

# Solution of the NLO BFKL equation

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- Leading-Log-Approximation at high-energy: general case.
- BFKL in  $\mathcal{N}=4$  SYM theory.
- QCD: construction of the NLO eigenfunctions.
- Solution of the NLO BFKL equation.
- General Form of the Solution of Higher-Order BFKL equation.
- DGLAP anomalous dimension from solution of BFKL.
- Outlook.

## DGLAP evolution equation

- Resum  $\alpha_s \ln \frac{Q^2}{\Lambda_{\text{QCD}}}$
- Eigenfunctions at any order: Powers of  $x_B$

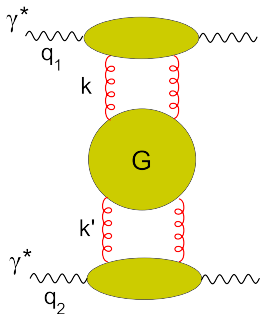
## DGLAP evolution equation

- Resum  $\alpha_s \ln \frac{Q^2}{\Lambda_{\text{QCD}}}$
- Eigenfunctions at any order: Powers of  $x_B$

## BFKL evolution equation

- LO: resum  $(\alpha_s \ln s)^n$ . NLO: resum  $\alpha_s (\alpha_s \ln s)^n$
- Eigenfunctions:
  - LO:  $(k^2)^{\gamma-1}$
  - NLO and higher order: Obtained recently (result of this talk).

# Balitsky-Fadin-Kuraev-Lipatov equation



$$\frac{\partial}{\partial Y} G(k, k', Y) = \int d^2 q K(k, q) G(q, k', Y)$$

$$G(k, k', Y = 0) = \frac{1}{2\pi k} \delta(k - k')$$

$$k \equiv |\vec{k}_\perp| \quad \text{and} \quad k' \equiv |\vec{k}'_\perp|$$

$$Y = \ln \frac{s}{k k'} \quad \text{and} \quad s = (q_1 + q_2)^2$$

- Resum  $(\alpha_s Y)^n \rightarrow$  LO BFKL eq.
- Resum  $\alpha_s (\alpha_s Y)^n \rightarrow$  NLO BFKL eq.
- The kernel is real and symmetric:  $K(k, k') = K(k', k) \Rightarrow K(k, k')$  is Hermitian and the eigenvalues are real.

$$\frac{\partial}{\partial Y} G(k, k', Y) = \int d^2 q K^{\text{LO}}(k, q) G(q, k', Y)$$

$$\int d^2 q K^{\text{LO}}(k, q) (q^2)^{1-\gamma} = \bar{\alpha}_\mu \chi_0(\gamma) (k^2)^{1-\gamma} \quad \bar{\alpha}_\mu \equiv \frac{\alpha_\mu N_c}{\pi}$$

- $(k^2)^{1-\gamma}$  are eigenfunctions.
- For  $\gamma = \frac{1}{2} + i\nu$  and  $\nu$  real parameter  $\Rightarrow (k^2)^{1-\gamma}$  form a complete set.
- $\Rightarrow$  LO eigenvalues  $\chi_0(\nu) = 2\psi(1) - \psi(\frac{1}{2} + i\nu) - \psi(\frac{1}{2} - i\nu)$  are real and sym.  $\nu \leftrightarrow -\nu$
- LO BFKL is Conformal invariant.

$$G(k, k', Y) = \int \frac{d\nu}{2\pi^2 k k'} \left( \frac{k^2}{k'^2} \right)^{i\nu} e^{\bar{\alpha}_\mu \chi_0(\nu) Y}$$

- In  $\mathcal{N} = 4$  SYM theory the coupling constant does not run.
- $\Rightarrow (k^2)^{-\frac{1}{2}+i\nu}$  are eigenfunctions at any order.

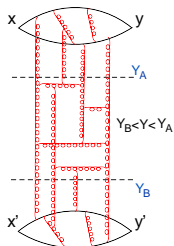
$$K(q, k) = \bar{\alpha}_\mu K^{\text{LO}}(q, k) + \bar{\alpha}_\mu^2 K^{\text{NLO}}(q, k) + \dots$$

$$\int d^2 q K(q, k) (q^2)^{-\frac{1}{2}+i\nu} = [\alpha_s \chi_0(\nu) + \alpha_s^2 \chi_1(\nu) \dots] (k^2)^{-\frac{1}{2}+i\nu}$$

$$G(k, k', Y) = \int \frac{d\nu}{2\pi^2 k k'} e^{[\alpha_s \chi_0(\nu) + \alpha_s^2 \chi_1(\nu) \dots]} \left( \frac{k^2}{k'^2} \right)^{i\nu}$$

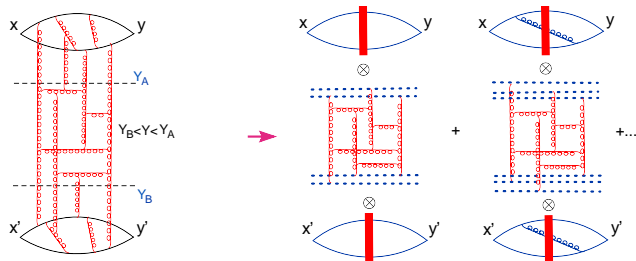
- The eigenvalues  $\bar{\alpha}_\mu \chi_0(\nu) + \bar{\alpha}_\mu^2 \chi_1(\nu) + \dots$  are real and symmetric for  $\nu \leftrightarrow -\nu$ .

Factorization in rapidity and composite conformal operator (see Balitsky talk)

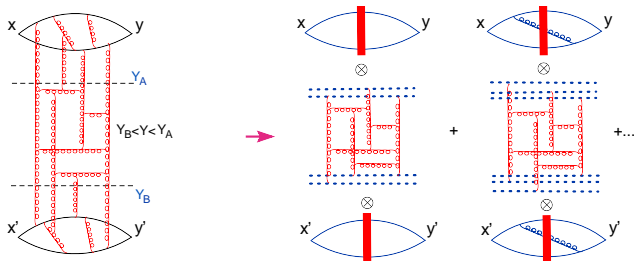




Factorization in rapidity and composite conformal operator (see Balitsky talk)



Factorization in rapidity and composite conformal operator (see Balitsky talk)



$$\begin{aligned}
 & (x-y)^4 (x'-y')^4 \langle T \{ \hat{\mathcal{O}}(x) \hat{\mathcal{O}}^\dagger(y) \hat{\mathcal{O}}(x') \hat{\mathcal{O}}^\dagger(y') \} \rangle \\
 &= \int d^2 z_{1\perp} d^2 z_{2\perp} d^2 z'_{1\perp} d^2 z'_{2\perp} \mathbf{IF}^{a_0}(x, y; z_1, z_2) [\mathbf{DD}]^{a_0, b_0}(z_1, z_2; z'_1, z'_2) \mathbf{IF}^{b_0}(x', y'; z'_1, z'_2)
 \end{aligned}$$

$a_0 = \frac{x^+ y^+}{(x-y)^2}, b_0 = \frac{x'^- y'^-}{(x'-y')^2} \Leftrightarrow$  impact factors do not scale with energy  
 $\Rightarrow$  all energy dependence is contained in  $[\mathbf{DD}]^{a_0, b_0}$

$$K^{\text{LO+NLO}}(k, q) \equiv \bar{\alpha}_\mu K^{\text{LO}}(k, q) + \bar{\alpha}_\mu^2 K^{\text{NLO}}(k, q)$$

$$\int d^2q K^{\text{LO+NLO}}(k, q) q^{2\gamma-2} = \left[ \bar{\alpha}_\mu \chi_0(\gamma) - \bar{\alpha}_\mu^2 \beta_2 \chi_0(\gamma) \ln \frac{k^2}{\mu^2} + \bar{\alpha}_\mu^2 \frac{\delta(\gamma)}{4} \right] k^{2\gamma-2}$$

$$\bar{\alpha}_\mu = \frac{\alpha_\mu N_c}{\pi}, \quad \beta_2 = \frac{11 N_c - 2 N_f}{12 N_c}$$

- $-\bar{\alpha}_\mu^2 \beta_2 \chi_0(\gamma) \ln \frac{k^2}{\mu^2}$  1-loop running coupling.
- $\delta(\gamma) = -2 \beta_2 \chi_0'(\gamma) + 4 \chi_1(\gamma)$  NLO Conformal terms Fadin-Lipatov (1998)
- $\chi_1(\gamma)$  Real and symmetric in  $\gamma \leftrightarrow 1 - \gamma$ .
- $\frac{d}{d\gamma} \chi_0(\gamma) \equiv \chi_0'(\gamma)$  imaginary and not symmetric.

Perturbative eigenfunctions  $H_\gamma(k) = k^{2\gamma-2} + \bar{\alpha}_\mu F_\gamma(k)$

- we have to determine  $F_\gamma(k)$  so that

$$\int d^2q K^{\text{LO+NLO}}(k, q) H_\gamma(q) = \Delta(\gamma) H_\gamma(k)$$

- $\Delta(\gamma)$  eigenvalues to be determined.

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- $\Delta(\gamma)$  eigenvalues to be determined.
- **Ansatz**

$$F_\gamma(k) = k^{2\gamma-2} \left[ c_0(\gamma) + c_1(\gamma) \ln \frac{k^2}{\mu^2} + c_2(\gamma) \ln^2 \frac{k^2}{\mu^2} + c_3(\gamma) \ln^3 \frac{k^2}{\mu^2} + \dots \right]$$

$c_n(\gamma)$  complex valued functions

- Truncate the series at  $n = 2 \Rightarrow$

$$F_\gamma(k) = k^{2\gamma-2} \left[ c_0(\gamma) + c_1(\gamma) \ln \frac{k^2}{\mu^2} + c_2(\gamma) \ln^2 \frac{k^2}{\mu^2} \right]$$

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- $\Rightarrow$  For  $c_2(\gamma) = \frac{\beta_2 \chi_0(\gamma)}{2 \chi_0'(\gamma)}$  and for any  $c_0(\gamma)$  and  $c_1(\gamma)$  the eigenfunction is

$$H_\gamma(k) = k^{2\gamma-2} \left[ 1 + \bar{\alpha}_\mu \left( \frac{\beta_2 \chi_0(\gamma)}{2 \chi_0'(\gamma)} \ln^2 \frac{k^2}{\mu^2} + c_1(\gamma) \ln \frac{k^2}{\mu^2} + c_0(\gamma) \right) \right]$$

and eigenvalues

$$\Delta(\gamma) = \bar{\alpha}_\mu \chi_0(\gamma) + \bar{\alpha}_\mu^2 \left( -\frac{1}{2} \beta_2 \chi_0'(\gamma) + \chi_1(\gamma) + c_1(\gamma) \chi_0'(\gamma) + \frac{\beta_2 \chi_0(\gamma)}{2 \chi_0'(\gamma)} \right)$$

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d\gamma}{2\pi i} H_\gamma(k) H_\gamma^*(k') = \delta(k^2 - k'^2)$$

- Completeness relation has to be satisfied order-by-order in  $\alpha_\mu$
- LO is satisfied with  $\sigma = \frac{1}{2}$  and  $\gamma = \frac{1}{2} + i\nu$  with  $\nu$  real parameter.
- $\Rightarrow$  we have to impose  $\alpha_\mu$ -order to be 0  $\Rightarrow$

$$\text{Re}[c_1(\nu)] = \frac{\beta_2}{2} \frac{\partial}{\partial \nu} \frac{\chi_0(\nu)}{\chi_0'(\nu)}$$

$$\frac{\partial}{\partial \nu} \text{Im}[c_1(\nu)] + 2 \text{Re}[c_0(\nu)] = 0$$



# Completeness of the NLO $H$ -eigenfunctions

- provided that  $\frac{\partial}{\partial \nu} \text{Im}[c_1(\nu)] + 2 \text{Re}[c_0(\nu)] = 0$  the eigenfunctions are

$$H_{\frac{1}{2}+i\nu}(k) = k^{-1+2i\nu} \left[ 1 + \bar{\alpha}_\mu \left( i \frac{\beta_2 \chi_0(\nu)}{2\chi_0'(\nu)} \ln^2 \frac{k^2}{\mu^2} + \frac{\beta_2}{2} \left( \frac{\partial}{\partial \nu} \frac{\chi_0(\nu)}{\chi_0'(\nu)} \right) \ln \frac{k^2}{\mu^2} + i \text{Im}[c_1(\nu)] \ln \frac{k^2}{\mu^2} + \text{Re}[c_0(\nu)] \right) \right]$$

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- and the eigenvalues are real ( $\text{Im}[c_1(\nu)] \chi'_0(\nu)$  is real function of  $\nu$ )

$$\Delta(\nu) = \bar{\alpha}_\mu \chi_0(\nu) + \bar{\alpha}_\mu^2 \left[ \chi_1(\nu) + \text{Im}[c_1(\nu)] \chi'_0(\nu) \right]$$

# Completeness of the NLO $H$ -eigenfunctions

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- and the eigenvalues are real ( $\text{Im}[c_1(\nu)] \chi'_0(\nu)$  is real function of  $\nu$ )

$$\Delta(\nu) = \bar{\alpha}_\mu \chi_0(\nu) + \bar{\alpha}_\mu^2 \left[ \chi_1(\nu) + \text{Im}[c_1(\nu)] \chi'_0(\nu) \right]$$

- The imaginary term  $-2\beta \bar{\alpha}_\mu^2 \chi'_0(\nu)$  has canceled.
- We have freedom to choose  $\text{Re}[c_0(\nu)]$  and  $\text{Im}[c_1(\nu)]$
- Are we in trouble? No!** This freedom will not affect the solution. It is just an artifact of the **phase** and  **$\nu$ -reparametrization** freedom of the  $H_{\frac{1}{2}+i\nu}$  function.

$$H_{\frac{1}{2}+i\nu}(k) = k^{-1+2i\nu} \left[ 1 + \bar{\alpha}_\mu \beta_2 \left( i \frac{\chi_0(\nu)}{2\chi'_0(\nu)} \ln^2 \frac{k^2}{\mu^2} + \frac{1}{2} \left( \frac{\partial}{\partial \nu} \frac{\chi_0(\nu)}{\chi'_0(\nu)} \right) \ln \frac{k^2}{\mu^2} + i\text{Im}[c_1(\nu)] \ln \frac{k^2}{\mu^2} + c_0(\nu) \right) \right]$$

The eigenfunctions satisfy the orthogonality relation

$$\int d^2k H_{\frac{1}{2}+i\nu} [H_{\frac{1}{2}+i\nu'}]^* = 2\pi^2 \delta(\nu - \nu')$$

- Imposing orthogonality condition will not introduce any new constraint to eliminate the freedom to chose  $\text{Re}[c_0(\nu)]$  and  $\text{Im}[c_1(\nu)]$

- Phase freedom of the  $H$ -functions:

$$H_{\frac{1}{2}+i\nu}(k) \rightarrow e^{-i\bar{\alpha}_\mu (\text{Im}[c_0(\nu)] - \text{Im}[c_1(\nu)] \ln \mu^2)} H_{\frac{1}{2}+i\nu}(k)$$

- $\nu$ -reparametrization:  $\nu' = \nu + \bar{\alpha}_\mu \text{Im}[c_1(\nu)]$

⇒

The Phase freedom and  $\nu$ -reparametrization allow us to put  $c_0(\nu) = 0$  and  $\text{Im}[c_1(\nu)] = 0$ .

## ■ NLO eigenfunctions

$$H_{\frac{1}{2}+i\nu}(k) = k^{-1+2i\nu} \left[ 1 + \bar{\alpha}_\mu \beta_2 \left( i \frac{\chi_0(\nu)}{2 \chi_0'(\nu)} \ln^2 \frac{k^2}{\mu^2} + \frac{1}{2} \left( \frac{\partial}{\partial \nu} \frac{\chi_0(\nu)}{\chi_0'(\nu)} \right) \ln \frac{k^2}{\mu^2} \right) \right]$$

## ■ NLO eigenvalues

$$\Delta(\nu) = \bar{\alpha}_\mu \chi_0(\nu) + \bar{\alpha}_\mu^2 \chi_1(\nu)$$

## Solution of NLO BFKL equation

### ■ NLO eigenfunctions

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### ■ NLO eigenvalues

$$\Delta(\nu) = \bar{\alpha}_\mu \chi_0(\nu) + \bar{\alpha}_\mu^2 \chi_1(\nu)$$

## Solution of NLO BFKL equation

$$G(k, k', Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2} e^{[\bar{\alpha}_\mu \chi_0(\nu) + \bar{\alpha}_\mu^2 \chi_1(\nu)] Y} H_{\frac{1}{2}+i\nu}(k) \left[ H_{\frac{1}{2}+i\nu}(k') \right]^*$$

- The  $H_\gamma(k)$  eigenfunctions diagonalize the LO+NLO BFKL kernel

$$\bar{\alpha}_\mu K^{\text{LO}}(k, q) + \bar{\alpha}_\mu^2 K^{\text{NLO}}(k, q) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\gamma}{2\pi^2 i} \Delta(\gamma) H_\gamma(k) H_\gamma^*(q)$$

- LO+NLO BFKL kernel is  $\mu$ -independent up to  $\mathcal{O}(\alpha_\mu^3) \Rightarrow$
- So is its diagonalization through  $H_\gamma(k)$  eigenfunctions.



$$\begin{aligned} G(k, k', Y) &= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2 k k'} e^{[\bar{\alpha}_\mu \chi_0(\nu) + \bar{\alpha}_\mu^2 \chi_1(\nu)] Y} H_\gamma(k) H_\gamma^*(q) \\ &= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2 k k'} e^{[\bar{\alpha}_\mu \chi_0(\nu) + \bar{\alpha}_\mu^2 \chi_1(\nu)] Y} \left(\frac{k^2}{k'^2}\right)^{i\nu} \left(1 - \bar{\alpha}_\mu^2 \beta_2 \chi_0(\nu) Y \ln \frac{kk'}{\mu^2}\right) \end{aligned}$$

■  $\Rightarrow G(k, k', Y)$  is  $\mu$ -independent up to order  $\mathcal{O}(\alpha_\mu^3)$ .

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■  $\Rightarrow G(k, k', Y)$  is  $\mu$ -independent up to order  $\mathcal{O}(\alpha_\mu^3)$ .

■ At NLO we may write the solution as

$$G(k, k', Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2 k k'} e^{[\bar{\alpha}_s(k k') \chi_0(\nu) + \bar{\alpha}_s^2(k k') \chi_1(\nu)] Y} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

■ At this order the scale  $\bar{\alpha}_s^\lambda(k^2) \bar{\alpha}_s^\lambda(k'^2) \bar{\alpha}_s^{1-2\lambda}(k k')$  (for real  $\lambda$ ) works as well.

# General Form of the Solution of All-Order BFKL equation

Ansatz

$$G(k, k', Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2} e^{[\bar{\alpha}_s(k k') \chi_0(\nu) + \bar{\alpha}_s^2(k k') \chi_1(\nu) + \bar{\alpha}_s^3(k k') \chi_2(\nu) + \dots]} Y \left( \frac{k^2}{k'^2} \right)^{i\nu}$$

- $\chi_2(\nu)$  and higher-order coefficients indicated by the ellipsis in the exponent are the scale-invariant (conformal) ( $\nu \leftrightarrow -\nu$ )-even (real-valued) parts of the prefactor function generated by the action of the next-to-next-to-leading-order (NNLO) (and higher-order) kernels on the LO eigenfunctions.

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- To check the ansatz we have to plug it in the evolution eq. and we need the two-loop beta-function  $\beta_3$ :

$$\mu^2 \frac{d\bar{\alpha}_\mu}{d\mu^2} = -\beta_2 \bar{\alpha}_\mu^2 + \beta_3 \bar{\alpha}_\mu^3$$

and

$$\int d^2q K^{\text{LO+NLO+NNLO}}(k, q) q^{-1+2i\nu} = \left\{ \bar{\alpha}_\mu \chi_0(\nu) \left[ 1 - \bar{\alpha}_\mu \beta_2 \ln \frac{k^2}{\mu^2} + \bar{\alpha}_\mu^2 \beta_2^2 \ln^2 \frac{k^2}{\mu^2} + \bar{\alpha}_\mu^2 \beta_3 \ln \frac{k^2}{\mu^2} \right] + \bar{\alpha}_\mu^2 \left[ \frac{i}{2} \beta_2 \chi_0'(\nu) + \chi_1(\nu) \right] \left[ 1 - 2 \bar{\alpha}_\mu \beta_2 \ln \frac{k^2}{\mu^2} \right] + \bar{\alpha}_\mu^3 [\chi_2(\nu) + i \delta_2(\nu)] \right\} k^{-1+2i\nu}$$

# General Form of the Solution of All-Order BFKL equation

- The ansatz does not work, but it allows us to recover the following NNLO solution

$$G(k, k', Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2 k k'} e^{[\bar{\alpha}_s(k k') \chi_0(\nu) + \bar{\alpha}_s^2(k k') \chi_1(\nu) + \bar{\alpha}_s^3(k k') \chi_2(\nu)] Y} \left( \frac{k^2}{k'^2} \right)^{i\nu} \\ \times \left\{ 1 + (\bar{\alpha}_\mu \beta_2)^2 \left[ -\frac{1}{24} (\bar{\alpha}_\mu Y)^3 \chi_0(\nu)^2 \chi_0''(\nu) + \frac{1}{4} (\bar{\alpha}_\mu Y)^2 \chi_0(\nu) \left( \frac{\chi_0'(\nu)^2}{2\chi_0(\nu)} - \chi_0''(\nu) \right) + \bar{\alpha}_\mu Y \frac{\chi_0''(\nu)}{4} \right] \right\}$$

provided that the imaginary part  $i\delta_2(\gamma)$  is

$$i\delta_2(\nu) = -\frac{i}{2} \chi_0'(\nu) \beta_3 + i \chi_1'(\nu) \beta_2$$

This solution satisfies also the initial condition: the solution is unique so it is the right one.

# General Form of the Solution of All-Order BFKL equation

- The ansatz does not work, but it allows us to recover the following NNLO solution

$$G(k, k', Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2 k k'} e^{[\bar{\alpha}_s(k k') \chi_0(\nu) + \bar{\alpha}_s^2(k k') \chi_1(\nu) + \bar{\alpha}_s^3(k k') \chi_2(\nu)] Y} \left( \frac{k^2}{k'^2} \right)^{i\nu} \\ \times \left\{ 1 + (\bar{\alpha}_\mu \beta_2)^2 \left[ -\frac{1}{24} (\bar{\alpha}_\mu Y)^3 \chi_0(\nu)^2 \chi_0''(\nu) + \frac{1}{4} (\bar{\alpha}_\mu Y)^2 \chi_0(\nu) \left( \frac{\chi_0'(\nu)^2}{2\chi_0(\nu)} - \chi_0''(\nu) \right) + \bar{\alpha}_\mu Y \frac{\chi_0''(\nu)}{4} \right] \right\}$$

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This solution satisfies also the initial condition: the solution is unique so it is the right one.

- The imaginary part  $i\delta_2(\nu)$  has to be confirmed from the explicit calculation of the NNLO eigenfunction.
- So, let us calculate explicitly the NNLO eigenfunction.

# General Form of the Solution of All-Order BFKL equation

The eigenfunction of the NNLO BFKL equation is

$$H_{\frac{1}{2}+i\nu}(k) = k^{-1+2i\nu} \left[ 1 + \bar{\alpha}_\mu \beta_2 \left( i \frac{\chi_0(\nu)}{2\chi'_0(\nu)} \ln^2 \frac{k^2}{\mu^2} + \frac{1}{2} \left( \frac{\partial}{\partial \nu} \frac{\chi_0(\nu)}{\chi'_0(\nu)} \right) \ln \frac{k^2}{\mu^2} \right) + \bar{\alpha}_\mu^2 f_2 \left( \frac{k}{\mu}, \nu \right) + \dots \right]$$

- The function  $f_2(k/\mu, \nu)$  denotes the NNLO corrections to the eigenfunctions.
- Ansatz for  $f_2(k/\mu, \nu)$ :

$$f_2(k/\mu, \nu) = c_0^{(2)}(\nu) + c_1^{(2)}(\nu) \ln \frac{k^2}{\mu^2} + c_2^{(2)}(\nu) \ln^2 \frac{k^2}{\mu^2} + c_3^{(2)}(\nu) \ln^3 \frac{k^2}{\mu^2} + \dots$$

# NNLO BFKL Solution: the NNLO eigenfunction

- This time we truncated the series up to  $c_4^{(2)}$  and proceeding similarly to the NLO case and we get

$$\text{Re}[c_1^{(2)}] = \beta_2 \left( \frac{\chi_1'(\nu)}{2\chi_0'(\nu)} - \frac{\chi_1(\nu)\chi_0''(\nu)}{\chi_0'^2(\nu)} - \frac{\chi_1''(\nu)\chi_0(\nu)}{2\chi_0'^2(\nu)} + \frac{\chi_0(\nu)\chi_1'(\nu)\chi_0''(\nu)}{\chi_0'^3(\nu)} \right) - \beta_3 \left( -\frac{1}{2} - \frac{\chi_0(\nu)\chi_0''(\nu)}{2\chi_0'^2(\nu)} \right)$$

$$c_2^{(2)} = \beta_2^2 \left( \frac{5}{8} \frac{\chi_0''^2(\nu)\chi_0^2(\nu)}{\chi_0'^4(\nu)} - \frac{\chi_0(\nu)\chi_0''(\nu)}{4\chi_0'^2(\nu)} - \frac{\chi_0^2(\nu)\chi_0''(\nu)}{4\chi_0'^3(\nu)} - \frac{1}{8} \right) - i\beta_2 \left( \frac{\chi_1(\nu)}{\chi_0'(\nu)} - \frac{\chi_0(\nu)\chi_1'(\nu)}{2\chi_0'(\nu)} \right) - i\beta_3 \frac{\chi_0(\nu)}{\chi_0'(\nu)}$$

$$c_3^{(2)} = i\beta_2^2 \left( -\frac{5}{12} \frac{\chi_0''(\nu)\chi_0^2(\nu)}{\chi_0'^3(\nu)} + \frac{\chi_0(\nu)}{4\chi_0'(\nu)} \right)$$

$$c_4^{(2)} = -\beta_2^2 \frac{\chi_0^2(\nu)}{8\chi_0'^2(\nu)}$$

The  $\text{Im}[c_1^{(2)}(\nu)]$  is fixed to be 0 using again the  $\nu$ -reparametrization.

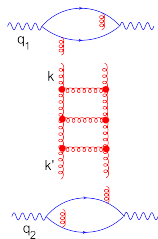


- From explicit calculation of  $f_2(\gamma)$  we not only confirm the imaginary part

$$i\delta_2(\nu) = -\frac{i}{2}\chi_0'(\nu)\beta_3 + i\chi_1'(\nu)\beta_2$$

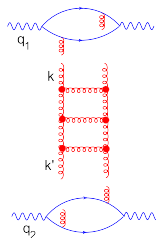
- but also confirm the NNLO BFKL solution obtained above in an indirect way

$$G(k, k', Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2 k k'} e^{[\bar{\alpha}_s(k k') \chi_0(\nu) + \bar{\alpha}_s^2(k k') \chi_1(\nu) + \bar{\alpha}_s^3(k k') \chi_2(\nu)] Y} \left(\frac{k^2}{k'^2}\right)^{i\nu} \\ \times \left\{ 1 + (\bar{\alpha}_\mu \beta_2)^2 \left[ -\frac{1}{24} (\bar{\alpha}_\mu Y)^3 \chi_0(\nu)^2 \chi_0''(\nu) + \frac{1}{4} (\bar{\alpha}_\mu Y)^2 \chi_0(\nu) \left( \frac{\chi_0'(\nu)^2}{2\chi_0(\nu)} - \chi_0''(\nu) \right) + \bar{\alpha}_\mu Y \frac{\chi_0''(\nu)}{4} \right] \right\}$$

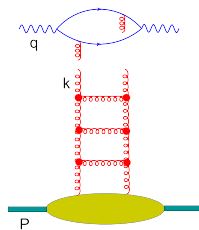


$K_{\text{BFKL}}$  is  $k \leftrightarrow k'$  symmetric

$$Y^{\text{sym}} = \ln \frac{s}{k k'}$$



VS.

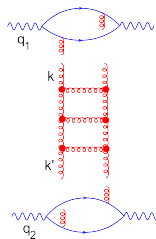


$K_{\text{BFKL}}$  is  $k \leftrightarrow k'$  symmetric

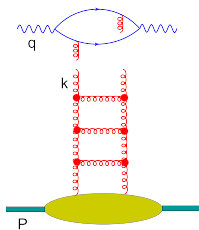
$$Y^{\text{sym}} = \ln \frac{s}{k k'}$$

NLO  $K_{\text{BFKL}}^{\text{DIS}}$  is not Symmetric

$$Y^{\text{DIS}} = \ln \frac{s}{k^2} \simeq \ln \frac{1}{x_B}$$



VS.



$K_{\text{BFKL}}$  is  $k \leftrightarrow k'$  symmetric

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NLO  $K_{\text{BFKL}}^{\text{DIS}}$  is not Symmetric

$$Y^{\text{DIS}} = \ln \frac{s}{k^2} \simeq \ln \frac{1}{x_B}$$

$$K_{\text{NLO}}^{\text{DIS}} = K_{\text{NLO}}^{\text{sym}} - \frac{1}{2} \int d^{D-2} q' K_{\text{LO}}(q_1, q') \ln \frac{q'^2}{q^2} K_{\text{LO}}(q, q_2) \quad \text{Fadin - Lipatov (1998)}$$

- $K_{\text{BFKL}}^{\text{DIS}}$  is not symmetric  $\Rightarrow$  eigenvalues not  $\gamma \leftrightarrow 1 - \gamma$  symmetric.
- $\Rightarrow$  Eigenvalues get an extra term:  $\Delta^{\text{DIS}}(\gamma) = \Delta^{\text{sym}}(\gamma) - \frac{1}{2} \bar{\alpha}_\mu^2 \chi_0(\gamma) \chi'(\gamma)$
- Reproduced lower order and predicted (and later confirmed) the 3-loop DGLAP anomalous dimension (Fadin-Lipatov (1998))

$$\begin{aligned}
 G(k, k', Y) &= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2} e^{[\bar{\alpha}_\mu \chi_0(\nu) + \bar{\alpha}_\mu^2 \chi_1(\nu)] (Y^{\text{DIS}} + \ln \frac{k}{k'})} H_{\frac{1}{2}+i\nu}(k) \left[ H_{\frac{1}{2}+i\nu}(k') \right]^* \\
 &\simeq \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2} e^{[\bar{\alpha}_\mu \chi_0(\nu) + \bar{\alpha}_\mu^2 \chi_1(\nu)] Y^{\text{DIS}}} H_{\frac{1}{2}+i\nu}(k) \left[ H_{\frac{1}{2}+i\nu}(k') \right]^* \left( 1 + \bar{\alpha}_\mu \chi_0(\nu) \ln \frac{k}{k'} \right)
 \end{aligned}$$

⇒ perform partial integration and exponentiate the  $Y^{\text{DIS}}$ -dependent terms ⇒

$$\begin{aligned}
 G(k, k', Y^{\text{DIS}}) &= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2} e^{[\bar{\alpha}_\mu \chi_0(\nu) + \bar{\alpha}_\mu^2 (\chi_1(\nu) + 2i\chi_0(\nu)\chi'_0(\nu))] Y^{\text{DIS}}} H_{\frac{1}{2}+i\nu}(k) \left[ H_{\frac{1}{2}+i\nu}(k') \right]^* \\
 &\quad \times \left( 1 + \frac{i}{2} \bar{\alpha}_\mu \chi'_0(\nu) \right)
 \end{aligned}$$

■ ⇒  $\Delta^{\text{DIS}}(\gamma) = \bar{\alpha}_\mu \chi_0(\gamma) + \bar{\alpha}_\mu^2 \chi_1(\gamma) - \frac{1}{2} \bar{\alpha}_\mu^2 \chi_0(\gamma) \chi'_0(\gamma)$

■ Agrees with DGLAP 3-loop anomalous dimension.

- The NLO BFKL eigenfunctions have been constructed: they are complete and orthogonal.
- The NLO BFKL solution has been presented.
- The NNLO Eigenfunctions has also been presented.
  - The NNLO solution has also been presented up to the still unknown conformal contribution  $\chi_2(\nu)$ .
- Procedure to obtain the solution to the BFKL equation to any order has been presented.
- With the shift  $Y^{\text{sym}} \rightarrow Y^{\text{DIS}} + \ln \frac{k}{k'}$  one can obtain the solution with non-symmetric kernel for DIS from the symmetric one.

- Analytic  $\gamma^* - \gamma^*$  amplitude at NLO.
- Extend the  $\mathcal{N} = 4$  SYM NLO four-point function to QCD case:  
 $\text{T}\{J^\mu(x)J^\nu(y)J^\rho(x')J^\lambda(y')\}$  at NLO.