NLO BFKL and anomalous dimensions of light-ray operators

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Outline

- Light-ray operators.
- Regge limit in the coordinate space.
- "BFKL" representation of 4-point correlation function in $\mathcal{N} = 4$ SYM.
- "DGLAP" representation of 4-point correlation function.
- Anomalous dimensions from DGAP vs BFKL representations.

Problem

Gluon operators of leading twist

$$\mathcal{O}_n^g \equiv F^a_{-i}\nabla_-^{n-2}F^{ai}_-$$

Anomalous dimension (in gluodynamics)

$$\gamma_n = \frac{2}{\pi}\alpha_s N_c\Big[-\frac{1}{n(n-1)} - \frac{1}{(n+1)(n+2)} + \psi(n+1) + \gamma_E - \frac{11}{12}\Big] + O(\alpha_s^2)$$

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BFKL gives $\gamma(n\alpha_s)$ at the non-physical point $n \to 1$

$$\gamma_n \;=\; \Big[A_n^{\text{LO BFKL}} + \omega B_n^{\text{NLO BFKL}} + \ldots\Big]\Big(\frac{\alpha_s N_c}{\pi\omega}\Big)^n \qquad \omega \equiv n-1$$

LO: Jaroszewicz (1980), NLO: Lipatov, Fadin, Camici, Ciafaloni (1998)

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Q: which one? A: gluon light-ray (LR) operator

Light-ray operators and parton densities

Gluon light-ray (LR) operator of twist 2

$$F^a_{-i}(x'_+ + x_\perp)[x'_+, x_+]^{ab} F^{b\ i}_-(x_+ + x_\perp)$$

Forward matrix element - gluon parton density

$$z^\mu z^\nu \langle p|F^a_{\mu\xi}(z)[z,0]^{ab} F^{b\xi}_\nu(0)|p\rangle^\mu \stackrel{z^2=0}{=} 2(pz)^2 \int_0^1 dx_B\, x_B D_g(x_B,\mu)\cos(pz)x_B$$

Evolution equation (in gluodynamics)

$$\begin{aligned}
&\mu^2 \frac{d}{d\mu^2} F^a_{-i}(x'_+ + x_\perp)[x'_+, x_+]^{ab} F^{b\ i}_-(x_+ + x_\perp) \\
&= \int_{x_+}^{x'_+} dz'_+ \int_{x_+}^{z'_+} dz_+\ K(x'_+, x_+; z'_+, z_+; \alpha_s) F^a_{-i}(z'_+ + x_\perp)[z'_+, z_+]^{ab} F^{b\ i}_-(z_+ + x_\perp)
\end{aligned}$$

"Forward" LR operator

$$F(L_+, x_\perp) = \int dx_+\ F^a_{-i}(L_+ + x_+ + x_\perp)[L_+ + x_+, x_+]^{ab} F^{bi}_-(x_+ + x_\perp)$$

Light-ray operators and parton densities

Expansion in ("forward") local operators

$$F(L_+,x_\perp)\ =\ \sum_{n=2}^\infty \frac{L_+^{n-2}}{(n-2)!}\mathcal{O}_n^g(x_\perp),\qquad \mathcal{O}_n^g\ \equiv\ \int dx_+ F^a_{-i}\nabla_-^{n-2}F_-^{ai}(x_+,x_\perp)$$

Evolution equation for $F(L_+,x_\perp)$

$$\mu\frac{d}{d\mu}F(L_+,x_\perp)\ =\ \int_0^1 du\ K_{gg}(u,\alpha_s)F(uL_+,x_\perp)$$

$$\Rightarrow\ \ \gamma_n(\alpha_s)\ =\ -\int_0^1 du\ u^{n-2}K_{gg}(u,\alpha_s)\qquad \mu\frac{d}{d\mu}\mathcal{O}_n^g\ =\ -\gamma_n(\alpha_s)\mathcal{O}_n^g$$

$u^{-1}K_{gg}$ - DGLAP kernel

$$u^{-1}K_{gg}(u)=\frac{2\alpha_sN_c}{\pi}\Big(\bar{u}u+\Big[\frac{1}{\bar{u}u}\Big]_+-2+\frac{11}{12}\delta(\bar{u})\Big)+\text{higher orders in }\alpha_s$$

Light-ray operators

Conformal LR operator ($j = \frac{1}{2} + i\nu$)

$$F^\mu(L_+, x_\perp) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} (L_+)^{-\frac{3}{2}+i\nu} \mathcal{F}^\mu_{\frac{1}{2}+i\nu}(x_\perp)$$

$$\mathcal{F}^\mu_j(x_\perp) = \int_0^\infty dL_+ \, L_+^{1-j} F^\mu(L_+, x_\perp)$$

Evolution equation for “forward” conformal light-ray operators

$$\Rightarrow \mu^2 \frac{d}{d\mu^2} \mathcal{F}_j(z_\perp) = \int_0^1 du \, K_{gg}(u, \alpha_s) u^{j-2} \mathcal{F}_j(z_\perp)$$

$\Rightarrow \gamma_j(\alpha_s)$ is an analytical continuation of $\gamma_n(\alpha_s)$

Conformal four-point amplitude in $\mathcal{N}=4$ SYM

4-point correlation function (CF)

$$A(x,y,x',y') = [(x-y)^2(x'-y')^2]^{2+\gamma_k} N_c^2 \langle \mathcal{O}(x)\mathcal{O}(y)\mathcal{O}(x')\mathcal{O}(y')\rangle$$

$\mathcal{O} = \phi_I^a \phi_I^a$ - Konishi operator (γ_k - anomalous dimension)

In a conformal theory the amplitude is a function of two conformal ratios

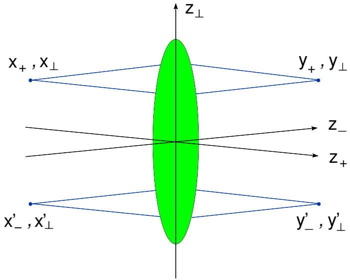
$$A = F(R,R')$$
$$R = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2}, \qquad R' = \frac{(x-y)^2(x'-y')^2}{(x-y')^2(x'-y)^2}$$

At large N_c

$$A(x,y,x',y') = A(g^2N_c) \qquad\qquad g^2N_c = \lambda \ - \ \text{`t Hooft coupling}$$

Regge limit in the coordinate space

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+,\ y_- \to \rho' y_-,\ y'_- \to \rho' y-_-$ $\qquad \rho, \rho' \to \infty$



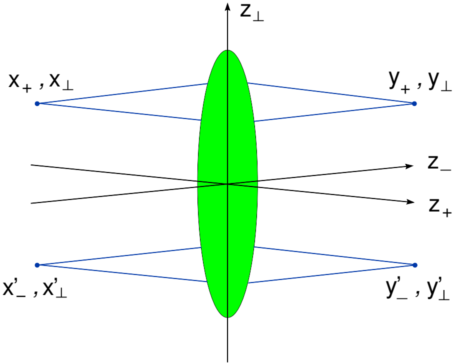
Full 4-dim conformal group: $A \;=\; F(R, r)$

$$R \;=\; \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2} \;\to\; \frac{\rho^2\rho'^2 x_+ x'_+ y_- y'_-}{(x-x')^2_\perp (y-y')^2_\perp} \;\to\; \infty$$

$$r \;=\; \frac{[(x-y)^2(x'-y')^2 - (x'-y)^2(x-y')^2]^2}{(x-x')^2(y-y')^2(x-y)^2(x'-y')^2}$$

$$\to\; \frac{[(x'-y')^2_\perp x_+ y_- + x'_+ y'_-(x-y)^2_\perp + x_+ y'_-(x'-y)^2_\perp + x'_+ y_-(x-y')^2_\perp]^2}{(x-x')^2_\perp (y-y')^2_\perp x_+ x'_+ y_- y'_-}$$

Regge limit: $x_+ \to \rho x_+, \; x'_+ \to \rho x'_+, y_- \to \rho' y_-, \; y'_- \to \rho' y-_-$
$\rho, \rho' \to \infty$



Regge limit symmetry: 2-dim conformal group $SL(2, C)$ formed from P_1, P_2, M^{12}, D, K_1 and K_2 which leave the plane $(0, 0, z_\perp)$ invariant.

Pomeron in a conformal theory

$$A(x,y;x',y') \stackrel{s\to\infty}{=} \frac{i}{2}\int d\nu\, f_+(\aleph(\lambda,\nu))F(\lambda,\nu)\Omega(r,\nu)R^{\aleph(\lambda,\nu)/2}$$

L. Cornalba (2007)

$f_+(\omega) = \frac{e^{i\pi\omega}-1}{\sin\pi\omega}$ - signature factor

$$\Omega(r,\nu) = \frac{\sin\nu\rho}{\sinh\rho}, \qquad \cosh\rho = \frac{\sqrt{r}}{2}$$

- solution of the eqn $(\Box_{H_3} + \nu^2 + 1)\Omega(r,\nu) = 0$

The dynamics is described by:
$\aleph(\lambda,\nu)$ - pomeron intercept,
and
$F(\lambda,\nu)$ - "pomeron residue".

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Pomeron intercept:

$$\aleph(\nu,\lambda) = \frac{\lambda}{4\pi^2}\Big[\chi(\nu) + \frac{\lambda}{16\pi^2}\delta(\nu)\Big] + O(\lambda^3)$$

$\chi(\nu) = 2\psi(1) - \psi(\frac{1}{2}+i\nu) - \psi(\frac{1}{2}-i\nu) = -S_1(-\frac{1}{2}+i\nu) + \text{c.c.}$ - BFKL intercept,

$$\delta(\nu) = -\frac{3}{2}\zeta(3)+\pi^2\ln 2+\frac{\pi^2}{3}S_1(-\frac{1}{2}+i\nu)+2S_3(-\frac{1}{2}+i\nu)+\pi^2S_{-1}(-\frac{1}{2}+i\nu)-4S_{-2,1}(-\frac{1}{2}+i\nu)+\text{c.c.}$$

- NLO BFKL intercept $S_i(x)$ - harmonic sums Lipatov, Kotikov (2000)

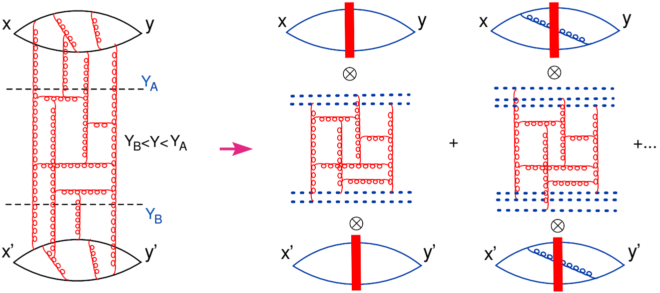
"Pomeron residue" $F(\nu,\lambda)$:

$$F(\nu,\lambda) = \lambda^2F_0(\nu) + \lambda^3F_1(\nu) + \ldots$$

$F_0(\nu) = \frac{\pi\sinh\pi\nu}{4\nu\cosh^3\pi\nu}$ Cornalba, Costa, Penedones (2007)

$F_1(\nu)$ = see below G. Chirilli and I.B. (2009)

NLO Amplitude in $\mathcal{N}$=4 SYM theory: factorization in rapidity



$$[(x-y)^2(x'-y')^2]^{2+\gamma_k}\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

$a_0 = \frac{x_+y_+}{(x-y)^2}$, $b_0 = \frac{x'_-y'_-}{(x'-y')^2}$ $\Leftrightarrow$ impact factors do not scale with energy

$\Rightarrow$ all energy dependence is contained in $[\mathrm{DD}]^{a_0,b_0}$ $(a_0b_0 = R)$

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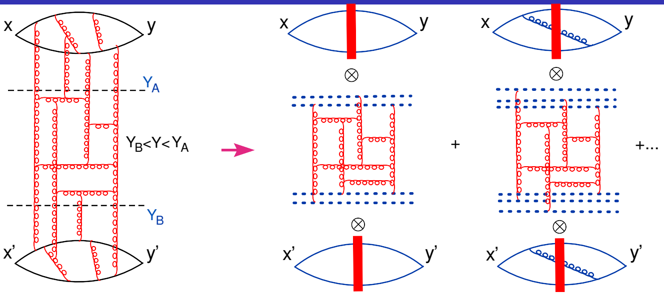

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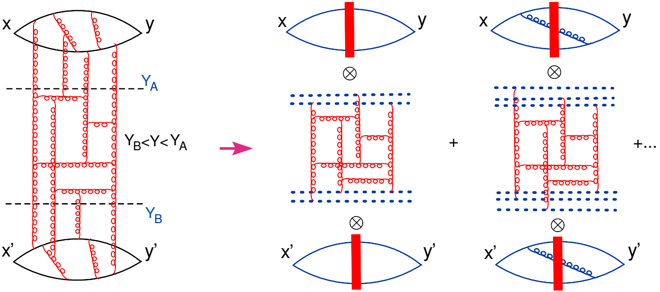
Dipole-dipole scattering

$$\chi(\gamma)\equiv 2C-\psi(\gamma)-\psi(1-\gamma)$$

$$[DD]=\int d\nu\int dz_0\,\Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^{\frac12+i\nu}\Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^{\frac12-i\nu}D(\frac12+i\nu;\lambda)R^{\omega(\nu)/2}$$

$$D(\gamma;\lambda)\;=\;\frac{\Gamma(-\gamma)\Gamma(\gamma-1)}{\Gamma(1+\gamma)\Gamma(2-\gamma)}\Big\{1-\frac{\lambda}{4\pi^2}\Big[\frac{\chi(\gamma)}{\gamma(1-\gamma)}-\frac{\pi^2}{3}\Big]+O(\lambda^2)\Big\}$$



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Result : (G.A. Chirilli and I.B., 2010)

$$F(\nu) = \frac{N_c^2}{N_c^2-1}\frac{4\pi^4\alpha_s^2}{\cosh^2\pi\nu}\left\{1+\frac{\alpha_s N_c}{\pi}\left[\frac{\pi^2}{2}-\frac{2\pi^2}{\cosh^2\pi\nu}-\frac{8}{1+4\nu^2}\right]+O(\alpha_s^2)\right\}$$

$$[x_{12}^2x_{34}^2]^{2+\gamma_K}\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{i}{2}\int d\nu\,\frac{\tanh\pi\nu}{\nu}F(\nu)\Omega(r,\nu)R^{\frac{1}{2}\aleph(\nu)}\tilde{f}_+(\aleph(\nu))$$

$\aleph(\nu)$ - pomeron intercept, $\tilde{f}_+(\omega)=(e^{i\pi\omega}-1)/\sin\pi\omega$ - signature factor

$$\Omega(r,\nu) = \frac{\nu}{2\pi^2}\frac{\sin 2\nu\rho}{\sinh\rho}, \quad \cosh\rho=\frac{\sqrt{r}}{2}$$

In the double limit (Regge) + ($x_{12}^2\to 0$)

$$R=\frac{x_{13}^2x_{24}^2}{x_{12}^2x_{34}^2} \to \frac{x_{1_+}x_{2_+}x_{3_-}x_{4_-}}{x_{12_\perp}^2x_{34_\perp}^2}$$

$$r \to \frac{x_{12_+}^2(x_{3_-}x_{14_\perp}^2-x_{4_-}x_{13_\perp}^2)^2}{x_{1_+}x_{2_+}x_{3_-}x_{4_-}x_{12_\perp}^2x_{34_\perp}^2}$$

$$\Omega(r,\nu) \to \frac{\nu}{2\pi^2 i}\left(r^{-\frac{1}{2}+i\nu}-r^{-\frac{1}{2}-i\nu}\right)$$

Cornalba's formula as $x^2_{12\perp}\to 0$

$$[x^2_{12}x^2_{34}]^{2+\gamma_K}\int dx_{2_+}dx_{3_-}\ \langle\mathcal{O}(L_+ + x_{2_+},x_{1\perp})\mathcal{O}(x_{2_+},x_{2\perp})\mathcal{O}(L_- + x_{3_-},x_{2\perp})\mathcal{O}(x_{3_-},x_{3\perp})\rangle$$

$$= \frac{i\alpha_s^2}{8}\pi^2L_+L_-\int_0^1 dudv\int d\nu\frac{\tanh\pi\nu}{\nu\cosh^2\pi\nu}\Big(\frac{x^2_{12\perp}x^2_{34\perp}\bar{u}u\bar{v}v}{[x^2_{13\perp}v+x^2_{14}\bar{v}]^2}\Big)^{\frac{1}{2}+i\nu}\Big(\frac{L_+^2L_-^2\bar{u}u\bar{v}v}{x^2_{12\perp}x^2_{34\perp}}\Big)^{\aleph(\nu)/2}f_+$$

$$= \frac{i\alpha_s^2}{8}\pi^2\int_0^1 dv\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}d\xi\, f\big(\aleph(\xi)\big) \qquad \xi\equiv\frac{1}{2}+i\nu$$

$$\times\ \frac{(\bar{v}v)^{1-\xi+\frac{\aleph(\xi)}{2}}\cos\pi\xi}{(\xi-\frac{1}{2})\sin^3\pi\xi}\frac{B\big(2-\xi+\frac{\aleph(\xi)}{2}\big)}{[x^2_{13\perp}v+x^2_{14}\bar{v}]^{2+\aleph(\xi)}}\Big(\frac{x^2_{12\perp}x^2_{34\perp}}{[x^2_{13\perp}v+x^2_{14}\bar{v}]^2}\Big)^{-\xi-\frac{\aleph(\xi)}{2}}(L_+L_-)^{1+\aleph(\xi)}$$

= "BFKL" representation of the amplitude in the (Regge) + ($x^2_{12}\to 0$) limit

SU_4 singlet operators. (Korchemsky et al, 2003)

$$\begin{aligned} S_{1n}(z) &= \mathcal{O}_g^n(z) + \frac{n-1}{8}\mathcal{O}_\lambda^n(z) - \frac{n(n-1)}{8}\mathcal{O}_\phi^n(z) \\ S_{2n}(z) &= \mathcal{O}_g^n(z) - \frac{1}{8}\mathcal{O}_\lambda^n(z) + \frac{n(n+1)}{24}\mathcal{O}_\phi^n(z) \\ S_{3n}(z) &= \mathcal{O}_g^n(z) - \frac{n+2}{4}\mathcal{O}_\lambda^n(z) - \frac{(n+1)(n+2)}{8}\mathcal{O}_\phi^n(z) \end{aligned}$$

$$\begin{aligned} \mathcal{O}_\lambda^n(x_\perp) &= \int dx_+ \ i\bar{\lambda}^a \nabla^{n-1}\sigma_-\lambda^a(x_+, x_\perp) \\ \mathcal{O}_\phi^n(z) &= \int dx_+ \ \bar{\phi}^{a,I}\nabla^n \phi^{a,I}(x_+, x_\perp) \end{aligned}$$

All operators have the same anomalous dimension

$$\gamma_n^{S_1}(\alpha_s) \equiv \gamma_n(\alpha_s) = \frac{2\alpha_s}{\pi}N_c[\psi(n-1) + C] + O(\alpha_s^2), \qquad \gamma_n^{S_2} = \gamma_{n+1}^{S_1}, \qquad \gamma_n^{S_3} = \gamma_{n+2}^{S_1}$$

Supermultiplet of LR operators

Gluino and scalar LR operators

$$\Lambda(L_+,x_\perp) = \frac{i}{2}\int dx'_+ \, [\bar\lambda^a(L_+ + x_+ + x_\perp)[x'_+ + x_+, x_+]^{ab}\sigma_-\nabla_-\lambda^b(x_+ + x_\perp) + \text{c.c}]$$

$$\Phi(L_+,x_\perp) = \int dx'_+ \, \phi^{a,I}(L_+ + x_+ + x_\perp)[x'_+ + x_+, x_+]^{ab}\nabla_-^2\phi^{b,I}(x_+ + x_\perp)$$

$$\Lambda_j(x_\perp) = \int_0^\infty dL_+ \, L_+^{-j+1}\Lambda(L_+,x_\perp), \qquad \Phi_j(x_\perp) = \int_0^\infty dL_+ \, L_+^{-j+1}\Phi(L_+,x_\perp)$$

SU_4 singlet LR operators.

$$S_{1j}(x_\perp) = F_j(x_\perp) + \frac{j-1}{8}\Lambda_j(x_\perp) - \frac{j(j-1)}{8}\Phi_j(x_\perp)$$

$$S_{2j}(x_\perp) = F_j(x_\perp) - \frac{1}{8}\Lambda_j(x_\perp) + \frac{j(j+1)}{24}\Phi_j(x_\perp)$$

$$S_{3j}(x_\perp) = F_j(x_\perp) - \frac{j+2}{4}\Lambda_j(x_\perp) - \frac{(j+1)(j+2)}{8}\Phi_j(x_\perp)$$

All operators have the same anomalous dimension

$$\gamma_j^{S_1}(\alpha_s) \equiv \gamma_j(\alpha_s) = \frac{2\alpha_s}{\pi}N_c[\psi(j-1) + C] + O(\alpha_s^2), \qquad \gamma_j^{S_2} = \gamma_{j+1}^{S_1}, \qquad \gamma_j^{S_3} = \gamma_{j+2}^{S_1}$$

Three-point CF of LR operator and two locals

Three-point correlation function of local operators ($\mathcal{O} = \phi_I^a \phi_I^a$ - Konishi operator)

$$(\mu^2 z_{23}^2)^{-\gamma_K} \langle S_{1n}(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3)\rangle = \sum \frac{c_n[1+(-1)^n]}{z_{12}^2 z_{13}^2 z_{23}^2}\Big(\frac{z_{12-}}{z_{12}^2} - \frac{z_{13-}}{z_{13}^2}\Big)^n \Big(\frac{\mu^{-2} z_{23}^2}{z_{12}^2 z_{13}^2}\Big)^{\frac{1}{2}\gamma(n,\alpha_s)}$$

$\Rightarrow$

$$x_{23_\perp}^2 (\mu^2 x_{23_\perp}^2)^{\gamma_K} \int dx_{1_+} \langle S_{1n}(x_{1_+}, x_{1_\perp})\mathcal{O}(x_{2_-}, x_{2_\perp})\mathcal{O}(x_{3_-}, x_{3_\perp})\rangle$$

$$= -2\pi i \frac{c_n[1+(-1)^n]}{x_{23_\perp}^4} \frac{\Gamma(1+2n+\gamma_n)}{\Gamma^2(1+n+\frac{1}{2}\gamma_n)} \Big(\frac{x_{23_\perp}^2}{x_{2-}x_{3-}}\Big)^{1+\frac{\gamma_n}{2}} \Big(\frac{x_{12_\perp}^2}{x_{2-}} + \frac{x_{13_\perp}^2}{x_{3-}}\Big)^{-1-n-\gamma_n}$$

CF of light-ray operator and two local operators

$$x_{23_\perp}^2 (\mu^2 x_{23_\perp}^2)^{\gamma_K} \langle S_{1j}(x_{1_\perp})\mathcal{O}(x_{2_-}, x_{2_\perp})\mathcal{O}(x_{3_-}, x_{3_\perp})\rangle$$

$$= -2\pi i \frac{c_j[1+e^{i\pi j}]}{x_{23_\perp}^4} \frac{\Gamma(1+2j+\gamma_j)}{\Gamma^2(1+j+\frac{1}{2}\gamma_j)} \Big(\frac{x_{23_\perp}^2}{x_{2-}x_{3-}}\Big)^{1+\frac{\gamma_j}{2}} \Big(\frac{x_{12_\perp}^2}{x_{2-}} + \frac{x_{13_\perp}^2}{x_{3-}}\Big)^{-1-j-\gamma_j}$$

Three-point CF of LR operator and two locals

Proof: compare conformal Ward identities for $\int dx_+ S_{1n}(x_+,x_\perp)$ and $S_{1j}(x_\perp)$

$$i[D,\int dx_+ S_{1n}(x_+,x_\perp)] = \int dx_+ \big(x_+\frac{\partial}{\partial x_+} + n + 2 + (x-x_0)^\perp_i \frac{\partial}{\partial x^\perp_i}\big) S_{1n} = (n+1)\int dx_+ S_{1n}(x_+,x_\perp)$$

vs

$$i[D,\Phi(x_+,x_\perp)\nabla^2_-\Phi(x'_+,x_\perp)] \ = \ \Big[4 + x_+\frac{\partial}{\partial x_+} + x'_+\frac{\partial}{\partial x'_+} + (x-x_0)^\perp_i\frac{\partial}{\partial x^\perp_i}\Big]\Phi(x_+,x_\perp)\nabla^2_-\Phi(x'_+,x_\perp)$$

$$\Rightarrow \ i[D,\Phi_j(x_\perp)] \ = \ \big[j + 1 + (x-x_0)^\perp_i\frac{\partial}{\partial x^\perp_i}\big]\Phi_j(x_\perp)$$

Similarly

$$i[K_+,\int dx_+ \Phi(x_\perp)\nabla^n_-\Phi(x_\perp)] \ = \ 4\big[n + (x-x_0)_i\frac{\partial}{\partial x^\perp_i}\big]\int dx_+ \ x_+\Phi(x_\perp)\nabla^n_-\Phi(x_\perp)$$

vs

$$i[K_+,\int dx_+\int_0^\infty ds_+ \ (s_+)^{-j+1}\nabla_-\Phi(x_+ + \frac{s_+}{2} + x_\perp)\nabla_-\Phi(x_+ - \frac{s_+}{2} + x_\perp)]$$

$$= \ 4\big[j + (x-x_0)^\perp_i\frac{\partial}{\partial x^\perp_i}\big]\int dx_+ \ x_+\int_0^\infty ds_+ \ (s_+)^{-j+1}\Phi(x_+ + \frac{s_+}{2} + x_\perp)\Phi(x_+ - \frac{s_+}{2} + x_\perp)$$

Light-cone OPE

CF of light-ray operator and two local operators “in forward kinematics”

$$x^2_{23_\perp}(\mu^2 x^2_{23_\perp})^{\gamma_K}\int dx_{3_-}\langle S^\mu_j(x_{1_\perp})\mathcal{O}(L_- + x_{3_-}, x_{2_\perp})\mathcal{O}(x_{3_-}, x_{3_\perp})\rangle$$
$$= \int_0^1 du\, \frac{c(j,\alpha_s)[1+e^{i\pi j}]L_-^j(\bar{u}u)^j}{[x^2_{12_\perp}u + x^2_{13_\perp}\bar{u}]^{1+j}}\Big(\frac{\bar{u}u\mu^{-2}x^2_{23_\perp}}{[x^2_{12_\perp}u + x^2_{13_\perp}\bar{u}]^2}\Big)^{\frac{1}{2}\gamma(j,\alpha_s)}$$

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Limit $x_{23_\perp} \equiv \Delta \to 0$

$$\Delta^2(\mu^2\Delta^2)^{\gamma_K}\int dx_{3_-}\langle S_j^+(x_{1_\perp})\mathcal{O}(L_- + x_{3_-}, x_{2_\perp})\mathcal{O}(x_{3_-}, x_{3_\perp})\rangle$$
$$= \frac{\Gamma^2\big(1+j+\frac{\gamma}{2}\big)}{\Gamma(2+2j+\gamma)}\ \frac{c(j,\alpha_s)[1+e^{i\pi j}]L_-^j}{(x_{12_\perp}^2)^{1+j}(\mu^2 x_{12_\perp}^2)^{-\gamma(j,\alpha_s)}}(\mu\Delta)^{\gamma(j,\alpha_s)}$$

Light-cone OPE

Light-ray operators in (+) and (-) directions:

$$\Phi^+(L_+, x_\perp) = \int dx'_+ \, \nabla_-\phi^{a,I}(L_+ + x_+ + x_\perp)[x'_+ + x_+, x_+]^{ab}\nabla_-\phi^{b,I}(x_+ + x_\perp)$$

$$\Phi^+_j(x_\perp) = \int_0^\infty dL_+ \, L_+^{-j+1}\Phi(L_+, x_\perp)$$

$$\Phi^-(L_-, x_\perp) = \int dx'_- \, \nabla_+\phi^{a,I}(L_- + x_- + x_\perp)[x'_- + x_-, x_-]^{ab}\nabla_+\phi^{b,I}(x_- + x_\perp)$$

$$\Phi^-_j(x_\perp) = \int_0^\infty dL_- \, L_+^{-j+1}\Phi(L_-, x_\perp)$$

and similarly for Λ_j's and G_j's

$$\text{CF of two LRs :} \quad \langle S^+_{j=\frac{3}{2}+i\nu}(x_{1_\perp}) S^-_{j'=\frac{3}{2}+i\nu'}(x_{3_\perp})\rangle = \frac{\delta(\nu-\nu')a(j,\alpha_s)}{(x^2_{13_\perp})^{j+1}(x^2_{13_\perp}\mu^2)^{\gamma(j,\alpha_s)}}$$

⇒ Light-cone OPE

$$x_{12_\perp}^2(\mu^2 x_{12_\perp}^2)^{\gamma_K}\int dx_{1_+}\ \mathcal{O}(L_+ + x_{2_+}, x_{1_\perp})\mathcal{O}(x_{2_+}, x_{2_\perp})$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj\ c(j,\alpha_s)[1+e^{i\pi j}]L_+^j(\mu^2 x_{12_\perp}^2)^{\gamma(j,\alpha_s)}S_j^+(x_{1_\perp})$$

⇒ Light-cone OPE

$$x^2_{12_\perp}(\mu^2 x^2_{12_\perp})^{\gamma_K}\int dx_{1_+}\ \mathcal{O}(L_+ + x_{2_+}, x_{1_\perp})\mathcal{O}(x_{2_+}, x_{2_\perp})$$
$$= \int_{\frac12 - i\infty}^{\frac12 + i\infty} dj\ c(j,\alpha_s)[1+e^{i\pi j}]L_+^j(\mu^2 x^2_{12_\perp})^{\gamma(j,\alpha_s)} S_j^+(x_{1_\perp})$$

⇒ “DGLAP” result (at $x^2_{12} \to 0$)

$$\mu^{-4}(\mu^4 x^2_{12}x^2_{34})^{2+\gamma_k}\int dx_{2_+}dx_{3_-}\ \langle\mathcal{O}(L_+ + x_{2_+}, x_{1_\perp})\mathcal{O}(x_{2_+}, x_{2_\perp})\mathcal{O}(L_- + x_{4_-}, x_{3_\perp})\mathcal{O}(x_{4_-}, x_{4_\perp})$$

“DGLAP” represenation of 4-point CFs

⇒ Light-cone OPE

$$x^2_{12_\perp}(\mu^2 x^2_{12_\perp})^{\gamma_K}\int dx_{1_+}\ \mathcal{O}(L_+ + x_{2_+}, x_{1_\perp})\mathcal{O}(x_{2_+}, x_{2_\perp})$$
$$= \int_{\frac12 - i\infty}^{\frac12 + i\infty} dj\ c(j,\alpha_s)[1+e^{i\pi j}]L^j_+(\mu^2 x^2_{12_\perp})^{\gamma(j,\alpha_s)} S^+_j(x_{1_\perp})$$

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$$= \int_{\frac12 - i\infty}^{\frac12 + i\infty} dj\ C(j,\alpha_s)[1+e^{i\pi j}]L^j_+(\mu^2 x^2_{12_\perp})^{\frac12\gamma(j,\alpha_s)}\langle S^\mu_j(x_{2_\perp})\mathcal{O}(L_- + x_{4_-}, x_{3_\perp})\mathcal{O}(x_{4_-}, x_{4_\perp})\rangle$$

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$$= \int_{\frac12 - i\infty}^{\frac12 + i\infty} dj\ C(j,\alpha_s)\int_0^1 dv\ \frac{[1+e^{i\pi j}](L_+L_-)^j}{[x^2_{12_\perp}v + x^2_{13_\perp}\bar v]^{1+j}}\left(\frac{x^2_{12}x^2_{34}}{[x^2_{13_\perp}v + x^2_{14_\perp}\bar v]^2}\right)^{\frac12\gamma(j,\alpha_s)}(\bar v v)^{j+\frac12\gamma(j,\alpha_s)}$$

DGLAP *vs* BFKL for 4-point CFs

"DGLAP" result (leading twist)

$$\mu^{-4}(\mu^4 x_{12}^2 x_{34}^2)^{2+\gamma_k}\int dx_{2_+}dx_{3_-}\ \langle\mathcal{O}(L_+ + x_{2_+},x_{1_\perp})\mathcal{O}(x_{2_+},x_{2_\perp})\mathcal{O}(L_- + x_{4_-},x_{3_\perp})\mathcal{O}(x_{4_-},x_{4_\perp})\rangle$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj\ C(j,\alpha_s)\int_0^1 dv\ \frac{[1+e^{i\pi j}](L_+L_-)^j}{[x_{12_\perp}^2 v + x_{13_\perp}^2 \bar{v}]^{1+j}}\Big(\frac{x_{12}^2 x_{34}^2}{[x_{13_\perp}^2 v + x_{14_\perp}^2 \bar{v}]^2}\Big)^{\frac{1}{2}\gamma(j,\alpha_s)}(\bar{v}v)^{j+\frac{1}{2}\gamma(j,\alpha_s)}$$

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$$= \int_{\frac12 - i\infty}^{\frac12 + i\infty} dj\ C(j,\alpha_s)\int_0^1 dv\ \frac{[1+e^{i\pi j}](L_+L_-)^j}{[x_{12_\perp}^2 v + x_{13_\perp}^2 \bar v]^{1+j}}\Big(\frac{x_{12}^2 x_{34}^2}{[x_{13_\perp}^2 v + x_{14_\perp}^2 \bar v]^2}\Big)^{\frac12\gamma(j,\alpha_s)}(\bar v v)^{j+\frac12\gamma(j,\alpha_s)}$$

BFKL result ($\aleph(\xi,\alpha_s) = \frac{\alpha_s N_c}{\pi}\chi(\xi) + ...$ - pomeron intercept)

$$[x_{12}^2 x_{34}^2]^{2+\gamma_K}\int dx_{2_+}dx_{3_-}\ \langle\mathcal{O}(L_+ + x_{2_+},x_{1\perp})\mathcal{O}(x_{2_+},x_{2\perp})\mathcal{O}(L_- + x_{3_-},x_{2\perp})\mathcal{O}(x_{3_-},x_{3\perp})\rangle$$

$$= \frac{i\alpha_s^2}{8}\pi^2\int_0^1 dv\int_{\frac12 - i\infty}^{\frac12 + i\infty} d\xi\, f\big(\aleph(\xi)\big) \qquad \xi \equiv \frac12 + i\nu$$

$$\times\ \frac{(\bar v v)^{1-\xi+\frac{\aleph(\xi)}{2}}\cos\pi\xi}{(\xi-\frac12)\sin^3\pi\xi}\ \frac{B\big(2-\xi+\frac{\aleph(\xi)}{2}\big)}{[x_{13\perp}^2 v + x_{14}^2\bar v]^{2+\aleph(\xi)}}\Big(\frac{x_{12\perp}^2 x_{34\perp}^2}{[x_{13\perp}^2 v + x_{14}^2 \bar v]^2}\Big)^{-\xi-\frac{\aleph(\xi)}{2}}(L_+L_-)^{1+\aleph(\xi)}$$

DGLAP *vs* BFKL for 4-point CFs

"DGLAP" result (leading twist)

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$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj\ C(j,\alpha_s)\int_0^1 dv\ \frac{[1+e^{i\pi j}](L_+L_-)^j}{[x_{12_\perp}^2 v + x_{13_\perp}^2 \bar{v}]^{1+j}}\Big(\frac{x_{12}^2 x_{34}^2}{[x_{13_\perp}^2 v + x_{14_\perp}^2 \bar{v}]^2}\Big)^{\frac{1}{2}\gamma(j,\alpha_s)}(\bar{v}v)^{j+\frac{1}{2}\gamma(j,\alpha_s)}$$

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$$\times\ \frac{(\bar{v}v)^{1-\xi+\frac{\aleph(\xi)}{2}}\cos\pi\xi}{(\xi-\frac{1}{2})\sin^3\pi\xi}\ \frac{B\big(2-\xi+\frac{\aleph(\xi)}{2}\big)}{[x_{13\perp}^2 v + x_{14}^2\bar{v}]^{2+\aleph(\xi)}}\Big(\frac{x_{12\perp}^2 x_{34\perp}^2}{[x_{13\perp}^2 v + x_{14}^2\bar{v}]^2}\Big)^{-\xi-\frac{\aleph(\xi)}{2}}(L_+L_-)^{1+\aleph(\xi)}$$

We compare these results around $\xi \sim j-1 \sim 0$

$$\Rightarrow\ 1+\aleph(\xi,\alpha_s) = j\ \text{ and }\ \gamma(j,\alpha_s) = -2\xi - \aleph(\xi)$$

$$\aleph(\xi) - 2\aleph(\xi)\aleph'(\xi)\ \simeq\ \frac{\alpha_s N_c}{\pi\xi} + \frac{\zeta(3)\alpha_s^2}{\xi} + ...\ \ \rightarrow\ \ \gamma_j = -2\frac{\alpha_s N_c}{\pi(j-1)}\ +\ [0+\zeta(3)(j-1)]\Big(\frac{\alpha_s N_c}{\pi(j-1)}\Big)^3 + ...$$

Result

$$\gamma_j = -2\frac{\alpha_s N_c}{\pi(j-1)} \; + \; [0 + \zeta(3)(j-1)]\Big(\frac{\alpha_s N_c}{\pi(j-1)}\Big)\Big)^3 + ...$$

is an anomalous dimension of light-ray operator $F\nabla^{j-2}F(x_\perp)$

Conclusions and Outlook

1. Conclusions:
   - NLO BFKL gives anomalous dimensions of light-ray operators $F_{+i}\nabla^{\omega-1}F_{+}{}^{i}$ as $\omega \to 0$ (in all orders in α_s)

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1. Conclusions:
   - NLO BFKL gives anomalous dimensions of light-ray operators $F_{+i}\nabla^{\omega-1}F_{+}{}^{i}$ as $\omega \to 0$ (in all orders in α_s)
2. Outlook
   - In QCD
   - 3-point CF of $\langle F\nabla^{\omega_1-1}F(x_{1_\perp})\ F\nabla^{\omega_2-1}F(x_{2_\perp})\ F\nabla^{\omega_3-1}F(x_{3_\perp})\rangle$ as $\omega_1,\omega_2,\omega_3 \to 0$ (joint project with V. Kazakov and E. Sobko):