Renormalization and gauge symmetry of 2PI approximation schemes

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Motivation

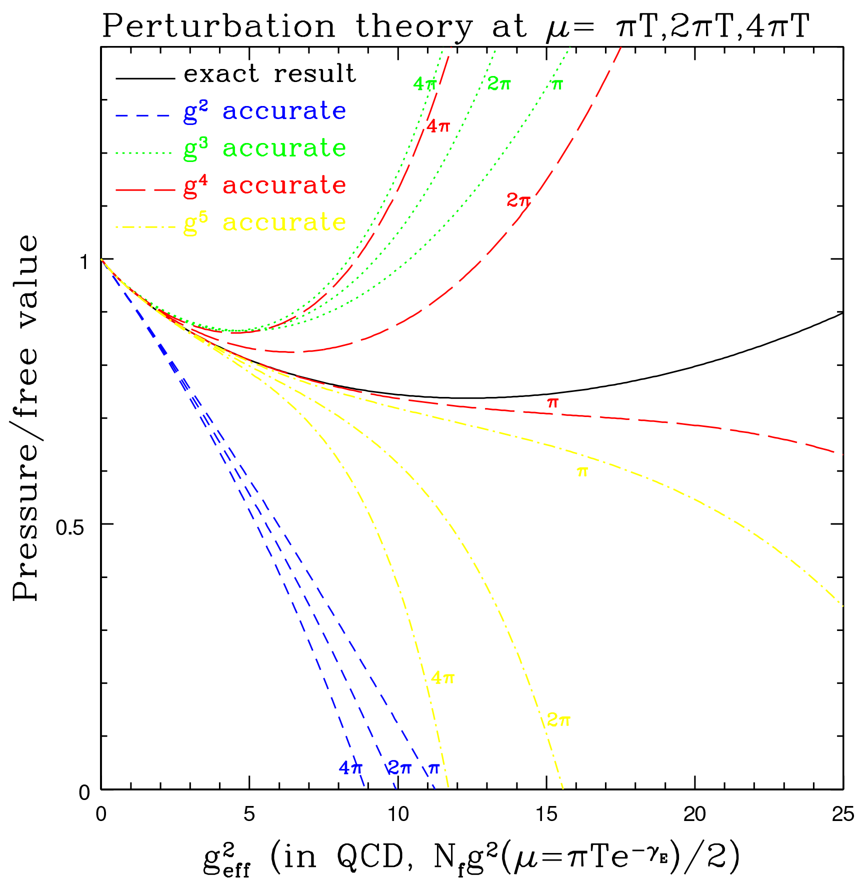
Non-perturbative QFT approaches

- Lattice Quantum Field Theory
- Separation of scales:
  - ⋆ Effective theories.
  - ⋆ Functional Renormalization Group.
- Resummations:
  - ⋆ Schwinger-Dyson equations.
  - ⋆ NPI approximation schemes.
- . . .

Equilibrium

Motivation
Definitions
Symmetry
Renormalization
Gauge-fixing (in)dependence

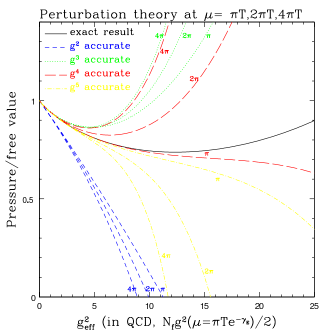
Very poor convergence of perturbation theory (ex: large N_f QCD)



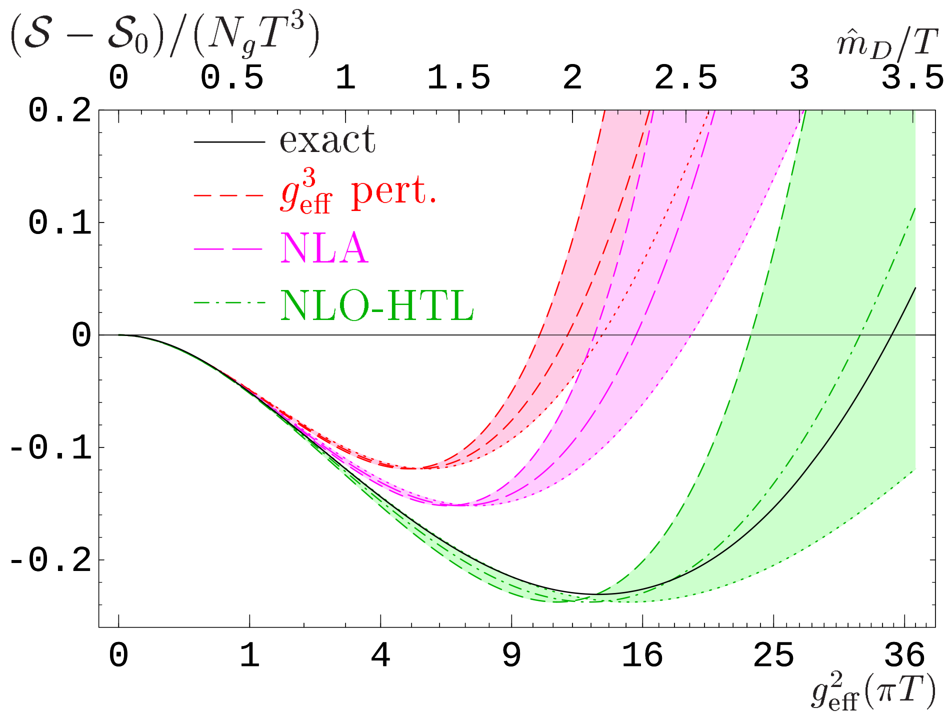
Moore, JHEP 0210 (2002) – Ipp, Moore, Rebhan, JHEP 0306 (2003)

Equilibrium

Perturbations



HTL resummation using the 2PI approach



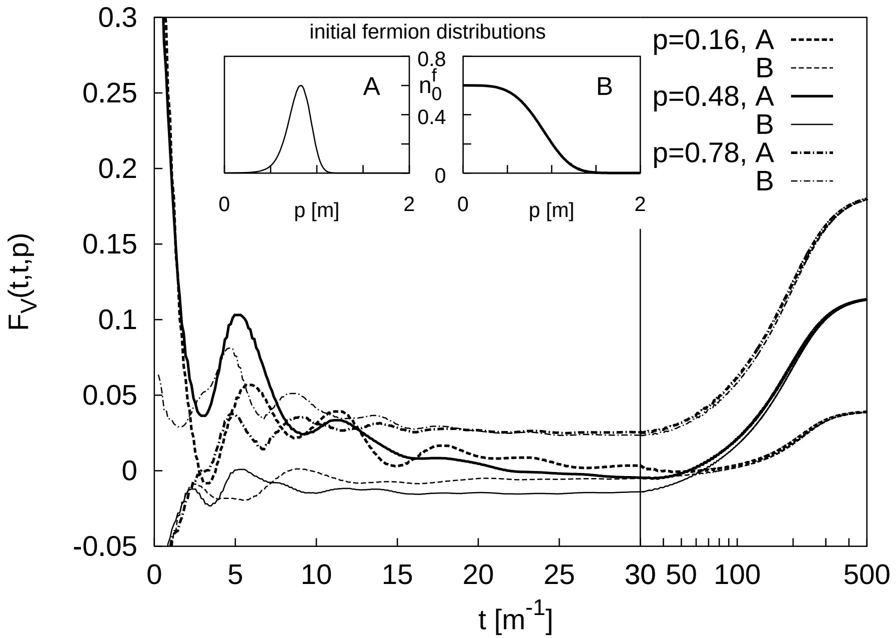
Blaizot, Ipp, Rebhan, Reinosa, PRD 72 (2005)

Out-of-equilibrium

Long time behavior can usually not be accesed from perturbative calculations

Secular terms $\sim (gt)^n \sim 1$ as $t \sim \dfrac{1}{g}$.

In contrast, the 2PI scheme shows a non-secular evolution from early to late times



Berges, Borsányi, Serreau, Nucl.Phys. B660 (2003)

Sensible approximation scheme?

- One expects from a “sensible” approximation scheme to respect the basic properties of the theory under consideration

symmetry and renormalizability

- Is this the case for 2PI approximation schemes?
  - ⋆ For theories involving a global symmetry? Yes.
  - ⋆ For theories involving a gauge symmetry? It depends:
    - For approximated self-consistent resummations (HTL): yes.
    - For fully self-consistent resummations (OOE): still open.

Outline

- What are two-particle-irreducible (2PI) approximation schemes?
- Do they comply with renormalizability and (gauge) symmetry requirements?
  - ⋆ Are they renormalizable? *in collaboration with J. Serreau*
  - ⋆ Are they compatible with the underlying gauge symmetry?
    - – Ward-Takahashi identities? *in collaboration with J. Serreau*
    - – Gauge-fixing (in)dependence? *in collaboration with Sz. Borsányi*

Definitions

The 2PI effective action

Consider for instance a scalar field theory, defined classically by

$$S[\varphi] = \int d^4x \left\{ \frac{1}{2}(\partial\varphi(x))^2 - \frac{m^2}{2}(\varphi(x))^2 - \frac{\lambda}{4!}(\varphi(x))^4 \right\}$$

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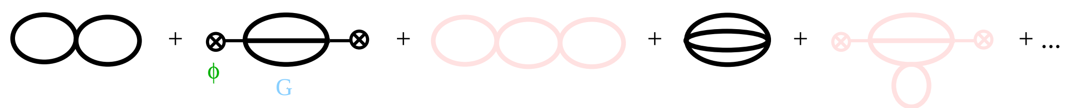
Quantum fluctuations can be encoded in the 2PI effective action

$$\Gamma_{2\mathrm{PI}}[\phi, G] = S[\phi] + \frac{i}{2}\mathrm{Tr}\,\log G^{-1} + \frac{i}{2}\mathrm{Tr}\, G_0^{-1}\, G + \Phi_{2\mathrm{PI}}[\phi, G]$$

$$\Phi_{2\mathrm{PI}}[\phi, G] =$$



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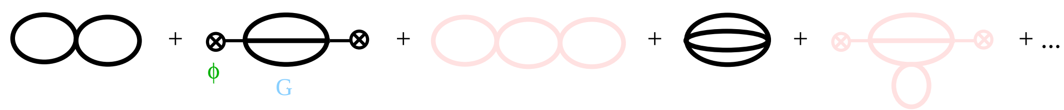

The physics is obtained from a variational principle

$$\frac{\delta}{\delta G}\,\Gamma_{\rm 2PI}[\phi, G]\Big|_{\bar{G}[\phi]} = 0$$

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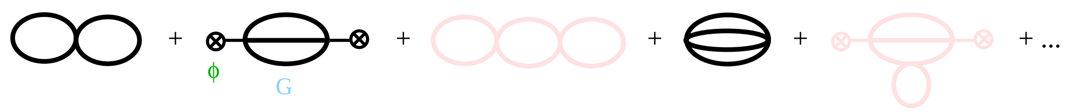

The physics is obtained from a variational principle

$$\bar{G}[\phi]^{-1} = G_0^{-1} - 2i\, \frac{\delta\Phi_{2\mathrm{PI}}}{\delta G}\bigg|_{\bar{G}[\phi]}$$

resummation encoded here!

2PI and 2PI-resummed vertices

Two possible definitions of the two-point function:

$$i\,\frac{\delta^2\Gamma_{2\mathrm{PI}}}{\delta\phi_2\,\delta\phi_1} \quad \text{or} \quad \bar{G}^{-1}_{12}$$

2PI and 2PI-resummed vertices

If no approximations, both definitions agree:

$$i\,\frac{\delta^2\Gamma_{\rm 2PI}}{\delta\phi_2\,\delta\phi_1} = \bar{G}^{-1}_{12}$$

2PI and 2PI-resummed vertices

In a given approximation, they become different:

$$i\,\frac{\delta^2\Gamma_{\rm 2PI}}{\delta\phi_2\,\delta\phi_1}\quad\neq\quad\bar{G}^{-1}_{12}$$

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A similar remark applies for higher vertices

$$i\,\frac{\delta^n\Gamma_{\rm 2PI}}{\delta\phi_n\cdots\delta\phi_1} \neq \frac{\delta^{n-2}\bar{G}^{-1}_{12}}{\delta\phi_n\cdots\delta\phi_3}$$

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2PI-resummed vertices $\quad i\,\frac{\delta^n\Gamma_{\rm 2PI}}{\delta\phi_n\cdots\delta\phi_1}\neq\frac{\delta^{n-2}\bar{G}^{-1}_{12}}{\delta\phi_n\cdots\delta\phi_3}\quad$ 2PI vertices

Approximation artefact: at a given order of approximation

$$i\,\frac{\delta^n\Gamma_{\rm 2PI}}{\delta\phi_n\cdots\delta\phi_1}-\frac{\delta^{n-2}\bar{G}^{-1}_{12}}{\delta\phi_n\cdots\delta\phi_3}=\mathcal{O}(\text{higher order contributions})$$

Symmetry

Quantum Electrodynamics

QED in the covariant gauge with gauge fixing parameter ξ

$$S_{\rm qed} = \underbrace{\int d^d x \left\{ \bar{\psi}\, [i\partial\!\!\!/ - eA\!\!\!/]\, \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right\}}_{S_{\rm qed} - S_{\rm gf[A]}} \underbrace{- \frac{1}{\xi} \int d^d x \, (\partial^\mu A_\mu)^2}_{S_{\rm gf}[A]}$$

invariance under $\delta_\alpha \psi(x) = i\alpha(x)\, \psi(x)$ and $\delta_\alpha A^\mu(x) = -(1/e)\partial^\mu_x \alpha(x)$

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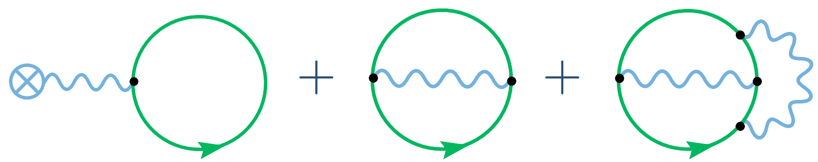
invariance under $\delta_\alpha\psi(x) = i\alpha(x)\,\psi(x)$ and $\delta_\alpha A^\mu(x) = -(1/e)\partial^\mu_x\alpha(x)$

The 2PI effective action is now a functional of ψ and A as well as D and G:

$$\Gamma_{\rm 2PI}[\psi,A,D,G] \;=\; S_{\rm qed} - i\,\mathrm{Tr}\left[\ln D^{-1} + D_0^{-1}\,D\right] + \Phi_{\rm 2PI}[\psi,A,D,G]$$
$$+\frac{i}{2}\mathrm{Tr}\left[\ln G^{-1} + G_0^{-1}\,G\right]$$

$$\Phi_{\rm 2PI}[\psi,A,D,G] \;=$$



$+\cdots$

Gauge invariance of $\Gamma_{2\text{PI}}[\psi, A, D, G]$

At any order of approximation, the 2PI effective action is invariant

$$\delta_\alpha\Big(\Gamma_{2\text{PI}}[\psi, A, D, G] - S_{\text{gf}}[A]\Big) = 0$$

under a gauge transformation of the fields

$$\delta_\alpha \psi(x) = i\alpha(x)\,\psi(x) \quad \text{and} \quad \delta_\alpha A^\mu(x) = -(1/e)\partial_x^\mu \alpha(x)$$

as well as a gauge transformation of the propagators

$$\delta_\alpha D(x) = i\alpha(x)\,D(x,y) - iD(x,y)\,\alpha(y) \quad \text{and} \quad \boxed{\delta_\alpha G(x,y) = 0}$$

[UR & J. Serreau, JHEP 0711:097 (2007)]

2PI Ward-Takahashi identities (1/3)

Remember: there are two distinct definitions of vertices:

$$\Gamma_{2\text{PI}}[\psi, A, D, G] \quad \rightarrow \quad \begin{cases} \bar{D}[\psi, A] \\ \bar{G}[\psi, A] \end{cases}$$

$$\downarrow \qquad\qquad \downarrow$$

$$i\frac{\delta^n \Gamma_{2\text{PI}}}{\delta A_{\mu_n} \cdots \delta A_{\mu_1}} \neq \frac{\delta^{n-2} \bar{G}^{-1}_{\mu_1 \mu_2}}{\delta A_{\mu_n} \cdots \delta A_{\mu_3}}$$

One thus expects two types of Ward-Takahashi identities.

2PI Ward-Takahashi identities (2/3)

2PI-resummed vertices fulfill usual WT identities [UR & J. Serreau, JHEP 0711:097 (2007)]

complete transversality: $$\partial^\mu \frac{\delta^4 \Gamma_{2\mathrm{PI}}}{\delta A^\mu \delta A^\nu \delta A^\rho \delta A^\sigma} = 0$$

2PI vertices fulfill partial WT identities [UR & J. Serreau, JHEP 0711:097 (2007)]

partial transversality: $$\partial^\rho \frac{\delta^2 \bar{G}^{-1}_{\mu\nu}}{\delta A^\rho \delta A^\sigma} = 0 \quad \text{but} \quad \partial^\mu \frac{\delta^2 \bar{G}^{-1}_{\mu\nu}}{\delta A^\rho \delta A^\sigma} \neq 0$$

2PI Ward-Takahashi identities (3/3)

2PI-resummed vertices fulfill usual WT identities [UR & J. Serreau, JHEP 0711:097 (2007)]

complete transversality: $\partial^\mu \left[i \frac{\delta^2 \Gamma_{2\mathrm{PI}}}{\delta A^\mu \delta A^\nu} - G^{-1}_{0,\mu\nu} \right] = 0$

2PI vertices fulfill partial WT identities [UR & J. Serreau, JHEP 0711:097 (2007)]

no transversality: $\partial^\mu \left[\bar{G}^{-1}_{\mu\nu} - G^{-1}_{0,\mu\nu} \right] \neq 0$

(Note however that the HTL self-energy $G^{-1}_{HTL} - G^{-1}_0$ is transverse).

Renormalization

The problem

The 2PI effective action is defined in presence of a regulator

in dimensional regularisation: $\int d^4x \longrightarrow \int d^dx$

Problem: as $d \longrightarrow 4$ the 2PI vertices diverge!

How to define a finite continuum limit?

- Define a continuum limit for 2PI-resummed vertices:

$$\frac{\delta^n \Gamma_{2\mathrm{PI}}}{\delta\phi_n \cdots \delta\phi_1} \quad \left(\text{ex: 2PI-resummed photon propagator } \frac{\delta^2 \Gamma_{2\mathrm{PI}}}{\delta A^\mu \delta A^\nu}\right)$$

- Simultaneously define a continuum limit for 2PI vertices:

$$\frac{\delta^{n-2} \bar{G}^{-1}_{12}}{\delta\phi_n \cdots \delta\phi_3} \quad \left(\text{ex: 2PI photon propagator } \bar{G}^{-1}_{\mu\nu}\right)$$

- Is the gauge invariance of $\Gamma_{2\mathrm{PI}}$ compatible with the continuum limit?

Perturbative renormalizability

QED is renormalizable: one can redefine the fields

$$A_b^\mu \equiv Z_3^{1/2}(d) A^\mu \qquad \psi \equiv Z_2^{1/2}(d)\psi_b \qquad \bar{\psi}_b \equiv Z_2^{1/2}(d)\bar{\psi}$$

as well as the parameters of the theory

$$Z_2(d) m_b \equiv Z_0(d) m \qquad Z_2(d) Z_3^{1/2}(d) e_b \equiv Z_1(d) e \qquad \frac{Z_3(d)}{\xi_b} = \frac{Z_4(d)}{\xi}$$

such that the vertices are convergent as $d \longrightarrow 4$.

2PI approximation scheme

In terms of renormalized fields, the 2PI effective action gets the additional contribution

$$\begin{aligned}\delta\Gamma_{\rm 2PI} &= \frac{\delta Z_3}{2}\int_x A_\mu(x)\Big(g^{\mu\nu}\partial_x^2-\partial_x^\mu\partial_x^\nu\Big)A_\nu(x)\\ &+\frac{\delta Z_3}{2}\int_x\Big[\Big(g^{\mu\nu}\partial_x^2-\partial_x^\mu\partial_x^\nu\Big)G_{\mu\nu}(x,y)\Big]_{x=y}+\cdots\end{aligned}$$

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The 2PI-resummed photon propagator gets the additional contribution

$$\frac{\delta^2\Gamma_{\rm 2PI}}{\delta A_\mu \delta A_\nu} = \delta Z_3\big(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\big) + \cdots$$

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$$\bar{G}^{-1}_{\mu\nu} = i\,\delta Z_3\big(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\big) + \cdots$$

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The 2PI-resummed photon propagator gets the additional contribution

$$\frac{\delta^2\Gamma_{2\mathrm{PI}}}{\delta A_\mu \delta A_\nu} = \delta Z_3\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right) + \cdots$$

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$$\delta Z_3 \neq \delta\bar{Z}_3$$

The 2PI photon propagator gets the additional contribution

$$\bar{G}^{-1}_{\mu\nu} = i\,\delta\bar{Z}_3\left(g^{\mu\nu}\partial^2-\partial^\mu\partial^\nu\right) + \cdots$$

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The 2PI-resummed photon propagator gets the additional contribution

$$\frac{\delta^2\Gamma_{\rm 2PI}}{\delta A_\mu\delta A_\nu} = \delta Z_3\left(g^{\mu\nu}\partial^2-\partial^\mu\partial^\nu\right)+\cdots$$

$$\delta Z_3 \neq \delta\bar Z_3,\ \delta\bar Z_L,\ \delta\bar M^2$$

The 2PI photon propagator gets the additional contribution

$$\bar G^{-1}_{\mu\nu} = i\,\delta\bar Z_3\left(g^{\mu\nu}\partial^2-\partial^\mu\partial^\nu\right) + i\,\delta\bar Z_L\,\partial^\mu\partial^\nu + i\,\delta\bar M^2 g^{\mu\nu}+\cdots$$

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All these new contributions do not affect the gauge invariance of the 2PI effective action

$$\delta_\alpha G = 0 \;\Rightarrow\; \delta_\alpha\Big(\Gamma_{\rm 2PI}+\delta\Gamma_{\rm 2PI}-S_{\rm gf}[A]\Big)=0$$

$\Rightarrow$ new counterterms allowed by symmetry [UR & J. Serreau, in preparation]

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$$
\begin{aligned}
\delta\Gamma_{\rm 2PI} = \; & \frac{\delta Z_3}{2}\int_x A_\mu(x)\Big(g^{\mu\nu}\partial_x^2-\partial_x^\mu\partial_x^\nu\Big)A_\nu(x) \\
& + \frac{\delta \bar Z_3}{2}\int_x \Big[\Big(g^{\mu\nu}\partial_x^2-\partial_x^\mu\partial_x^\nu\Big)G_{\mu\nu}(x,y)\Big]_{x=y} + \cdots \\
& + \frac{\delta \bar Z_L}{2}\int_x \Big[\partial_x^\mu\partial_x^\nu G_{\mu\nu}(x,y)\Big]_{x=y} + \frac{\delta \bar M^2}{2}\int_x G_\mu^\mu(x,x) \\
& + \frac{\delta \bar g_1}{8}\int_x G_\mu^\mu(x,x)G_\nu^\nu(x,x) + \frac{\delta \bar g_2}{4}\int_x G^{\mu\nu}(x,x)\,G_{\mu\nu}(x,x)
\end{aligned}
$$

not wanted! : $\int_x A_\mu(x)G^{\mu\nu}(x,y)A_\nu(x)\,, \quad \int_x A_\mu(x)A^\mu(x)G_\nu^\nu(x,x)$

All these new contributions do not affect the gauge invariance of the 2PI effective action

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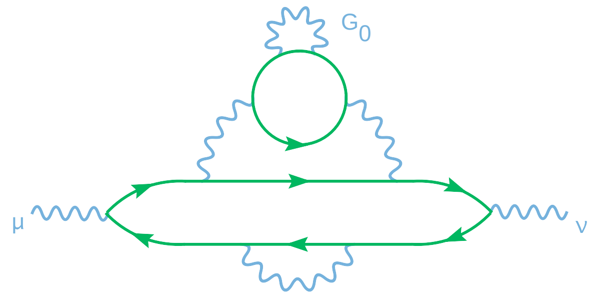
Four photon leg subgraphs

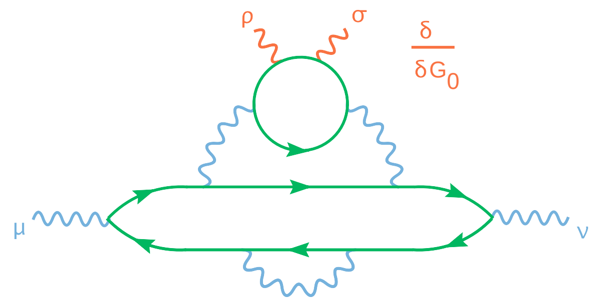
The 2PI photon propagator $\bar{G}^{-1}_{\mu\nu}$ contains subgraphs involving four photons legs

$$\bar{G}^{-1}_{\mu\nu} = \cdots +$$



$$+ \cdots$$

$$\frac{\delta}{\delta G_0^{\rho\sigma}} \left(\bar{G}^{-1}_{\mu\nu} \right) = \cdots +$$



$$+ \cdots \propto \bar{V}_{\mu\nu,\rho\sigma}$$

The same applies for the 2PI-resummed photon propagator $\delta^2\Gamma_{2\mathrm{PI}}/\delta A^\mu \delta A^\nu$

$$\frac{\delta}{\delta G_0^{\rho\sigma}} \left(\frac{\delta^2 \Gamma_{2\mathrm{PI}}}{\delta A^\mu \delta A^\nu} \right) \propto \frac{\delta^2 \bar{G}^{-1}_{\rho\sigma}}{\delta A^\mu \delta A^\nu}$$

Renormalized 2PI effective action

A similar analysis on other 2PI and 2PI-resummed vertices leads to

$$\begin{aligned}
\delta\Gamma_{2PI} = \; & \frac{\delta Z_3}{2}\int_x A_\mu(x)\Big(g^{\mu\nu}\partial_x^2 - \partial_x^\mu\partial_x^\nu\Big)A_\nu(x) \\
& + \delta Z_2 \int_x \bar\psi(x) i\partial\!\!\!/\psi(x) - m\,\delta Z_0 \int_x \bar\psi(x)\psi(x) \\
& - e\,\delta Z_1 \int_x \bar\psi(x) A\!\!\!/(x)\psi(x) \\
& + \frac{\delta\bar Z_3}{2}\int_x \Big[\Big(g^{\mu\nu}\partial_x^2 - \partial_x^\mu\partial_x^\nu\Big)G_{\mu\nu}(x,y)\Big]_{x=y} \\
& + \frac{\delta\bar Z_L}{2}\int_x \Big[\partial_x^\mu\partial_x^\nu G_{\mu\nu}(x,y)\Big]_{x=y} + \frac{\delta\bar M^2}{2}\int_x G_\mu^\mu(x,x) \\
& + \frac{\delta\bar g_1}{8}\int_x G_\mu^\mu(x,x)G_\nu^\nu(x,x) + \frac{\delta\bar g_2}{4}\int_x G^{\mu\nu}(x,x)\,G_{\mu\nu}(x,x) \\
& - \delta\bar Z_2 \int_x \mathrm{tr}\,\big[i\partial\!\!\!/_x D(x,y)\big]_{x=y} + \delta\bar m \int_x \mathrm{tr}\,D(x,x) \\
& + e\,\delta\bar Z_1 \int_x \mathrm{tr}\,\big[A\!\!\!/(x)D(x,x)\big]
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& - \delta\bar Z_2\int_x \text{tr}\,\big[i\partial\!\!\!/_x D(x,y)\big]_{x=y} + m\,\delta\bar Z_0\int_x \text{tr}D(x,x) \\
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& + e\,\delta\bar Z_1\int_x \mathrm{tr}\,\big[A\!\!\!/(x)D(x,x)\big]
\end{aligned}
$$

Renormalization conditions

A unique renormalization condition to fix δZ_1 and $\delta \bar{Z}_1$

$$e\,\gamma^{\mu}_{\bar{\alpha}\alpha} = i\,\frac{\delta^3\Gamma_{2\mathrm{PI}}}{\delta A_\mu \delta\psi_\alpha \delta\bar{\psi}_{\bar{\alpha}}}\Bigg|_{\text{at some ren. scale}} = \frac{\delta \bar{D}^{-1}_{\bar{\alpha}\alpha}}{\delta A_\mu}\Bigg|_{\text{at the same ren. scale}}$$

Similarly, one simultaneously fixes $(\delta Z_0, \delta\bar{Z}_0)$, $(\delta Z_2, \delta\bar{Z}_2)$ and $(\delta Z_3, \delta\bar{Z}_3)$.

$\delta g_1, \delta g_2, \delta Z_L, \delta M^2$ such that $\bar{V}_{\mu\nu,\rho\sigma}, \bar{G}^{-1}_{\mu\nu}$ are transverse at the ren. scale:

consistency conditions rather than renormalization conditions

Renormalized gauge invariance

The bare (unrenormalized) 2PI effective action is invariant

$$\delta_\alpha \Big(\Gamma_{2\mathrm{PI}} - S_{\mathrm{gf}}[A] \Big) = 0$$

under the gauge transformations of the fields

$$\delta_\alpha \psi(x) = i\alpha(x)\, \psi(x) \quad \text{and} \quad \delta_\alpha A^\mu(x) = -\frac{1}{e}\partial_x^\mu \alpha(x)$$

as well as the propagators

$$\delta_\alpha D(x) = i\alpha(x)\, D(x,y) - iD(x,y)\, \alpha(y) \quad \text{and} \quad \delta_\alpha G(x,y) = 0$$

Renormalized gauge invariance

The renormalized 2PI effective action is invariant

$$\delta_\alpha\Big(\Gamma_{2\text{PI}}+\delta\Gamma_{2\text{PI}}-S_{\text{gf}}[A]+\int\bar\psi\left[(\bar Z_2-Z_2)\,i\partial\!\!\!/-e\,(\bar Z_1-Z_1)A\!\!\!/\right]\psi\Big)=0$$

under the gauge transformations of the fields

$$\delta_\alpha\psi(x)=i\alpha(x)\,\psi(x)\quad\text{and}\quad\delta_\alpha A^\mu(x)=-\frac{\bar Z_2}{\bar Z_1}\frac{1}{e}\partial_x^\mu\alpha(x)$$

as well as the propagators

$$\delta_\alpha D(x)=i\alpha(x)\,D(x,y)-iD(x,y)\,\alpha(y)\quad\text{and}\quad\delta_\alpha G(x,y)=0$$

Renormalized gauge invariance

The renormalized 2PI effective action is invariant

$$\delta_\alpha \Big(\Gamma_{\rm 2PI} + \delta\Gamma_{\rm 2PI} - S_{\rm gf}[A]\Big) = 0$$

under the gauge transformations of the fields

$$\delta_\alpha \psi(x) = i\alpha(x)\,\psi(x) \quad \text{and} \quad \delta_\alpha A^\mu(x) = -\frac{\bar{Z}_2}{\bar{Z}_1}\frac{1}{e}\partial_x^\mu \alpha(x)$$

as well as the propagators

$$\delta_\alpha D(x) = i\alpha(x)\,D(x,y) - iD(x,y)\,\alpha(y) \quad \text{and} \quad \delta_\alpha G(x,y) = 0$$

Renormalized gauge invariance

The renormalized 2PI effective action is invariant

$$\delta_\alpha\Big(\Gamma_{2\mathrm{PI}}+\delta\Gamma_{2\mathrm{PI}}-S_{\mathrm{gf}}[A]\Big)=0$$

under the gauge transformations of the fields

$$\delta_\alpha\psi(x)=i\alpha(x)\,\psi(x)\quad\text{and}\quad\delta_\alpha A^\mu(x)=-\frac{1}{e}\partial_x^\mu\alpha(x)$$

as well as the propagators

$$\delta_\alpha D(x)=i\alpha(x)\,D(x,y)-iD(x,y)\,\alpha(y)\quad\text{and}\quad\delta_\alpha G(x,y)=0$$

Summary

- The 2PI effective action for QED is gauge invariant:

$$\delta_\alpha\Big(\Gamma_{\rm 2PI}-S_{\rm gf}[A]\Big)=0$$

- The 2PI effective action for QED is renormalizable:
  - ⋆ Doubling of the usual counterterms: δZ_3, $\delta\bar{Z}_3$, ...
  - ⋆ Additional counterterms: δg_1, δg_2, δZ_L and δM^2.
  - ⋆ Still, the number of ren. conditions is the same as usual.
  - ⋆ All these new features are allowed by the symmetry.

- The resulting renormalized 2PI effective action is gauge invariant:

$$\delta_\alpha\Big(\Gamma_{\rm 2PI}+\delta\Gamma_{\rm 2PI}-S_{\rm gf}[A]\Big)=0$$

- Sensible (UV convergent) approximation scheme.

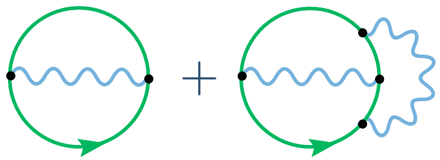
Outlook:

Gauge-fixing (in)dependence

The problem

Example: Consider the pressure of the system

$$\mathcal{P}_{\rm 2PI} = -\mathrm{Tr}\left[\ln \bar{D}^{-1} + D_0^{-1}\,\bar{D}\right] + \frac{1}{2}\mathrm{Tr}\left[\ln \bar{G}^{-1} + G_0^{-1}(\xi)\,\bar{G}\right]$$

$+$  $+ \ldots$

At a given order of approximation, there is a residual gauge-fixing dependence

At order e^2, $\quad \frac{d}{d\xi}\mathcal{P}_{\rm 2PI} = \mathcal{O}(e^4)$

At order e^4, $\quad \frac{d}{d\xi}\mathcal{P}_{\rm 2PI} = \mathcal{O}(e^6)$, and so on

Possible cures

One can try to cure this unconfortable feature:

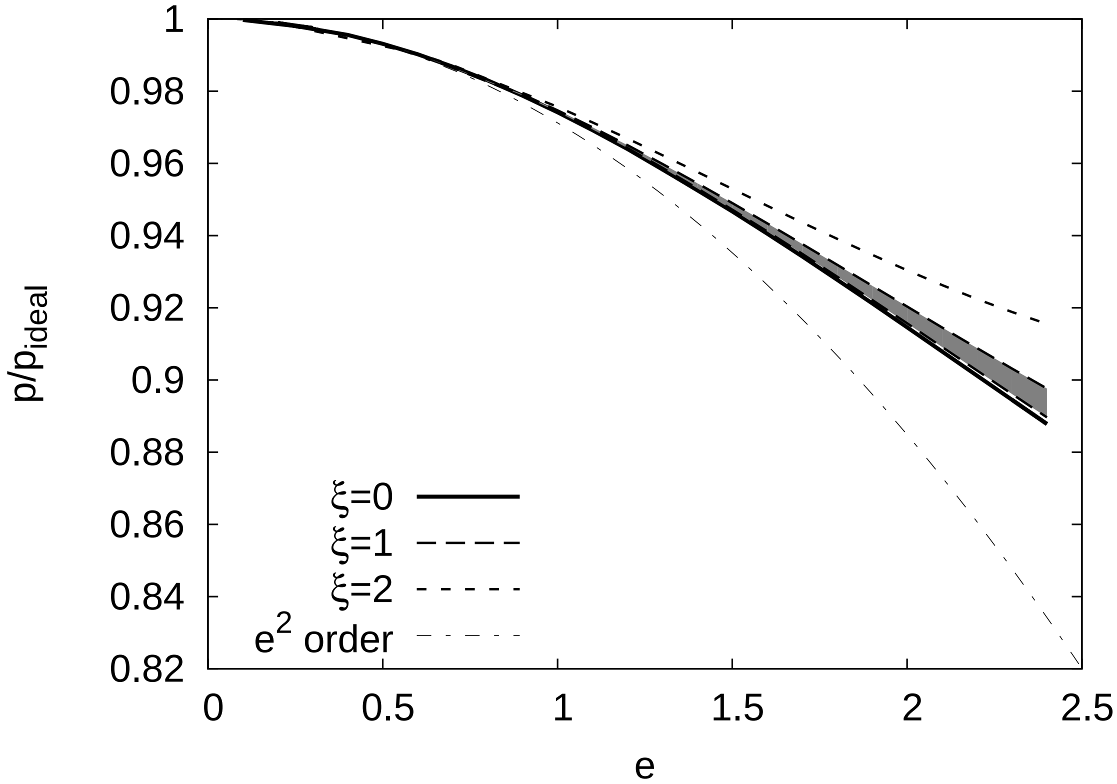
- Look for a new definition of observables within the 2PI scheme.
- Obtain $\bar{G}$ via a variational procedure with constraints.
- Use the freedom one has to reparametrize the fields [Leupold]
- Use approximately (rather than fully) self-consistent resummations [Blaizot, Iancu, Rebhan]

Or live with it and use it as a convergence criterium:

1. The gauge-fixing dependence is always an higher order effect and one expects this higher order contributions to become smaller and smaller.
2. At a given order of approximation one can try to find minimum sensitivity gauges.

Two-loop result (1/2)

In the range of converge $\xi \in [0, 2]$, the ξ-dependence is not dramatically big: comparable to the μ-dependence in the range $\mu \in [\pi T, 4\pi T]$

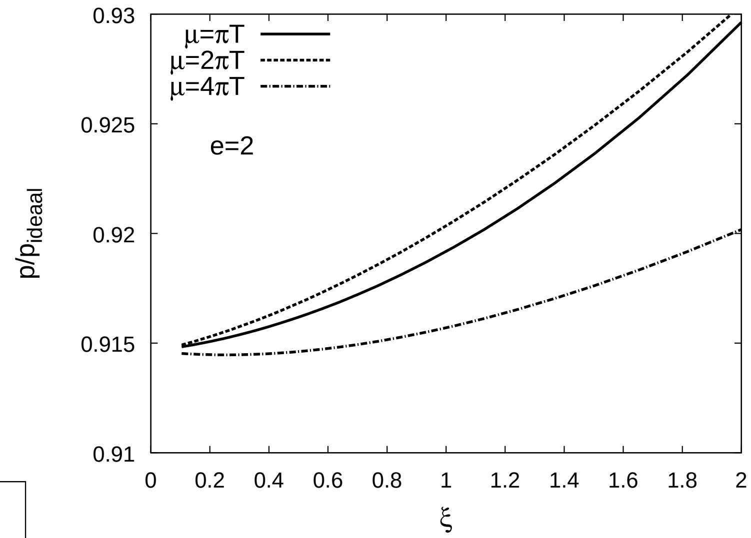


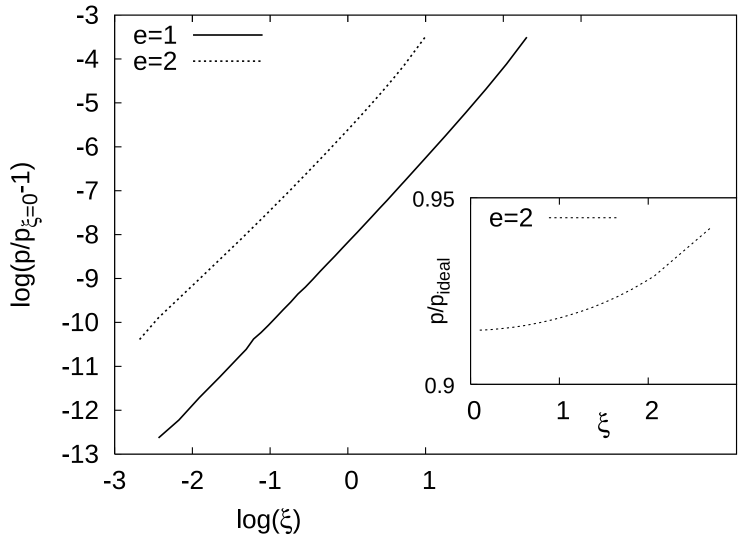
UR & Sz. Borsányi, PLB 661 (2008)

Two-loop result (2/2)

Minimum sensitivity obtained for $\xi = 0$ (Landau gauge):

μ dependence minimal for $\xi = 0$





$$\mathcal{P}_{\rm 2PI}(\xi) - \mathcal{P}_{\rm 2PI}(0) \sim \alpha(e,\mu)\,\xi^2$$