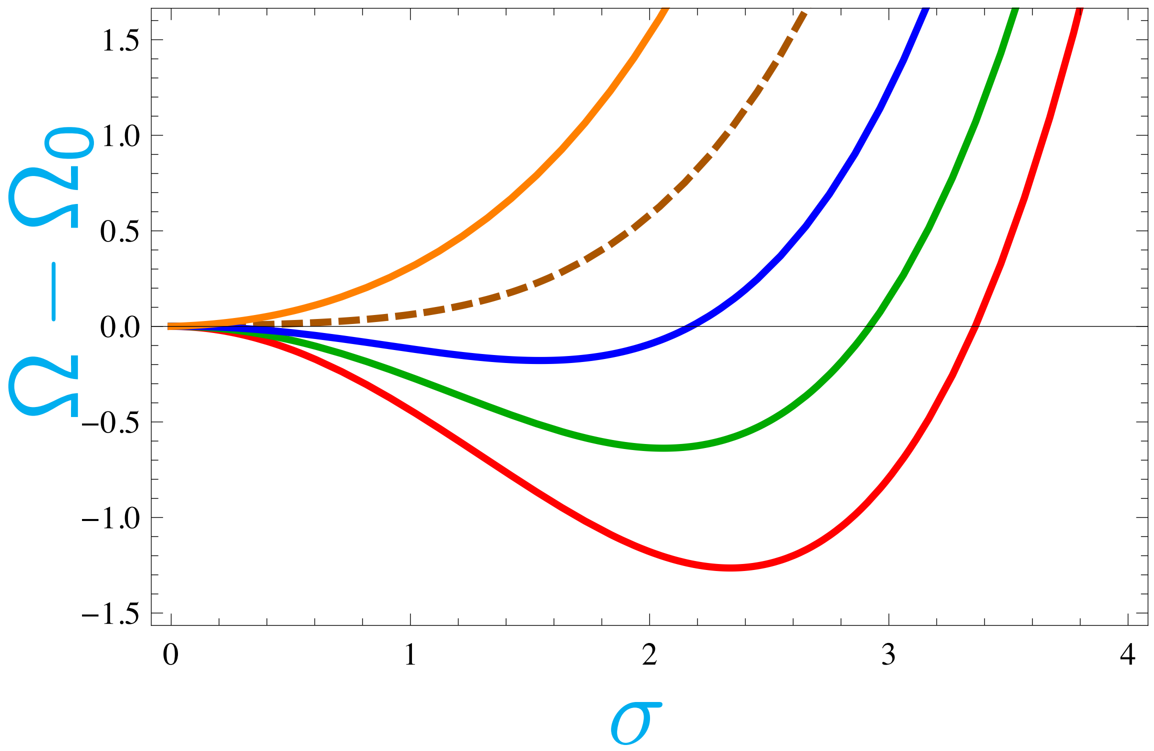
$\xi = ma$ is a “modulating” parameter for the vacuum energy

for free fields no modulations can occur

chiral symmetry breaking [interactions] can trigger modulations

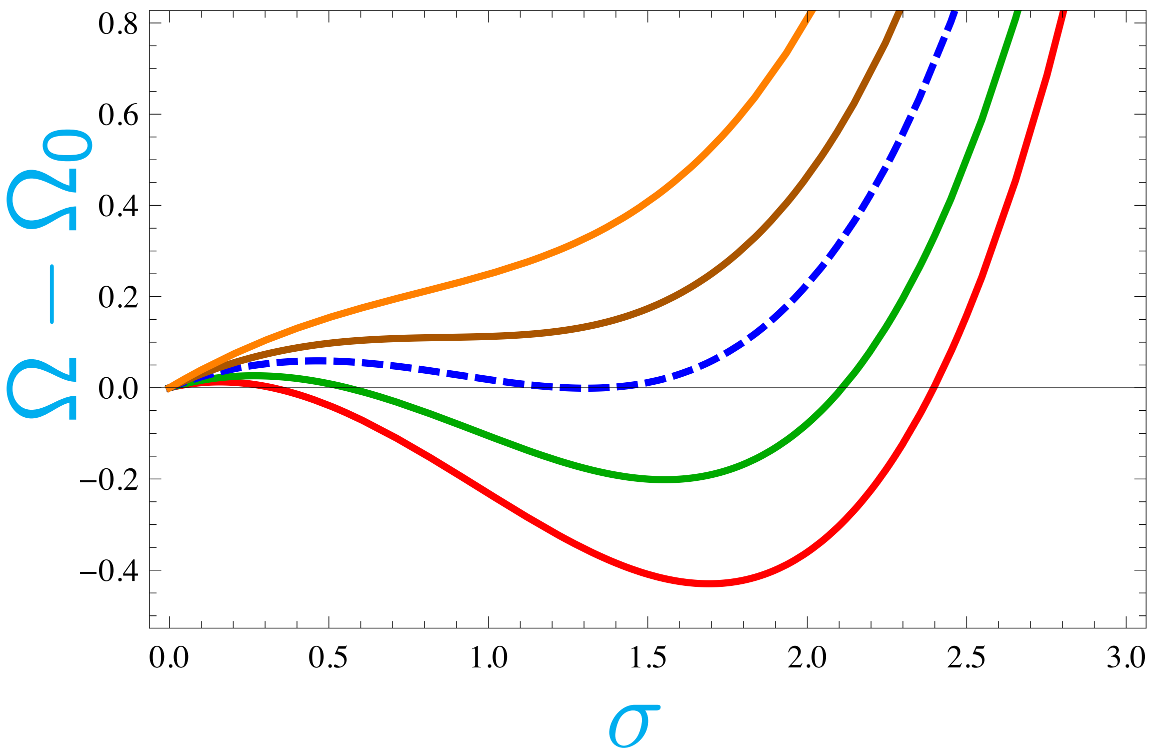
boundary effects “collaborate” with thermodynamical ones

Effective Potential



no boundaries

II order phase transition



boundaries

I order phase transition

Orders

[Ginzburg-Landau]

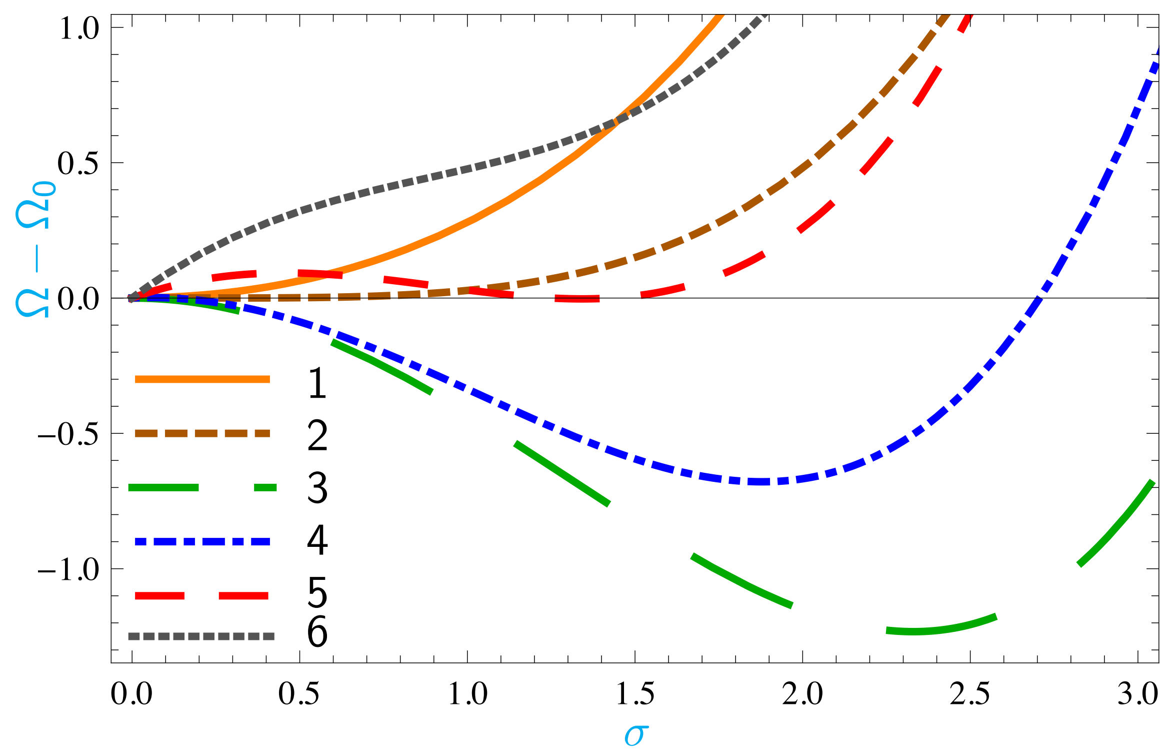
$$\Omega - \Omega_0 \quad = \quad c_0(a, T)\sigma^2 + c_1(a, T)\sigma^3 + c_2(a, T)\sigma^4 + \cdots$$

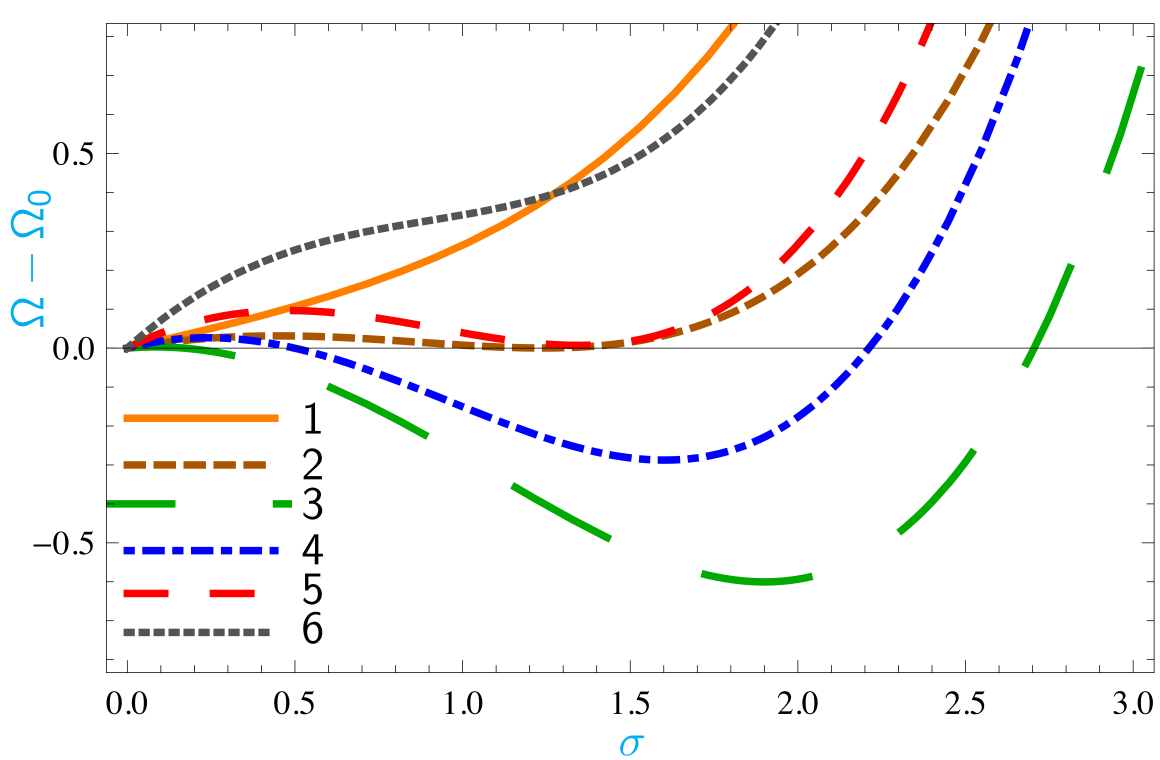
$$c_i(a, T) \sim \ \theta_{i/2}$$

in general depend on the geometry/topology and on the boundary conditions

- no boundaries, **σ⟷-σ** prohibits the appearance of odd powers of the condensate
- boundaries introduce another length scale and allow for odd powers
- [Schwinger-De Witt]

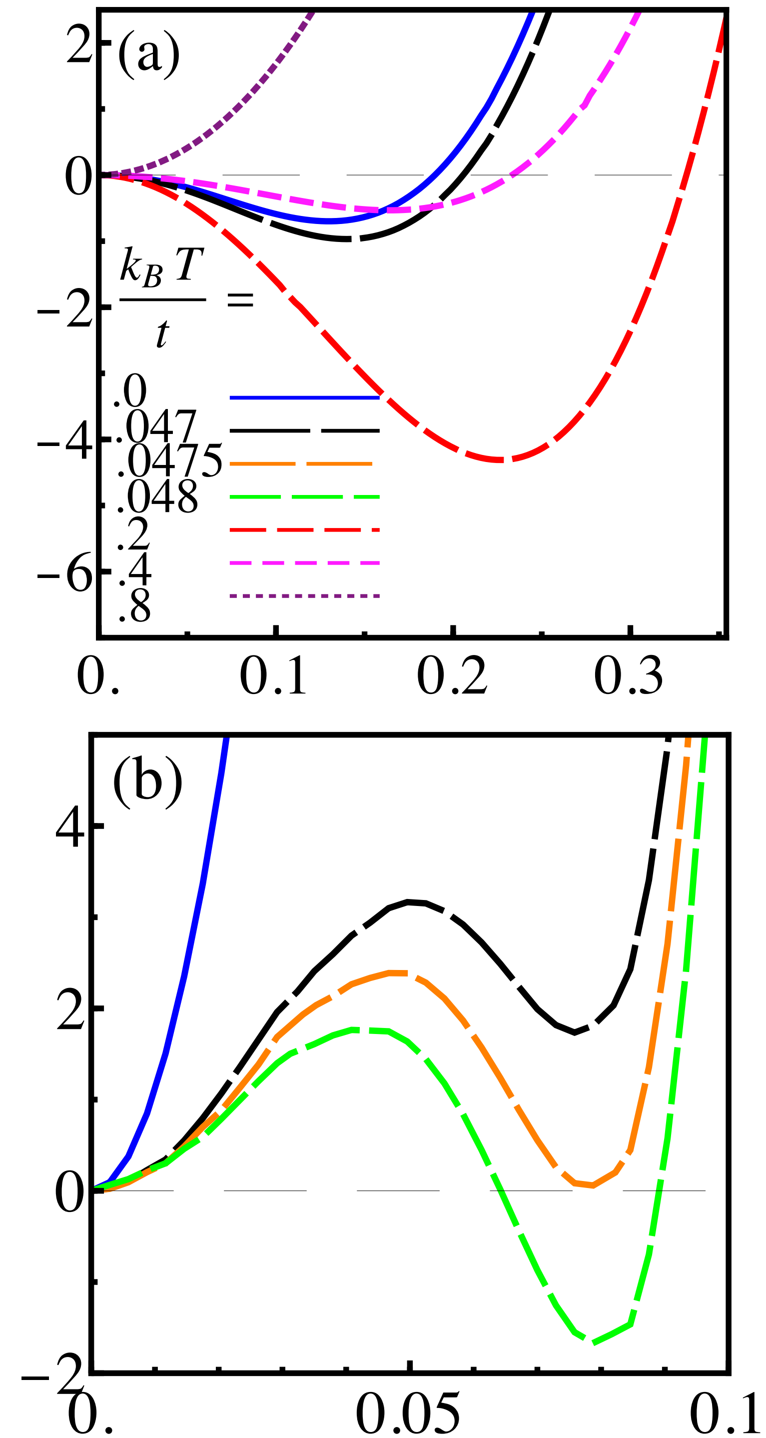
a-T





graphene by-layer with non-zero hopping

[Hosseini et al., Phys Rev. Lett. 2012]



black holes

Evaporation $T_{BH} \sim \frac{1}{m}$

$T_{BH} \ll m_e,$ photons, neutrinos, gravitons

$T_{BH} \sim m_e,$ electrons

$T_{BH} \sim \Lambda_{QCD},$ muons, pions, heavier hadrons

[heavier and expected to create a situation of approximate thermal equilibrium]

- Cosmology [**π**-bhs]
- Extra dimensions [**μ**-bhs]
- AdS/CFT [+branes]
- [hot] QCD

Hawking-Moss

- electroweak symmetry breaking
- As evaporation proceeds and T rises eventually a bubble of restored symmetry phase forms around the horizon
- In the Higgs model, the high temperature phase would be too localized implying [effectively] no symmetry restoration
- effect of trapped particles may change the above conclusion and a localized region of restored symmetry may in fact form

Chiral Symmetry & BHs

Schwarzschild bh of mass m surrounded by strongly interacting fermions

- Asymptotic temperature $T_{BH} = (8\pi m)^{-1}$
- Local Temperature $T_{loc} = T_{BH}/\sqrt{f} \quad f = 1 - 2m/r$

$T_{BH} > T_c$ chiral symmetry is restored

$T_{BH} < T_c$ chiral symmetry is broken

$T_{BH} < T_c$: T_{loc} crosses the critical temperature at some radius

a bubble of chirally restored symmetry phase should form

Chiral Symmetry & BHs

$$S = \int d^4x\sqrt{g}\left\{\bar{\psi}i\gamma^\mu\nabla_\mu\psi + \frac{\lambda}{2N}\left(\bar{\psi}\psi\right)^2\right\}$$

$$ds^2 = fdt^2 + f^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

- conformal transformation
- evaluation in the conformally transformed spacetime
- transform back

Chiral Symmetry & BHs

$$\Gamma = -\int d^4x\sqrt{g}\left(\frac{\sigma^2}{2\lambda}\right) + \hat{\Gamma} + \delta\Gamma$$

$$\hat{\Gamma} = \frac{\beta}{2(4\pi)^2}\sum_{\epsilon}\int d^3x\sqrt{\hat{g}}\left\{\frac{3\sigma_\epsilon^4}{4} - \left(\frac{\sigma_\epsilon^4}{2} + a_\epsilon\right)\ln\left(\frac{f\sigma_\epsilon^2}{\ell^2}\right)\right.$$

$$\left. + \; 16\frac{\sigma_\epsilon^2}{f\beta^2}\varpi_2(f^{\frac{1}{2}}\sigma_\epsilon) + 4a_\epsilon\varpi_0(f^{\frac{1}{2}}\sigma_\epsilon)\right\}$$

$$\varpi_\nu(u) := \sum_{n=1}^{\infty}(-1)^n n^{-\nu} K_\nu(n\beta u) \; ,$$

$$a_\epsilon := \frac{1}{180}\left(\hat{R}^2_{\mu\nu\tau\rho} - \hat{R}^2_{\mu\nu} - \hat{\Delta}\hat{R}\right) + \frac{1}{6}\hat{\Delta}\left(f\sigma_\epsilon^2\right)$$

Cocycle

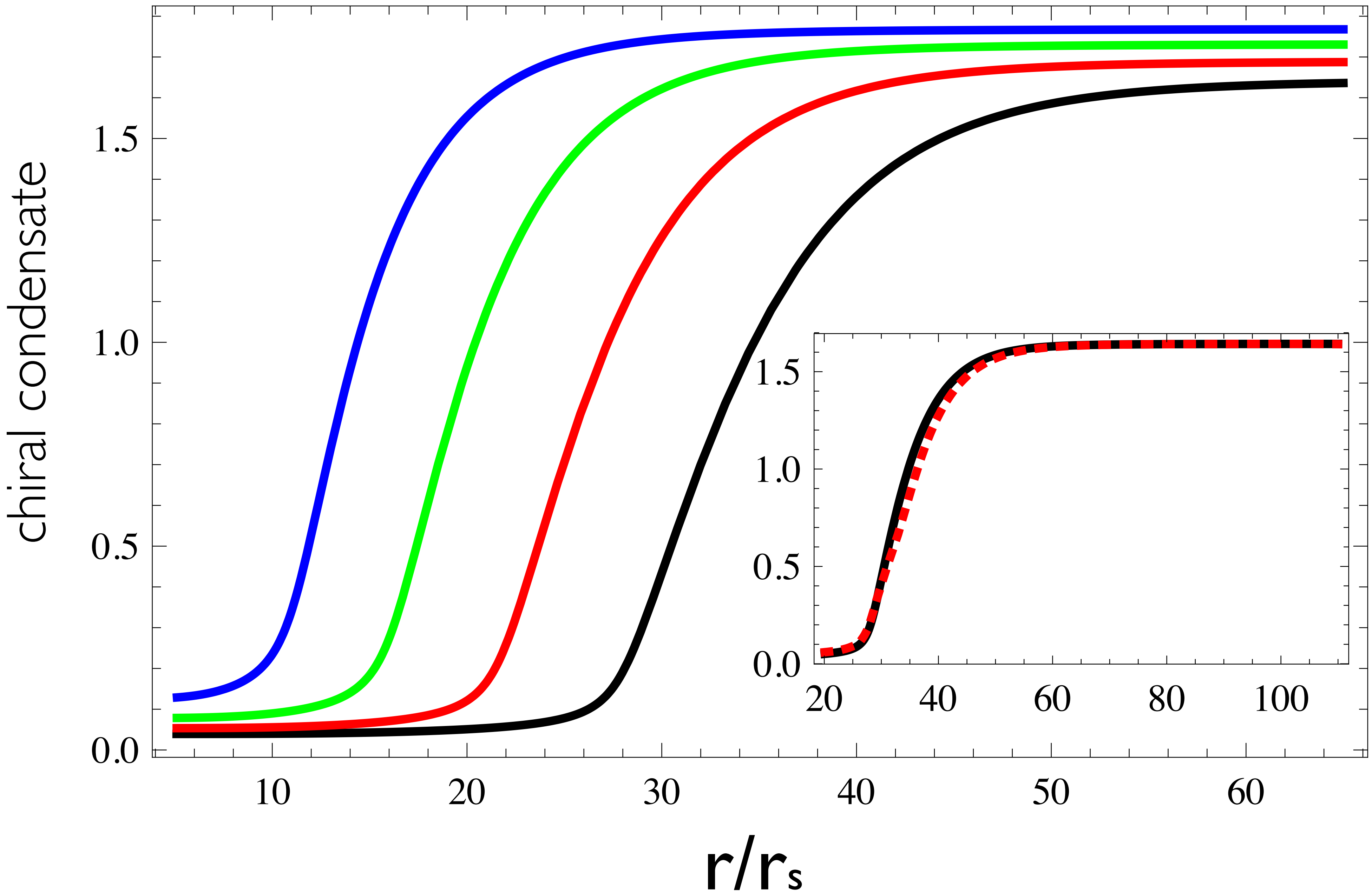
Co-cycle function [Dowker]

$$\delta\Gamma = \lim_{n\to 4}\left(C_n^{(2)}[\hat{g}] - C_n^{(2)}[g]\right)/(n-4)$$

$$\delta\Gamma \quad = \quad \frac{\beta}{2(4\pi)^2}\sum_{\epsilon=\pm}\int d^3x\sqrt{g}\left[\frac{\sigma_\epsilon^4}{2}\ln f - \frac{2\sigma_\epsilon^2}{f}\lim_{n\to 4}\frac{d\Lambda_n}{dn}\right]$$

$$\lim_{n\to 4} d\Lambda_n/dn = (f'^2 - 2ff'' + 4ff'/r)/24$$

Bubbles



- kink structure
- higher order corrections
- $r_{bubble} \sim r_s / \left(1 - T_{BH}^2 / T_c^2\right)$

Bubbles

- Chromosphere formation
- Hadronization and Hawking radiation [jets?]
- Black holes localized on branes

outlook

- External fields
- Back-Reaction
- Gluons
- Ginzburg - Landau ~ Schwinger - De Witt
- Lattice

curved space offers an interesting set-up to study
the physics of strongly interacting systems

thanks for the attention!