WILSON LOOPS, SHOCK WAVES AND RANDOM MATRICES

Maciej A. Nowak

M. Kac Complex Systems Research Center,
Jagiellonian University, Kraków

Quantum Field Theory in Extreme Environments:
60 th birthdays of Jean-Paul Blaizot and Larry McLerran
April 23-25, 2009, Saclay - Paris

(based on ongoing work with Jean-Paul Blaizot
PRL 101 (2008) 100102,
[hep-th] 0902.2223
......
)

OUTLINE:

- "Phase transition" in large N Yang-Mills
- Dynamics of eigenvalues of Wilson loop, shock waves and inverse spectral cascade
- Finite N as viscosity in the spectral flow
- Colored catastrophes and universality

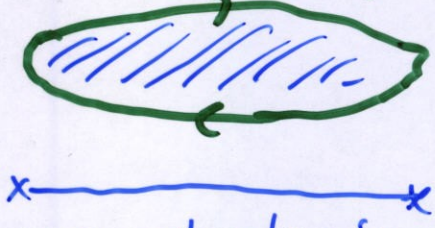
- Large N_c YM $\Longleftrightarrow$ "classical limit" ($\hbar \to 0$)

$$W(C) = \left\langle P \exp i \oint A_\mu dx^\mu \right\rangle_{YM}$$

- small loop — medium loop — large loop

pert. physics

critical area



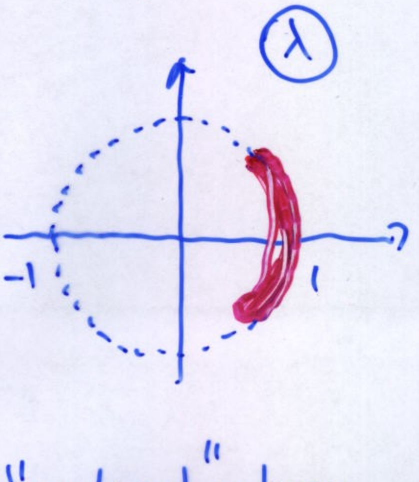
nonpert. physics

Narayanan & Neuberger, hep-th 0711.4551

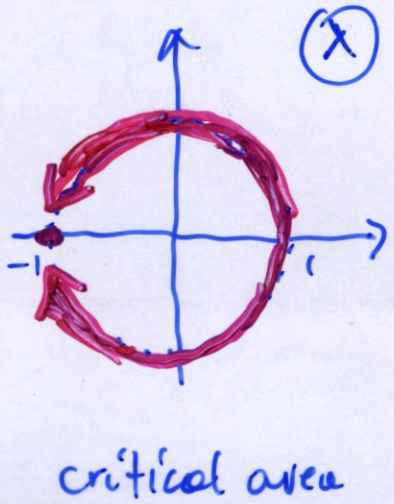
- Can we achieve "classical" picture for this transition?

- Eigenvalues of Wilson loop

$\lambda = e^{i\theta}$

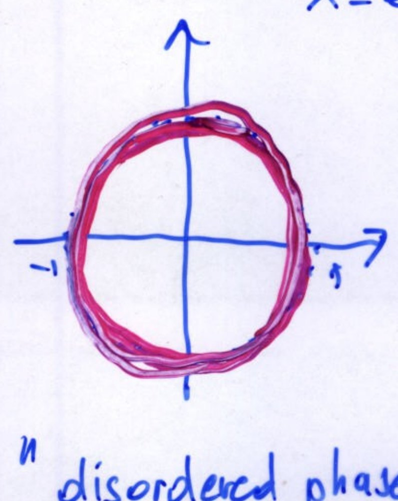


"ordered" phase



critical area of Wilson loop

$A^* = 4$

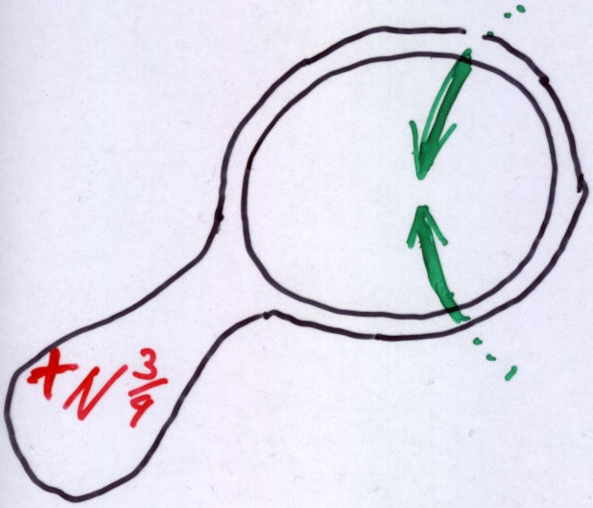


"disordered phase"

Durhuus – Olesen phase transition

WILSON LOOPS "20 years after..."

(Numerical studies on the lattice (Narayanan & Neuberger, 2006, 2007))



UNIVERSALITY AT THE CLOSURE OF THE GAP ($d=2,3,4$ YM)

$$Q_N = \langle \det(z - W(A)) \rangle$$

$$\left(\frac{4N}{3}\right)^{1/4} Q_N(z=e^{i\theta}, A) = \int_{-\infty}^{+\infty} du\, e^{-u^4 - \alpha u^2 + \xi u}$$

$$\alpha = (3N)^{1/2}\left(\frac{A^*}{A} - 1\right) \quad ; \quad \xi = 12^{1/4} N^{3/4} f(\theta)$$

- Wilson loops in the 80-90'ties ($d=2$, $N_c \to \infty$)

- spectral moments

$$W_n(A) = \langle \mathrm{tr}\, W^n(A) \rangle = \int_{-\pi}^{\pi} d\theta\, e^{in\theta} \rho(\theta, A) = \frac{1}{n} L^{1}_{(n-1)}(nA)\, e^{-n\frac{A}{2}}$$

$$= \frac{1}{n} \oint \frac{dz}{2\pi i} \left(1 + \frac{1}{z}\right)^n e^{-nA\left(z + \frac{1}{z}\right)} \xrightarrow[n\to\infty]{} \begin{cases} \frac{1}{n^{3/2}} \cdot \text{oscill} & A < A^* \\ \frac{1}{n^{4/3}} & A = A^* \\ \frac{1}{n^{3/2}} e^{-nf(A)} & A > A^* \end{cases}$$

- generating function $F(\theta, A)$: $\quad \rho(\theta, A) = \frac{1}{\pi} \mathrm{Im}\, F(\theta - i\epsilon, A)$

$$F(\theta, A) = \int_{-\infty}^{\infty} d\alpha \frac{\rho(\alpha, A)}{\theta - \alpha} = \frac{1}{2} \int_{-\pi}^{\pi} d\alpha\, \rho(\alpha, A) \cot\left(\frac{\theta - \alpha}{2}\right)$$

$$= i\left(\frac{1}{2} + \sum_{n=1}^{\infty} W_n(A)\, e^{-in\theta}\right)$$

- dynamical equation for $F(\theta, A)$

$$\boxed{\partial_A F(\theta, A) + F(\theta, A)\, \partial_\theta F(\theta, A) = 0}$$

$$\left(F = \pi H[\rho(\theta)] + i\pi \rho(\theta)\right)$$ "Kramers-Kronig"

Olesen, Durhuus, Rossi, Kazakov, Douglas, Gross, Kostov, Gopakumar, Matytsin, Migdal, Makeenko .....

Voiculescu (Free Random Variables) ....

Complex analogon of Burgers equation (?)

$$\partial_t f(\theta, t) + f(\theta, t)\, \partial_\theta f(\theta, t) = \nu_s\, \partial_{\theta\theta} f(\theta, t)$$

- Solution of complex Burgers

→ Method of characteristics = trick to convert PDE into ODE

$$\partial_A F + F \partial_\theta F = 0 \qquad F(A=0,\theta) = F_0(\theta)$$

$$\boxed{F(A,\theta) = F_0(\xi) \qquad \xi = \underset{\text{"position"}}{\theta} - \underset{\text{"time"}}{A}\,\underset{\text{"velocity"}}{F_0(\xi)}}$$

straight lines on which velocity is constant

– Singularity $\dfrac{\partial F}{\partial \theta} = \infty \qquad \dfrac{\partial F}{\partial \theta} = \dfrac{\partial F}{\partial \xi}\dfrac{d\xi}{d\theta} = \dfrac{\frac{\partial F}{\partial \xi}}{\frac{d\theta}{d\xi}}$

$$\boxed{0 = \frac{d\theta}{d\xi} = 1 + A F_0'(\xi)}$$

– Expanding around singularity

$$F_0(\xi)\Big|_{sing} = F_0(\xi_s) + \underbrace{\frac{dF_0}{d\xi}}_{= -\frac{1}{A}}(\xi-\xi_s) + \underbrace{\frac{1}{2}\frac{d^2F_0}{d\xi^2}(\xi-\xi_s)^2}_{\text{can be zero at inflixion point}} + \frac{1}{3!}\frac{d^3F}{d\xi^3}(\xi-\xi_s)^3$$

– Tracing complex singularities teaches us on Fourier components (moments)

$$(\theta-\theta_s)^\mu \longleftrightarrow w_n = |n|^{-(\mu+1)} e^{-n\,\mathrm{Im}\,\theta_s}\,\mathrm{Re}\, e^{in\,\mathrm{Re}\,\theta_s}$$

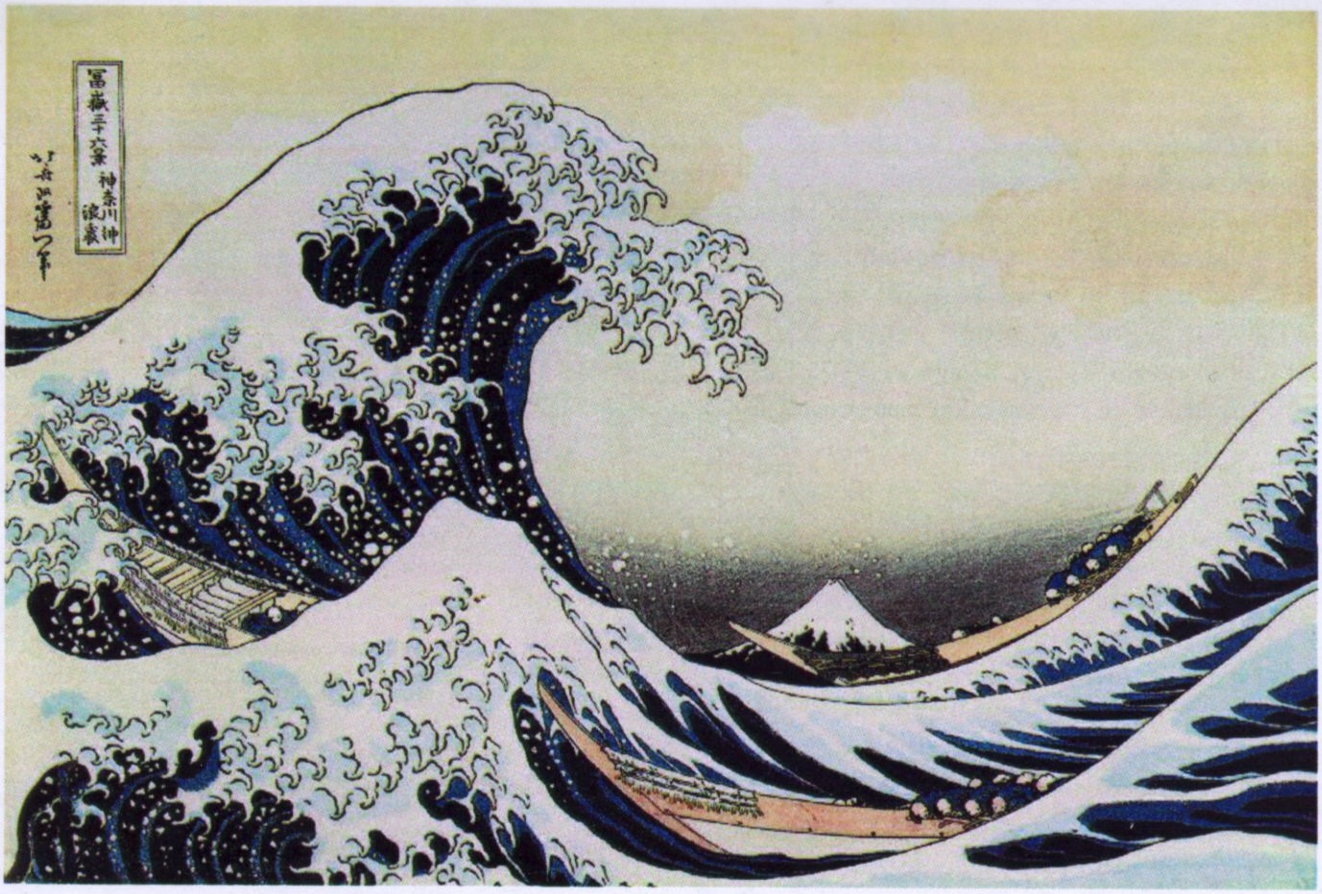
THREE REGIMES:

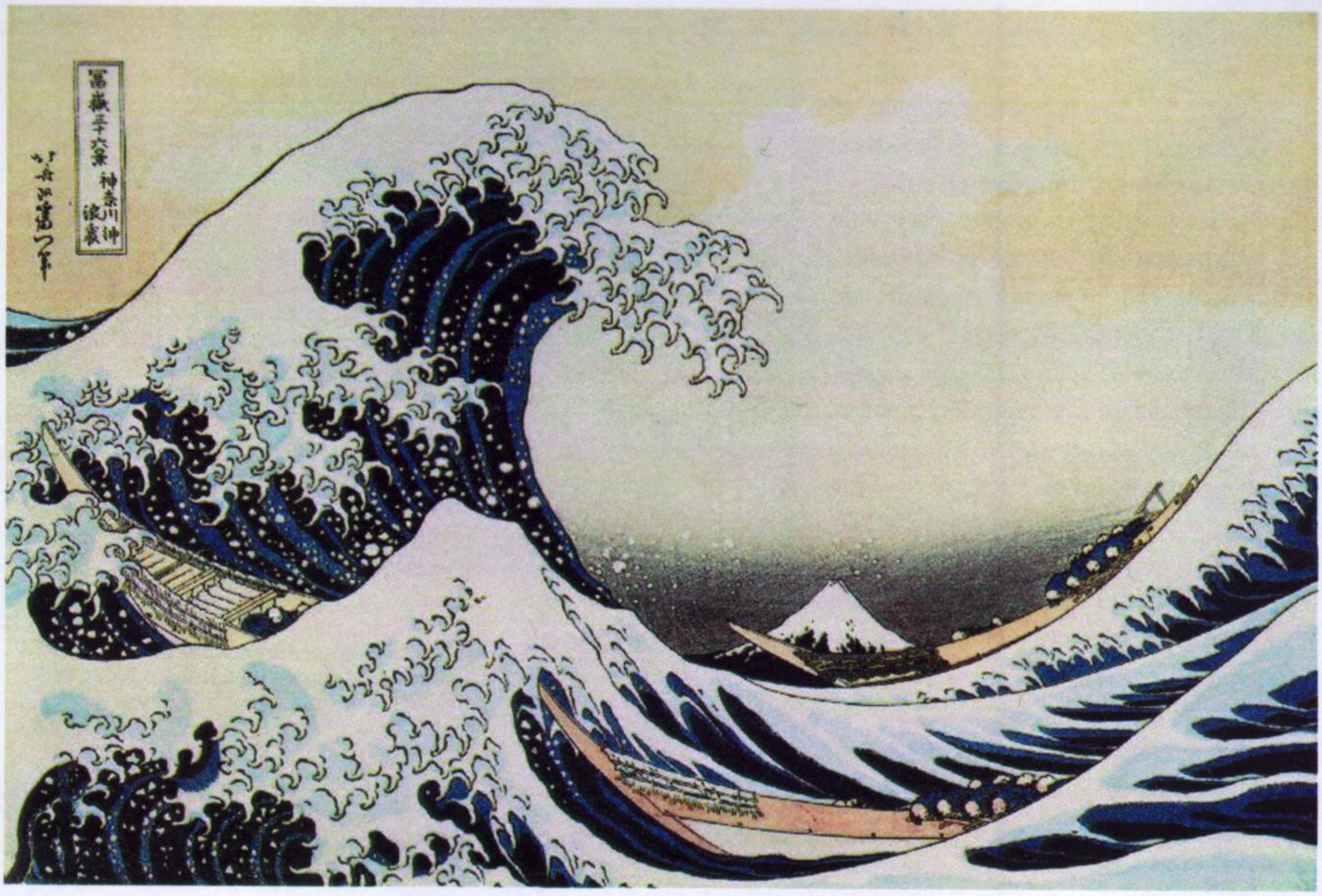
- GAPPED PHASE: LAMINAR "FLOW", REAL SQUARE ROOT SINGULARITY AT THE EDGE OF THE SPECTRUM (SPECTRAL MOMENTS $\boxed{w_n \sim n^{-(+\frac{1}{2}+1)} \times \text{OSCILLATIONS}}$)

- CRITICAL POINT: (TSUNAMI) TWO SQUARE ROOT SINGULARITIES COLLIDE AT $A=\pi$ FORMING CUBIC ROOT SINGULARITY (SPECTRAL MOMENTS $\boxed{w_n \sim n^{-(\frac{1}{3}+1)}}$)

- GAPLESS PHASE: SQUARE ROOT SINGULARITIES EVOLVE ALONG IMAGINARY AXIS (SPECTRAL MOMENTS $\boxed{w_n \sim n^{-(\frac{1}{2}+1)} \times \text{EXPONENTIAL DAMPING}}$)

HIGH FOURIER MODES ARE BEING "EATEN" BY LOWER ONES (INVERSE SPECTRAL TURBULENT CASCADE). AT $A=\infty$ ONLY CONSTANT MODE SURVIVES (DISORDER)


冨嶽三十六景
神奈川沖浪裏



"Behind the Great Wave at Kanagawa" (by Hokusai)
Color woodcut, Metropolitan Museum of Art, New York

ORIGIN OF THE UNIVERSALITY IN $d = 2, 3, 4$ $\boxed{???}$

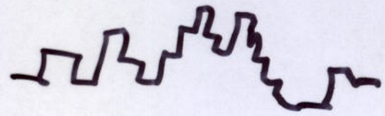
(FINITE N SCALING) $\boxed{???}$

.. "After considerable and fruitless effort to develop a Newtonian theory of ensembles, we discovered that the correct procedure is quite different and much simpler. The x_i should be interpreted as positions of particles in Brownian motion. ..."

Freeman J. Dyson, J. Math. Ph. 3 (1962) 1191

RANDOM WALKER VS RANDOM CROWD

- Random walker (Abelian additive case)



(i) $x \to x + \delta x$

$$dx(t) = e(x)\,dt + d\beta(t) \qquad \langle \delta x(t)^2 \rangle \sim dt$$

increment = drift + noise (Einstein's diffusion)

Langevin picture - realization of stochastic process

(ii) Smoluchowski - Fokker - Planck picture

$$\frac{\partial}{\partial t} p(x(t)) = \Big(\underset{\text{from drift}}{e(x)}\,\partial_x + \underset{\text{from noise}}{\tfrac{1}{2}\Delta} \Big)\, p(x(t))$$

- Random crowd (Non-Abelian additive case)

$$H \to H + \delta H$$

Matrix elements $\xrightarrow{\text{Jacobian!}}$ Matrix eigenvalues

$$V \equiv \prod_{i<j} (x_i - x_j) \sim e^{\sum_{i<j} \ln(x_i - x_j)}$$

Effective potential (or singular electric field $\sum \frac{1}{x_i - x_j}$)

$$\langle \delta x_i \rangle = E(x_i)\,dt \qquad \langle \delta x_i^2 \rangle \sim dt$$

interacting Brownian walk (nonlinearities)

Exact equation:

$$\frac{\partial \tilde{\rho}(x,t)}{\partial t} = \frac{1}{2} \frac{\partial^2 \tilde{\rho}(x,t)}{\partial x^2} - \frac{\partial}{\partial x} \int dy \, \frac{\bar{\rho}(x,y,t)}{x-y}$$

$$\langle AB \rangle \xrightarrow[N\to\infty]{} \langle A \rangle \langle B \rangle$$

$\bar{\rho}(x) = N\rho$

$\bar{\rho}(x,y) = N(N-1)\rho(x,y)$

$\tau = Nt$

$\bar{\rho}(x,y) = \bar{\rho}(x)\bar{\rho}(y) + \rho_{conn}(x,y)$

$$\frac{\partial \rho(x)}{\partial t} + \frac{\partial}{\partial x} \rho(x) \int dy \, \frac{\rho(y)}{x-y} = \underbrace{\frac{1}{2N} \frac{\partial^2 \rho(x)}{\partial x^2} + \int dy \, \frac{\rho_{conn}(x,y)}{x-y}}_{\text{vanishes when } N\to\infty}$$

- $N \to \infty$ $\qquad \partial_\tau G(z,\tau) + G(z,\tau)\,\partial_z G(z,\tau) = 0$

$$G(z,\tau) = \left\langle \frac{1}{N} \mathrm{Tr} \, \frac{1}{z - H(\tau)} \right\rangle = \int \frac{\rho(y,\tau)\,dy}{z-y}$$

- N finite (time dep. orthogonal polynomials) $\pi_k(z,\tau)$

$$f_k(z,\tau) \equiv 2\nu_s \, \partial_z \ln \pi_k(z,\tau) = \frac{1}{N} \sum_{i=1}^{k} \frac{1}{z - x_i(\tau)}$$

$$\partial_\tau f_k + f \partial_z f_k = -\nu_s \partial_z^2 f_k \qquad \boxed{\nu_s = \frac{1}{2N}}$$

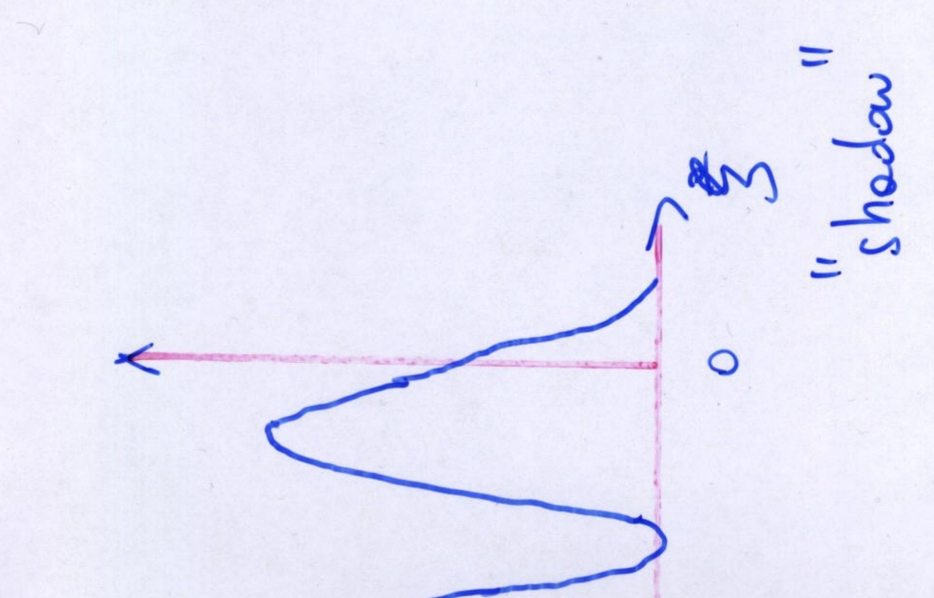
- Since $\langle \det(z - H(\tau)) \rangle = \pi_N(z,\tau)$

$$\frac{1}{N} \partial_z \ln \langle \det(z-H) \rangle \underset{N\to\infty}{\simeq} \frac{1}{N} \partial_z \langle \ln \det(z-H) \rangle = G(z)$$

Exact Burgers equation for characteristic polynomial

- At the edge: <u>Universal preshock</u>

(Airy function)


$|\Psi_{fold}(\xi)|^2$
ξ
0
"light"
"shadow"

Multiplicative random walks

- Abelian (trivial): $e^{x+y} = e^x e^y$ (lognormal pdf)
- Non-Abelian (Wilson loop): Non-trivial $e^{A+B} \neq e^A e^B$

BUT

- SFP EXACTLY SOLVABLE (CALOGERO-SUTHERLAND MODEL)
- HIERARCHY OF BURGERS EQUATIONS FOR CERTAIN "MONIC" POLYNOMIALS (BLAIZOT, JANIK, MAN)
- EXACT BURGERS EQUATION FOR DENSITY OF YANG-LEE ZEROES (NEUBERGER)

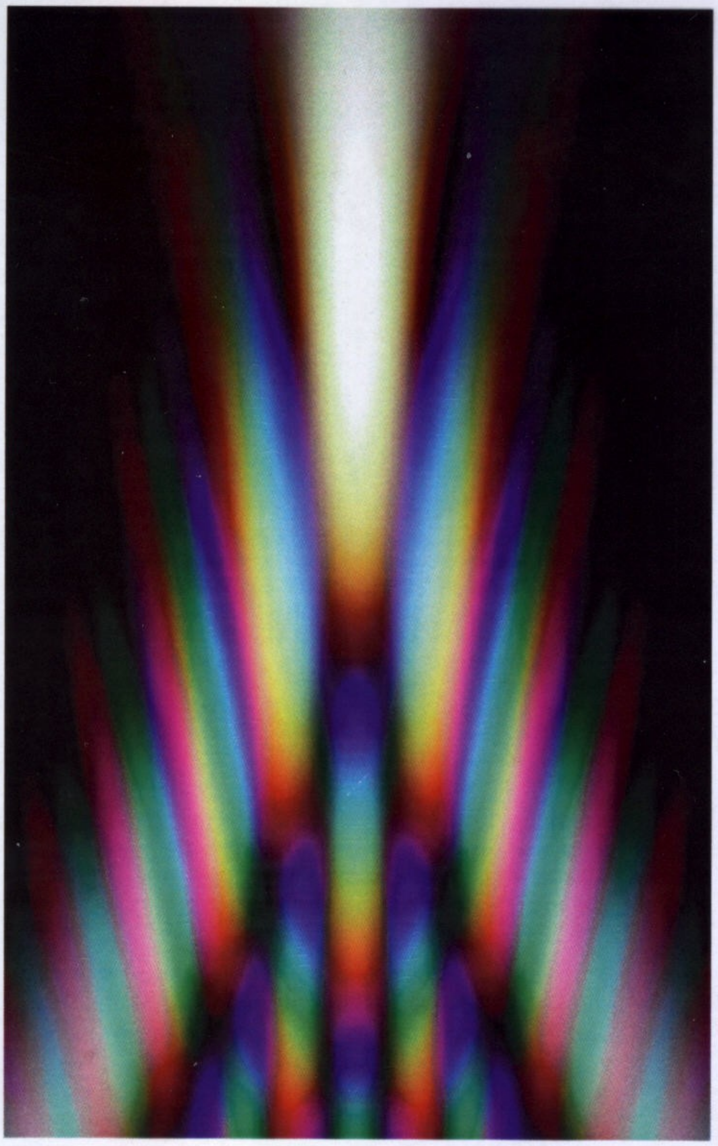
$$\phi_N(y,A) = \frac{1}{N}\frac{\partial}{\partial y}\ln\left(e^{N/2\left(\tau/2 - y\right)}\left\langle \det\left(e^y - W(A)\right)\right\rangle\right)$$

$$\frac{\partial \phi_N}{\partial A} + \phi_N \frac{\partial}{\partial y}\phi_N = \frac{1}{2N}\frac{\partial^2 \phi_N}{\partial y^2}$$

- Novel feature - due to boundary conditions, "left" and "right" Airy preshocks can collide forming "Pearcey preshock"

- UNIVERSALITY ≡ CENTRAL LIMIT THEOREM FOR LARGE UNITARY MATRICES

(sir M. Berry)



Modulus of cusp catastrophe (Pearcey)
Interference "fringes" scale in
an anisotrophic way ($N^{1/2}$ versus $N^{3/4}$)

ANALOGY WITH "CLASSICAL PHYSICS":

MORPHOLOGY OF SINGULARITY (THOM, BERRY, HOWLS)

GEOMETRIC OPTICS ($\lambda = 0$)

- TRAJECTORIES: RAYS OF LIGHT
- INTENSITY SURFACE: CAUSTICS

$N = \infty$ YM $\nu_S = \frac{1}{2N} = 0$

- TRAJECTORIES: CHARACTERISTICS
- SINGULARITIES OF SPECTRAL FLOW

WAVE OPTICS ($\lambda \to 0$)

FINITE N YM (VISCOSITY $\to 0$)

{ UNIVERSAL SCALING: ARNOLD (μ) and BERRY (σ) INDICES }

"WAVE PACKET" SCALING (INTERFERENCE)

- $\Psi \simeq \frac{1}{\lambda^{\mu}} \Psi\left(\frac{x}{\lambda^{\sigma_x}}, \frac{y}{\lambda^{\sigma_y}}\right)$
- fold $\mu = \frac{1}{6}$, $\sigma = \boxed{\frac{2}{3}}$, Airy
- cusp $\mu = \frac{1}{4}$, $\sigma_x = \boxed{\frac{1}{2}}$, $\sigma_y = \boxed{\frac{3}{4}}$

Pearcey

Y-L ~~ZEROES~~ SCALING (FINITE N)

- YL zeroes of Wilson loop
- $N^{\boxed{2/3}}$ scaling at the edge
- $N^{\boxed{1/2}}$ and $N^{\boxed{3/4}}$ scaling at the closure of the gap (Neuberger results for $d=2,3,4$)

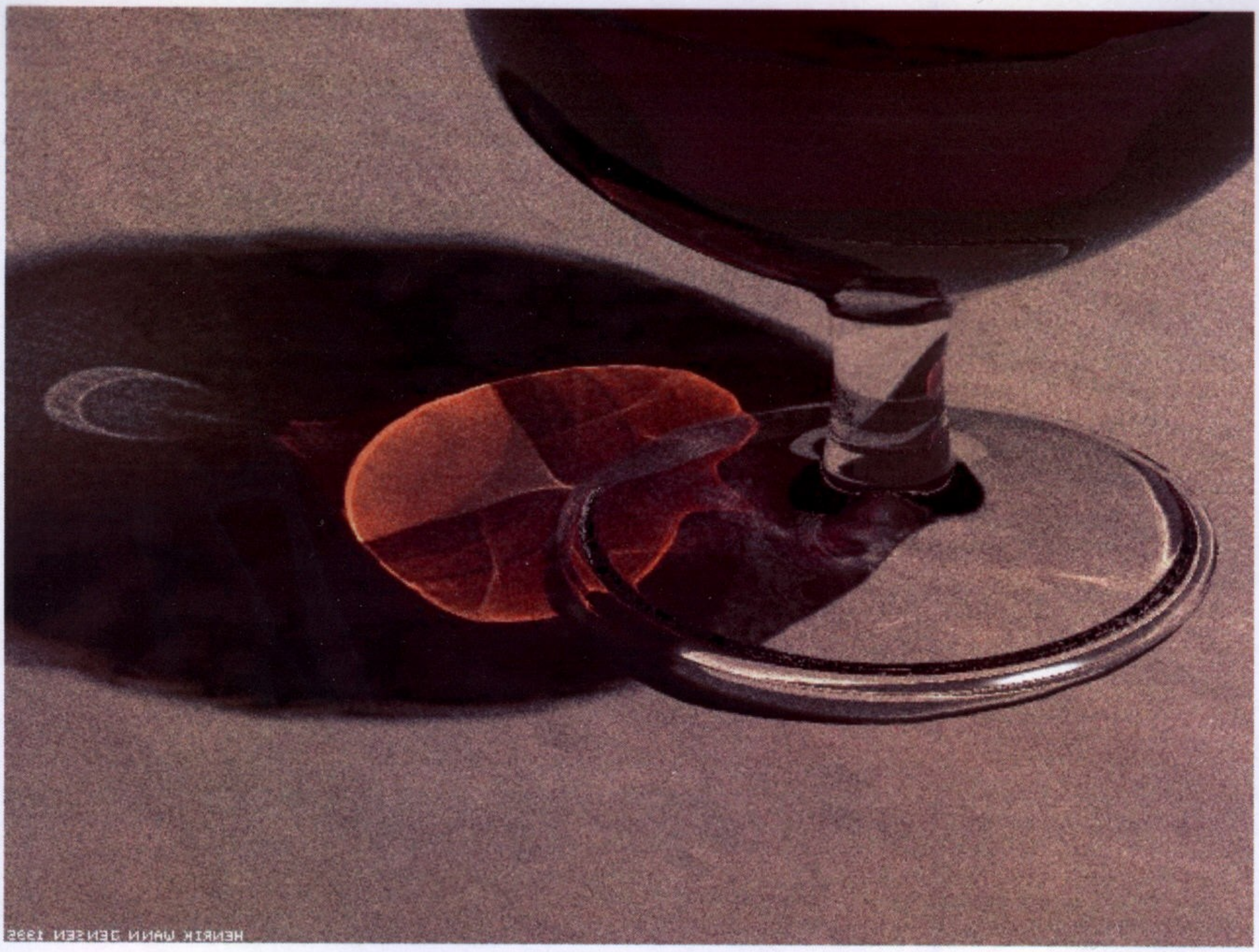
CONCLUSIONS:

- "DISORDERED PHASE" REPRESENTS NON-ABELIAN CLT (NON-LINEAR EFFECTS, SHOCK WAVES)
- FINITE N → VISCOSITY
- NEW PARADIGM: CONFINED PHASE IS "NATURAL", DECONFINEMENT IS A CHALLENGE
- INTERDISCIPLINARY ANALOGIES: (MESOSCOPIC SYSTEMS), (KPZ), (DIFFRACTION CATASTROPHES), (GEOSTROPHIC EQUATIONS..)

CHALLENGES

- DYNAMICS OF $d=3$, $d=4$ YM (RENORMALIZATION OF THE W. LOOP)
- FINITE T PICTURE
- HOW FAR CAN "PLATONIAN SHADOW" OF FULL Y-M DYNAMICS CAN HELP US TO UNDERSTAND YM?
- MATTER FIELDS
- SUPERSYMMETRIC GENERALIZATIONS

- - - ..


HENRIK WANN JENSEN 1995

HAPPY BIRTHDAY,
JEAN-PAUL AND LARRY!!!!