Initial particle production in nucleus-nucleus collisions, factorization, instabilities

François Gelis

CERN – TH division
Typical e+e- or pp collision
Why is pQCD predictive there?

- More precisely, why is pQCD predictive despite the fact that hadrons are non-perturbative bound states?

- **Factorization**:

  (Collinear) divergences in loop corrections can be absorbed into the (DGLAP) evolution of parton distributions and fragmentation functions.

- **Universality**: parton distributions are process independent.
Collinear factorization breaks down for momenta $p_\perp \lesssim Q_s$

99% of the multiplicity below $p_\perp \sim 2 \text{ GeV}$
Goals

- The Color Glass Condensate framework provides the technology for resumming all the $[Q_s/p_\perp]^n$ corrections

- Generalize the concept of “parton distribution”
  - Due to the high density of partons, observables depend on higher correlations (beyond the usual parton distributions, which are 2-point correlation functions)

- If divergences show up in loop corrections, one should be able to factor them out into the evolution of these distributions

- These distributions should be universal, with non-perturbative information relegated into the initial condition for the evolution

- There may possibly be extra divergences associated with the evolution of the final state
What do we mean by Initial Conditions?

- calculate the initial production of semi-hard particles
- prepare the stage for kinetic theory or hydrodynamics
Outline

■ Basic principles and bookkeeping
■ Inclusive gluon spectrum at leading order
■ 1-loop corrections, divergences and resummations
  - FG, Venugopalan, hep-ph/0601209, 0605246
  - Fukushima, FG, McLerran, hep-ph/0610416 (Kenji’s talk)
  - FG, Jeon, Venugopalan, in preparation
    + work in progress with Lappi, Venugopalan
Basic principles
Degrees of freedom and their interplay


- Soft modes have a large occupation number
  - they are described by a classical color field $A^\mu$ that obeys Yang-Mills’s equation:
    \[
    [D_\nu, F_{\nu\mu}]_a = J_\mu^A
    \]

- The source term $J_\mu^A$ comes from the faster partons. The hard modes, slowed down by time dilation, are described as frozen color sources $\rho_a$. Hence:
  \[
  J_\mu^A = \delta^\mu + \delta(x^-) \rho_a(x_\perp) \quad (x^- \equiv (t - z)/\sqrt{2})
  \]

- The color sources $\rho_a$ are random, and described by a distribution functional $W_Y[\rho]$, with $Y$ the rapidity that separates “soft” and “hard”. Evolution equation (JIMWLK):
  \[
  \frac{\partial W_Y[\rho]}{\partial Y} = \mathcal{H}[\rho] \ W_Y[\rho]
  \]
Description of hadronic collisions

- Compute the observable $O$ of interest for a configuration of the sources $\rho_1, \rho_2$. Note: the sources are $\sim 1/g$ weak coupling but strong interactions.

- At LO, this requires to solve the classical Yang-Mills equations in the presence of the following current:

$$J^\mu \equiv \delta^\mu_+ \delta(x^-) \rho_1(\vec{x}_\perp) + \delta^\mu_- \delta(x^+) \rho_2(\vec{x}_\perp)$$

(Note: the boundary condition depend on the observable)

- Average over the sources $\rho_1, \rho_2$

$$\langle O_Y \rangle = \int [D \rho_1] [D \rho_2] W_{Y_{\text{beam}}-Y}[\rho_1] W_{Y+Y_{\text{beam}}}[\rho_2] O[\rho_1, \rho_2]$$

- Can this procedure – and in particular the above factorization formula – be justified?
Main issues

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- **Dilute regime**: one source in each projectile interact
- **Dense regime**: non linearities are important
- Many gluons can be produced from the same diagram
- There can be many simultaneous disconnected diagrams
- Some of them may not produce anything (**vacuum diagrams**)
- All these diagrams can have loops (not at LO though)
In the **saturated regime**, the sources are of order $1/g$

The order of each **disconnected diagram** is given by:

$$\frac{1}{g^2} g \# \text{ produced gluons} g^2(\# \text{ loops})$$

The total order of a graph is the product of the orders of its disconnected subdiagrams ➤ quite messy...
Bookkeeping
Consider squared amplitudes (including interference terms) rather than the amplitudes themselves.
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See them as cuts through vacuum diagrams
Bookkeeping

- Consider **squared amplitudes** (including interference terms) rather than the amplitudes themselves.

- See them as **cuts through vacuum diagrams**.

- Consider **only the simply connected** ones, thanks to:

\[
\sum \left( \text{all the vacuum diagrams} \right) = \exp \left\{ \sum \left( \text{simply connected vacuum diagrams} \right) \right\}
\]

- Simpler power counting for connected vacuum diagrams:

\[
\frac{1}{g^2} g^2 (# \text{ loops})
\]
There is an operator $\mathcal{D}$ that acts on a pair of vacuum diagrams by removing two sources and attaching a cut propagator instead:
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$D$ can also act directly on single diagram, if it is already cut.

By repeated action of $\mathcal{D}$, one generates all the diagrams with an arbitrary number of cuts.

Thanks to this operator, one can write:

$$P_n = \frac{1}{n!} \mathcal{D}^n e^{iV} e^{-iV^*}, \quad iV = \sum \left( \begin{array}{c} \text{connected uncut vacuum diagrams} \\ \text{uncut vacuum diagrams} \end{array} \right)$$

$$\sum \left( \begin{array}{c} \text{all the cut vacuum diagrams} \\ \text{cut vacuum diagrams} \end{array} \right) = e^\mathcal{D} e^{iV} e^{-iV^*}$$
Inclusive gluon spectrum
First moment of the distribution

It is easy to express the average multiplicity as:

\[ \overline{N} = \sum_n n P_n = D \left\{ e^D e^{iV} e^{-iV^*} \right\} \]

\( \overline{N} \) is obtained by the action of \( D \) on the sum of all the cut vacuum diagrams. There are two kind of terms:

- \( D \) picks two sources in two distinct connected cut diagrams
- \( D \) picks two sources in the same connected cut diagram
Gluon multiplicity at LO

- At LO, only tree diagrams contribute to the second type of topologies can be neglected (it starts at 1-loop)

- In each blob, we must sum over all the tree diagrams, and over all the possible cuts:

\[
\overline{N}_{LO} = \sum_{\text{trees}} \sum_{\text{cuts}}
\]

- A major simplification comes from the following property:

\[\sim \sim \sim + \sim \sim = \text{retarded propagator}\]

- The sum of all the tree diagrams constructed with retarded propagators is the retarded solution of Yang-Mills equations:

\[
[D_\mu, F^{\mu\nu}] = J^\nu \quad \text{with} \quad A^\mu(x_0 = -\infty) = 0
\]
Gluon multiplicity at LO


\[
\frac{dN_{LO}}{dY d^2p_\perp} = \frac{1}{16\pi^3} \int_{x,y} e^{ip\cdot(x-y)} \Box_x \Box_y \sum_\lambda \epsilon_\lambda^\mu \epsilon_\lambda^\nu A_\mu(x)A_\nu(y)
\]

\[ A_\mu(x) = \text{retarded solution of Yang-Mills equations} \]

only tree diagrams at LO
Gluon multiplicity at LO


\[
\frac{d\overline{N}_{LO}}{dY d^2\vec{p}_\perp} = \frac{1}{16\pi^3} \int_{x,y} e^{i\vec{p} \cdot (x-y)} \Box x \Box y \sum_\lambda \epsilon_\mu^\lambda \epsilon_\nu^\lambda A_\mu(x) A_\nu(y)
\]

- \(A_\mu(x) = \) retarded solution of Yang-Mills equations
- can be cast into an initial value problem on the light-cone
Gluon multiplicity at LO

- Lattice artefacts at large momentum (they do not affect much the overall number of gluons)
- Important **softening at small** $k_\perp$ compared to pQCD (saturation)
Initial conditions and boost invariance

- **Gauge condition**: \( x^+ A^- + x^- A^+ = 0 \)

\[
\begin{align*}
A^i(x) &= \alpha^i(\tau, \eta, \vec{x}_\perp) \\
A^\pm(x) &= \pm x^\pm \beta(\tau, \eta, \vec{x}_\perp)
\end{align*}
\]

- Initial values at \( \tau = 0^+ \): \( \alpha^i(0^+, \eta, \vec{x}_\perp) \) and \( \beta(0^+, \eta, \vec{x}_\perp) \) do not depend on the rapidity \( \eta \)

- \( \alpha^i \) and \( \beta \) remain independent of \( \eta \) at all times (invariance under boosts in the \( z \) direction)

- numerical resolution performed in \( 1 + 2 \) dimensions
Local anisotropy

For a system finite in the $\eta$ direction, the gluons will have a longitudinal velocity tied to their space-time rapidity.
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- For a system finite in the $\eta$ direction, the gluons will have a longitudinal velocity tied to their space-time rapidity: $v_z = \tanh(\eta)$.

- At late times: if particles fly freely, only one longitudinal velocity can exist at a given $\eta$.

- The expansion of the system is the main obstacle to local isotropy.
NLO corrections
1-loop corrections to $\overline{N}$

- 1-loop diagrams for $\overline{N}$
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- This can be seen as a perturbation of the initial value problem encountered at LO, e.g.:
1-loop corrections to $N$

1-loop diagrams for $\overline{N}$

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The 1-loop correction to $\bar{N}$ can be written as a perturbation of the initial value problem encountered at LO:
1-loop corrections to $N$

The 1-loop correction to $\overline{N}$ can be written as a perturbation of the initial value problem encountered at LO:

$$\delta \overline{N} = \left[ \int_{\vec{x} \in \text{light cone}} \delta A(\vec{x}) \ T_{\vec{x}} \right] \overline{N}_{LO}$$

- $\overline{N}_{LO}$ is a functional of the initial fields $A_{\text{in}}(\vec{x})$ on the light-cone.
- $T_{\vec{x}}$ is the generator of shifts of the initial condition at the point $\vec{x}$ on the light-cone, i.e.: $T_{\vec{x}} \sim \delta / \delta A_{\text{in}}(\vec{x})$. 
The 1-loop correction to $\overline{N}$ can be written as a perturbation of the initial value problem encountered at LO:

$$\delta \overline{N} = \left[ \int_{\vec{x} \in \text{light cone}} \delta A(\vec{x}) \ T_{\vec{x}} + \int_{\vec{x}, \vec{y} \in \text{light cone}} \frac{1}{2} \Sigma(\vec{x}, \vec{y}) \ T_{\vec{x}} \ T_{\vec{y}} \right] \overline{N}_{LO}$$

- $\overline{N}_{LO}$ is a functional of the initial fields $A_{\text{in}}(\vec{x})$ on the light-cone
- $T_{\vec{x}}$ is the generator of shifts of the initial condition at the point $\vec{x}$ on the light-cone, i.e.: $T_{\vec{x}} \sim \delta / \delta A_{\text{in}}(\vec{x})$
- $\delta A(\vec{x})$ and $\Sigma(\vec{x}, \vec{y})$ are in principle calculable analytically
Divergences

- If taken at face value, this 1-loop correction is plagued by several divergences:

  - The two coefficients $\delta A(\vec{x})$ and $\Sigma(\vec{x}, \vec{y})$ are infinite, because of an unbounded integration over a rapidity variable.

  - At late times, $T_{\vec{x}} A(\tau, \vec{y})$ diverges exponentially,
    $$T_{\vec{x}} A(\tau, \vec{y}) \sim e^{\sqrt{\mu \tau}}$$
    because of an instability of the classical solution of Yang-Mills equations under rapidity dependent perturbations (Romatschke, Venugopalan (2005))
Initial state factorization

Anatomy of the full calculation:

\[ \begin{array}{c}
W_{Y_{\text{beam}}^{-}Y}[\rho_1] \\
N[ A_{\text{in}}(\rho_1, \rho_2) ] \\
W_{Y_{\text{beam}}^{+}Y}[\rho_2]
\end{array} \]
Initial state factorization

- Anatomy of the full calculation:

\[
\begin{align*}
W_{Y_{\text{beam}}-Y} [\rho_1] \\
N[ A_{\text{in}}(\rho_1, \rho_2) ] + \delta N \\
W_{Y_{\text{beam}}+Y} [\rho_2]
\end{align*}
\]

- When the observable \( \overline{N}[ A_{\text{in}}(\rho_1, \rho_2) ] \) is corrected by an extra gluon, one gets **divergences** of the form \( \alpha_s \int dY \) in \( \delta \overline{N} \)

\( \triangleright \) one would like to be able to absorb these divergences into the \( Y \) dependence of the source densities \( W_Y [\rho_{1,2}] \)
Initial state factorization

Anatomy of the full calculation:

When the observable $\overline{N}[A_{in}(\rho_1, \rho_2)]$ is corrected by an extra gluon, one gets divergences of the form $\alpha_s \int dY$ in $\delta \overline{N}$

One would like to be able to absorb these divergences into the $Y$ dependence of the source densities $W_Y[\rho_{1,2}]$

Equivalently, if one puts some arbitrary frontier $Y_0$ between the “observable” and the “source distributions”, the dependence on $Y_0$ should cancel between the various factors
The two kind of divergences don’t mix, because the divergent part of the coefficients is boost invariant.

Given their structure, the divergent coefficients seem related to the evolution of the sources in the initial state

In order to prove the factorization of these divergences in the initial state distributions of sources, one needs to establish:

\[
\begin{align*}
\left[ \delta N \right]_{\text{divergent coefficients}} &= (Y_0 - Y) \mathcal{H}^\dagger[\rho_1] + (Y - Y'_0) \mathcal{H}^\dagger[\rho_2] \end{align*}
\]

where \( \mathcal{H}[\rho] \) is the Hamiltonian that governs the rapidity dependence of the source distribution \( W_Y[\rho] \):

\[
\frac{\partial W_Y[\rho]}{\partial Y} = \mathcal{H}[\rho] W_Y[\rho]
\]

FG, Lappi, Venugopalan (work in progress)
Initial state factorization

Why is it plausible?

- Reminder:

\[
\begin{align*}
\left[ \delta \bar{N} \right]_{\text{divergent coefficients}} &= \left\{ \int_{\bar{x}} \left[ \delta A(\bar{x}) \right] \text{div} T_{\bar{x}} \\
&\quad + \frac{1}{2} \int_{\bar{x}, \bar{y}} \left[ \Sigma(\bar{x}, \bar{y}) \right] \text{div} T_{\bar{x}} T_{\bar{y}} \right\} \bar{N}_{LO}
\end{align*}
\]

- Compare with the evolution Hamiltonian:

\[
\mathcal{H}[\rho] = \int_{\bar{x}_\perp} \sigma(\bar{x}_\perp) \frac{\delta}{\delta \rho(\bar{x}_\perp)} + \frac{1}{2} \int_{\bar{x}_\perp, \bar{y}_\perp} \chi(\bar{x}_\perp, \bar{y}_\perp) \frac{\delta^2}{\delta \rho(\bar{x}_\perp) \delta \rho(\bar{y}_\perp)}
\]

The coefficients \(\sigma\) and \(\chi\) in the Hamiltonian are well known. There is a well defined calculation that will tell us if it works...
Romatschke, Venugopalan (2005)

- Rapidity dependent perturbations to the classical fields grow like \( \exp\left(\sqrt{\tau}\right) \) until the non-linearities become important:

\[
\max \frac{\tau^m}{g^4 L^2} = c_0 + c_1 \exp(0.427 \sqrt{g^2 \mu \tau})
\]

\[
\max \frac{\tau^m}{g^4 L^2} = c_0 + c_1 \exp(0.00544 g^2 \mu \tau)
\]
Unstable modes

- The coefficient $\delta A(\vec{x})$ is boost invariant. Hence, the divergences due to the unstable modes all come from the quadratic term in $\delta \overline{N}$:

$$
\begin{array}{l}
\left[ \delta \overline{N} \right]_{\text{unstable modes}} = \left\{ \frac{1}{2} \int_{\vec{x},\vec{y}} \Sigma(\vec{x},\vec{y}) T_{\vec{x}} T_{\vec{y}} \right\} \overline{N}_{\text{LO}} [A_{\text{in}}(\rho_1, \rho_2)] \\
\end{array}
$$

- When summed to all orders, these divergences exponentiate:

$$
\begin{array}{l}
\left[ \delta \overline{N} \right]_{\text{unstable modes}} = e^{\frac{1}{2} \int_{\vec{x},\vec{y}} \Sigma(\vec{x},\vec{y}) T_{\vec{x}} T_{\vec{y}}} \overline{N}_{\text{LO}} [A_{\text{in}}(\rho_1, \rho_2)] \\
\end{array}
$$
Unstable modes

This can be arranged in a more intuitive way:

\[
\begin{aligned}
\left[ \delta \overline{N} \right]_{\text{unstable modes}} &= \int [Da] \left( \frac{1}{2} \int \bar{a}, \bar{y} \right) \sum_{\bar{x}, \bar{y}} e^{i \int \bar{x} a(\bar{x}) T \bar{a}} \overline{N}_{\text{LO}} [A_{\text{in}}(\rho_1, \rho_2)] \\
&= \int [Da] \overline{Z}_{\text{fluct}}[a] \overline{N}_{\text{LO}} [A_{\text{in}}(\rho_1, \rho_2) + a]
\end{aligned}
\]

- summing these divergences simply requires to add Gaussian fluctuations to the initial condition for the classical problem
- See Kenji’s talk for a different perspective on this issue
  + evaluation of the spectrum of fluctuations

Interpretation:

Despite the fact that the fields are coupled to strong sources, the classical approximation alone is not good enough, because the classical solution has unstable modes that can be triggered by the quantum fluctuations.
Outstanding issues

- There are infrared and collinear singularities associated to splittings on the final leg of the diagram
  - Since they occur on the outer legs, they can be factored out in the usual fragmentation functions,
  - but they need to be disentangled from the other divergences...
  - What are “infrared and collinear safe” observables?
    - e.g. it would be nice to be able to handle the transition between the initial particle production and kinetic theory (or hydrodynamics) without the need of some non-perturbative input
    - for hydrodynamics, things should be easier because the density of energy-momentum tensor is probably more robust against soft or collinear splittings

- Integrating out the fluctuations may lead to ultra-violet divergences
  - Renormalization in a non-perturbative approximation?
  - Renormalization scale and RG equation?
Summary
Summary

- When the parton densities in the projectiles are large, the study of particle production becomes rather involved
  - non-perturbative techniques that resum all-twist contributions are needed

- At Leading Order, the inclusive gluon spectrum can be calculated from the classical solution with retarded boundary conditions on the light-cone

- At Next-to-Leading Order, the gluonic corrections can be seen as a perturbation of the initial value problem encountered at LO

- Resummation of the leading divergences to all orders:
  - Evolution with $Y$ of the distribution of sources
  - Quantum fluctuations on top of initial condition for the classical solution in the forward light-cone
Summary

Classical solution in 2+1 dimensions
Combining everything, one should write

\[
\frac{d\overline{N}}{dY \, d^2 \vec{p}_\perp} = \int [D\rho_1] \, [D\rho_2] \, W_{Y_{\text{beam}}-Y} [\rho_1] \, W_{Y_{\text{beam}}+Y} [\rho_2] \\
\times \int [Da] \, Z_{\text{fluct}} [a] \, \frac{d\overline{N}[A_{\text{in}}(\rho_1, \rho_2)+a]}{dY \, d^2 \vec{p}_\perp}
\]

This formula resums (all?) the divergences that occur at one loop.
Summary

- Tree level:
Summary

- Tree level:

- One loop:

  - The momentum $\vec{q}$ is integrated out
  - If $\alpha_s^{-1} \lesssim |y_p - y_q|$, the correction is absorbed in $W[\rho_{1,2}]$
  - If $|y_p - y_q| \lesssim \alpha_s^{-1}$: gluon splitting in the final state
Summary

- After summing the fluctuations, things may look like this:

> these splittings may help to fight against the expansion?

Note: the timescale for this process is \( \tau \sim Q_s^{-1} \ln^2(1/\alpha_s) \)
Extra bits
Parton evolution

▷ assume that the projectile is big, e.g. a nucleus, and has many valence quarks (only two are represented)
▷ on the contrary, consider a small probe, with few partons
▷ at low energy, only valence quarks are present in the hadron wave function
Parton evolution

▷ when energy increases, new partons are emitted

▷ the emission probability is \( \alpha_s \int \frac{dx}{x} \sim \alpha_s \ln(\frac{1}{x}) \), with \( x \) the longitudinal momentum fraction of the gluon

▷ at small-\( x \) (i.e. high energy), these logs need to be resummed
as long as the density of constituents remains small, the evolution is **linear**: the number of partons produced at a given step is proportional to the number of partons at the previous step (BFKL)
Eventually, the partons start overlapping in phase-space.

Parton recombination becomes favorable.

After this point, the evolution is non-linear: the number of partons created at a given step depends non-linearly on the number of partons present previously.
Saturation criterion

Gribov, Levin, Ryskin (1983)

- Number of gluons per unit area:
  \[ \rho \sim \frac{x G_A(x, Q^2)}{\pi R_A^2} \]

- Recombination cross-section:
  \[ \sigma_{gg\to g} \sim \frac{\alpha_s}{Q^2} \]

- Recombination happens if \( \rho \sigma_{gg\to g} \gtrsim 1 \), i.e. \( Q^2 \lesssim Q_s^2 \), with:
  \[ Q_s^2 \sim \frac{\alpha_s x G_A(x, Q_s^2)}{\pi R_A^2} \sim A^{1/3} \frac{1}{x^{0.3}} \]

- At saturation, the phase-space density is:
  \[ \frac{dN_g}{d^2 \vec{x}_\perp d^2 \vec{p}_\perp} \sim \frac{\rho}{Q^2} \sim \frac{1}{\alpha_s} \]
Saturation domain

\[ \log(x^{-1}) \]

\[ \log(Q^2) \]

\[ \Lambda_{QCD} \]
Diagrammatic interpretation

- One loop:
Diagrammatic interpretation

- One loop:

- Two loops:
Diagrammatic interpretation

- One loop:

- Two loops:

▷ The sum of tree diagrams for fluctuations on top of the classical field with initial condition $A_{\text{in}}$ gives the classical field with a shifted initial condition $A_{\text{in}} + a$

▷ If we keep only the fastest growing terms, we need only the leading two-point correlation of the initial fluctuation $a$
Definition

- One can encode the information about all the probabilities $P_n$ in a generating function defined as:

$$F(z) \equiv \sum_{n=0}^{\infty} P_n z^n$$

- From the expression of $P_n$ in terms of the operator $D$, we can write:

$$F(z) = e^{zD} e^{iV} e^{-iV^*}$$

- Reminder:
  - $e^{zD} e^{iV} e^{-iV^*}$ is the sum of all the cut vacuum diagrams
  - The cuts are produced by the action of $D$

- Therefore, $F(z)$ is the sum of all the cut vacuum diagrams in which each cut line is weighted by a factor $z$
What would it be good for ?

Let us pretend that we know the generating function $F(z)$. We could get the probability distribution as follows:

$$P_n = \frac{1}{2\pi} \int_0^{2\pi} d\theta \ e^{-in\theta} F(e^{i\theta})$$

Note: this is trivial to evaluate numerically by a FFT:
F(z) at Leading Order

- We have: \( F'(z) = D \left\{ e^{zD} e^{iV} e^{-iV^*} \right\} \)

- By the same arguments as in the case of \( \overline{N} \), we get:

\[
\frac{F'(z)}{F(z)} = +
\]

- The major difference is that the sum of cut graphs that must be evaluated have a factor \( z \) attached to each cut line

- At tree level (LO), we can write \( F'(z)/F(z) \) in terms of solutions of the classical Yang-Mills equations, but these solutions are not retarded anymore, because:

\[
\text{--------- + z ~x~ ~x~} \neq \text{retarded propagator}
\]
**F(z) at Leading Order**

- The derivative $F'/F$ has an expression which is formally identical to that of $\langle n_{\text{gluons}} \rangle$,

$$
\frac{F'(z)}{F(z)} = \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_p} \int_{x,y} e^{i p \cdot (x-y)} \sum_\lambda \epsilon^\mu_\lambda \epsilon^\nu_\lambda A^{(\pm)}_\mu(x) A^{(\pm)}_\nu(y),
$$

with $A^{(\pm)}_\mu(x)$ two solutions of the Yang-Mills equations

- If one decomposes these fields into plane-waves,

$$
A^{(\varepsilon)}_\mu(x) \equiv \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_p} \left\{ f^{(\varepsilon)}_+(x^0, \vec{p}) e^{-i p \cdot x} + f^{(\varepsilon)}_-(x^0, \vec{p}) e^{i p \cdot x} \right\}
$$

the boundary conditions are:

$$
f^{(+)}_+(\pm\infty, \vec{p}) = f^{(-)}_-(\pm\infty, \vec{p}) = 0
$$

$$
f^{(-)}_+(\pm\infty, \vec{p}) = z f^{(+)}_+(\pm\infty, \vec{p}), \quad f^{(+)}_-(\pm\infty, \vec{p}) = z f^{(-)}_-(\pm\infty, \vec{p})
$$

- There are boundary conditions both at $x_0 = -\infty$ and $x_0 = +\infty$ ➔ not an initial value problem ➔ hard++...
Remarks on factorization

- As we’ve seen before, the fact that the calculation of the first moment $\overline{N}$ can be formulated as an initial value problem is probably going to be crucial for proving factorization.

- If this is indeed the case, then factorization does not hold for the generating function $F(z)$, or equivalently for the individual probabilities $P_n$.

- Preliminary studies seem to indicate that the computation of higher moments of the distribution can also be reformulated as initial value problems.
  - the moments would be factorizable as well.
Quark production


\[ E_P \frac{d\langle n_{\text{quarks}} \rangle}{d^3 \vec{p}} = \frac{1}{16\pi^3} \int_{x,y} e^{i \vec{p} \cdot (x-y)} \phi_x \phi_y \langle \overline{\psi}(x) \psi(y) \rangle \]

- Dirac equation in the classical color field:
Quark production


\[ E_p \frac{d\langle n_{\text{quarks}} \rangle}{d^3 \vec{p}} = \frac{1}{16\pi^3} \int_{x,y} e^{i\vec{p} \cdot (x-y)} \phi_x \phi_y \langle \overline{\psi}(x) \psi(y) \rangle \]

- Dirac equation in the classical color field:
Spectra for various quark masses

![Graph showing spectra for various quark masses]

- For $m = 60$ MeV
- For $m = 300$ MeV
- For $m = 600$ MeV
- For $m = 1.5$ GeV
- For $m = 3$ GeV

$dN/dy d^2q_T$ [arbitrary units]

$m$ indicates quark mass in various units.

$dN/dy d^2q_T$ represents the differential number of events with respect to $y$ and $q_T$.
Exclusive processes

- So far, we have considered only **inclusive quantities** – i.e. the $P_n$ are defined as probabilities of producing particles **anywhere** in phase-space.

- What about events where a part of the phase-space remains unoccupied? e.g. **rapidity gaps**

![Diagram showing exclusive processes with empty region in the middle and rapidity scale on the right.](https://example.com/diagram.png)
Main issues

1. How do we calculate the probabilities $P_{n}^{\text{excl}}$ with an excluded region $\Omega$ in the phase-space? Can one calculate the total gap probability $P_{\text{gap}} = \sum_{n} P_{n}^{\text{excl}}$?

2. What is the appropriate distribution of sources $W^{\text{excl}}_{y}[\rho]$ to describe a projectile that has not broken up?

3. How does it evolve with rapidity?
   
   See: Hentschinski, Weigert, Schafer (2005)

4. Are there some factorization theorems, and for which quantities do they hold?
Exclusive probabilities

The probabilities $P_{n}^{\text{excl}}[\Omega]$, for producing $n$ particles – but none in the region $\Omega$ – can also be constructed from the vacuum diagrams, as follows:

$$P_{n}^{\text{excl}}[\Omega] = \frac{1}{n!} D_{\Omega}^{n} e^{iV} e^{-iV^{*}}$$

where $D_{\Omega}$ is an operator that removes two sources and links the corresponding points by a cut (on-shell) line, for which the integration is performed only in the region $\Omega$.

One can define a generating function,

$$F_{\Omega}(z) \equiv \sum_{n} P_{n}^{\text{excl}}[\Omega] z^{n},$$

whose derivative is given by the same diagram topologies as the derivative of the generating function for inclusive probabilities.
Exclusive probabilities

- Differences with the inclusive case:
  - In the diagrams that contribute to $\frac{F'_\Omega(z)}{F_\Omega(z)}$, the cut propagators are restricted to the region $\Omega$ of the phase-space.
    - At leading order, this only affects the boundary conditions for the classical fields in terms of which one can write $\frac{F'_\Omega(z)}{F_\Omega(z)}$.
    - Not more difficult than the inclusive case.

- Contrary to the inclusive case – where we know that $F'(1) = 1$ – the integration constant needed to go from $\frac{F'_\Omega(z)}{F_\Omega(z)}$ to $F_\Omega(z)$ is non-trivial. This is due to the fact that the sum of all the exclusive probabilities is smaller than unity.
  - $F_\Omega(1)$ is in fact the probability of not having particles in the region $\Omega$ – i.e. the gap probability.
We can write:

\[
F_\Omega(z) = F_\Omega(1) \exp \left\{ \int_1^z d\tau \frac{F'_\Omega(\tau)}{F_\Omega(\tau)} \right\}
\]

- the prefactor \(F_\Omega(1)\) will appear in all the exclusive probabilities

This prefactor is nothing but the famous "survival probability" for a rapidity gap.

- One can in principle calculate it by the general techniques developed for calculating inclusive probabilities:

\[
F_\Omega(1) = F_{1-\Omega}\text{incl}(0)
\]

- Note: it is incorrect to say that a certain process with a gap can be calculated by multiplying the probability of this process without the gap by the survival probability.
Factorization?

- Except for the case of Deep Inelastic Scattering, nothing is known regarding factorization for exclusive processes in a high density environment.

- For the overall framework to be consistent, one should have factorization between the gap probability, $F_\Omega (1)$, and the source density studied in Hentschinski, Weigert, Schafer (2005) (and the ordinary $W_Y [\rho]$ on the other side).

- The total gap probability is the “most inclusive” among the exclusive quantities one may think of. For what quantities – if any – does factorization work?