I: Hard Thermal Loops and Hard Dense Loops in Equilibrium QCD Thermodynamics

II: Hard Anisotropic Loops and Nonabelian Plasma Instabilities

III: Hard Expanding Loops

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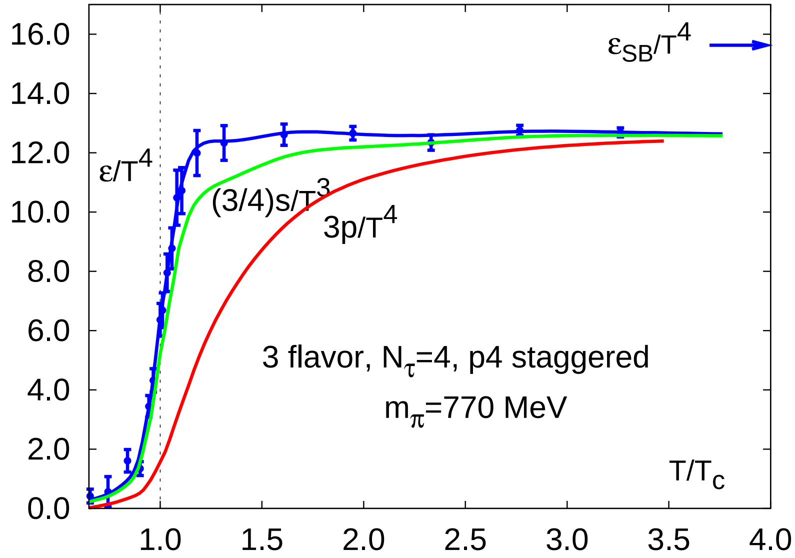
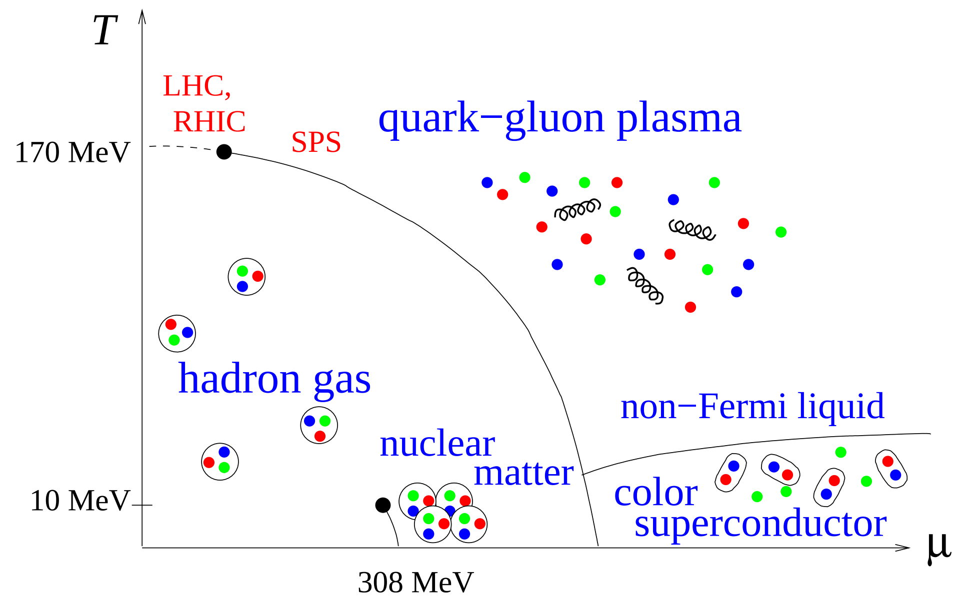
High-temperature/density QCD





Asymptotic freedom: $g \to 0$ for $T/\Lambda_{\rm QCD}$ or $\mu/\Lambda_{\rm QCD} \to \infty$

Q: Can one use weak-coupling techniques at T a few times T_c (μ a few times μ_c)?

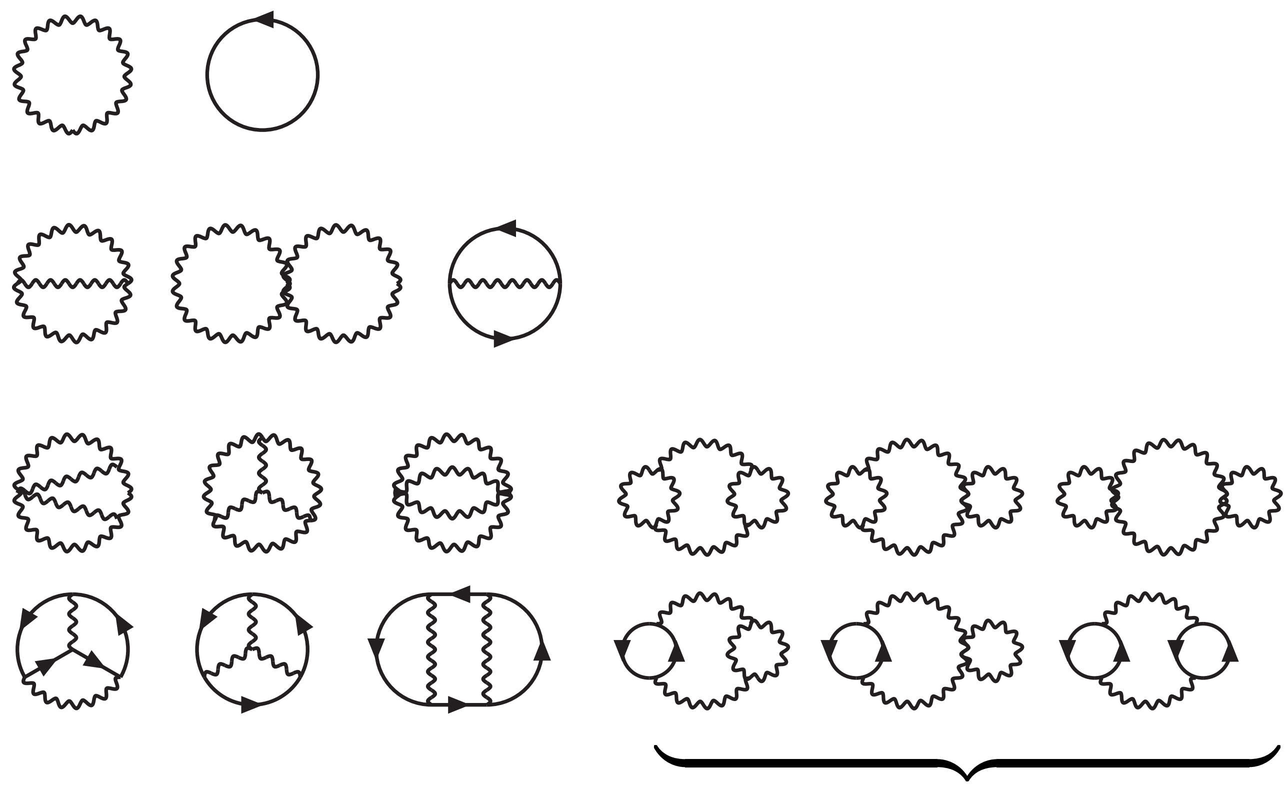
Matsubara frequencies $2\pi T > 2\pi T_c > 1$ GeV

RHIC: sQGP

LHC: sQGP or wQGP/pQCD?

Perturbative calculation of QCD thermodynamics to 3-loop order

$$P = (\beta V)^{-1} \ln Z =$$



IR singular

due to (perturbative) electrostatic screening!

Zero Matsubara mode contribution in outer loop:

$$T \int_{\lambda \to 0} d^3k \frac{[g^2T^2]^2}{(\mathbf{k}^2)^2} \sim \frac{g^4T^5}{\lambda} \to \infty$$

Needs resummation of screening mass: $T \int d^3k \frac{[g^2T^2]^2}{(\mathbf{k}^2+g^2T^2)^2} \sim \frac{g^4T^5}{gT} = g^3T^4$

Effective Field Theory for Zero Modes: 3d Yang-Mills+adj.scalar

Scales when $g \ll 1$:

T: *hard* — scale of Matsubara frequencies $\omega_n = \pi i n T$ (n even/odd for bosons/fermions)

gT: *soft*, g^2T: *ultrasoft*, ... only bosonic zero-modes $n = 0$ (no fermions)

Braaten & Nieto 1996: Pressure of QCD

$$P = P^{\rm hard} + P^{\rm soft}, \qquad P^{\rm hard} = T^4(c_1 + c_2 g^2 + c_3 g^4 + c_4 g^6 + \ldots)$$

with $P^{\rm soft}$ from effective 3-d theory EQCD (electrostatic QCD)

$$\mathcal{L}_E = \frac{1}{2}\,{\rm tr}\,F_{ij}^2 + \,{\rm tr}\,[D_i, A_0]^2 + m_E^2\,{\rm tr}\,A_0^2 + \frac{1}{2}\lambda_E(\,{\rm tr}\,A_0^2)^2 + \ldots,$$

perturbative matching:

$$g_E^2 = g^2 T + \ldots, \quad m_E^2 = (1 + N_f/6) g^2 T^2 + \ldots, \quad \lambda_E = \frac{9-N_f}{12\pi^2} g^4 T + \ldots,$$

$$P_{\rm soft}/T = \frac{2}{3\pi} m_E^3 - \frac{3}{8\pi^2}\left(4 \ln\frac{\Lambda_E}{2m_E} + 3\right) g_E^2 m_E^2 - \frac{9}{8\pi^3}\left(\frac{89}{24} - \frac{11}{6}\ln 2 + \frac{1}{6}\pi^2\right) g_E^4\, m_E + \ldots$$

Perturbative result for the 3-loop pressure

$$P_{\rm soft}/T = \frac{2}{3\pi} m_E^3 - \frac{3}{8\pi^2}\left(4\ln\frac{\Lambda_E}{2m_E}+3\right) g_E^2 m_E^2 - \frac{9}{8\pi^3}\left(\frac{89}{24}-\frac{11}{6}\ln 2+\frac{1}{6}\pi^2\right) g_E^4\, m_E$$

$$m_E \sim gT, \quad g_E^2 \simeq g^2 T \qquad\qquad \alpha_s \equiv \frac{g^2}{4\pi}$$

$$\begin{aligned} P \;=\;& \tfrac{8\pi^2}{45}T^4\Big\{\big(1+\tfrac{21}{32}N_f\big)-\tfrac{15}{4}\big(1+\tfrac{5}{12}N_f\big)\frac{\alpha_s}{\pi}+30\big[\big(1+\tfrac{1}{6}N_f\big)\big(\frac{\alpha_s}{\pi}\big)\big]^{3/2}+\mathcal{O}(\alpha_s^2)\Big\}. \\ & +\Big\{\tfrac{135}{2}\big(1+\tfrac{1}{6}N_f\big)\ln\big[\frac{\alpha_s}{\pi}\big(1+\tfrac{1}{6}N_f\big)\big]+237.2+15.97N_f-0.413N_f^2 \\ & -\tfrac{165}{8}\big(1+\tfrac{5}{12}N_f\big)\big(1-\tfrac{2}{33}N_f\big)\ln\frac{\bar\mu}{2\pi T}\Big\}\big(\frac{\alpha_s}{\pi}\big)^2\Big\}.+\mathcal{O}(\alpha_s^2)\Big\}. \\ & +\big(1+\tfrac{1}{6}N_f\big)^{1/2}\Big[-799.2-21.96N_f-1.926N_f^2 \\ & +\tfrac{495}{2}\big(1+\tfrac{1}{6}N_f\big)\big(1-\tfrac{2}{33}N_f\big)\ln\frac{\bar\mu}{2\pi T}\Big]\big(\frac{\alpha_s}{\pi}\big)^{5/2}+\mathcal{O}(\alpha_s^3\ln\alpha_s)\Big\}. \end{aligned}$$

Kapusta 1979

Toimela 1983

Arnold & Zhai 1995

Zhai & Kastening 1995

Braaten & Nieto 1996

Perturbative result for the 3-loop pressure

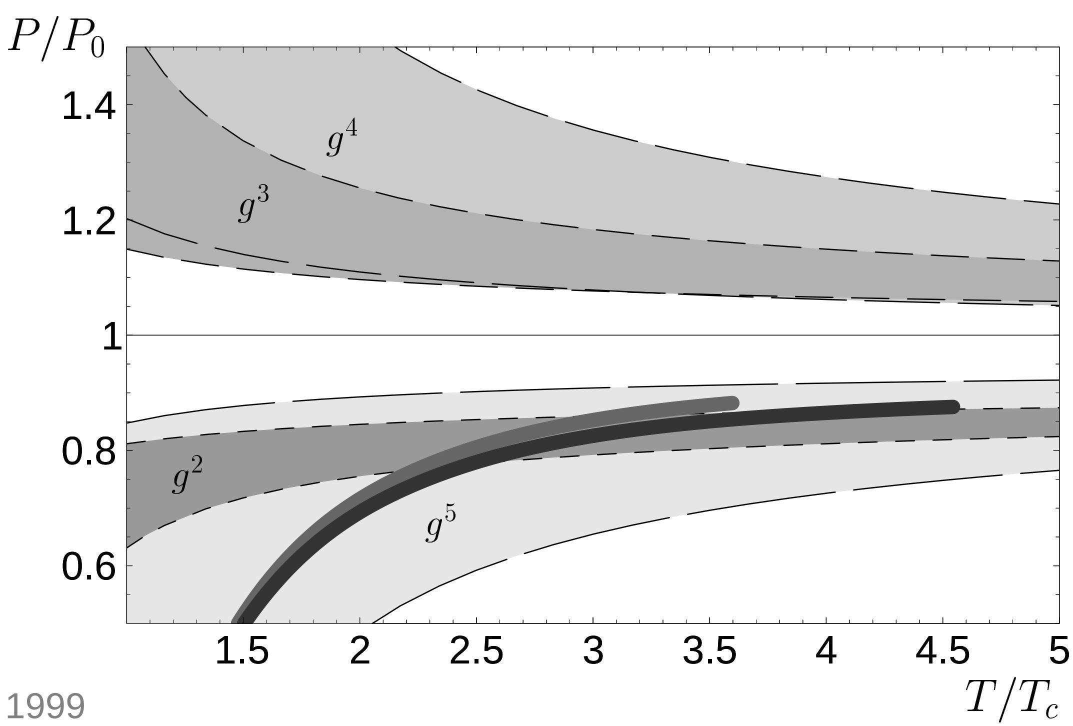
$$
\begin{aligned}
P \;=\; & \frac{8\pi^2}{45}T^4\Big\{(1+\tfrac{21}{32}N_f) - \tfrac{15}{4}(1+\tfrac{5}{12}N_f)\frac{\alpha_s}{\pi} + 30[(1+\tfrac{1}{6}N_f)(\frac{\alpha_s}{\pi})]^{3/2} \\
& +\Big\{\tfrac{135}{2}(1+\tfrac{1}{6}N_f)\ln[\frac{\alpha_s}{\pi}(1+\tfrac{1}{6}N_f)] + 237.2 + 15.97N_f - 0.413N_f^2 \\
& -\tfrac{165}{8}(1+\tfrac{5}{12}N_f)(1-\tfrac{2}{33}N_f)\ln\frac{\bar{\mu}}{2\pi T}\Big\}(\frac{\alpha_s}{\pi})^2 \\
& +(1+\tfrac{1}{6}N_f)^{1/2}\Big[-799.2 - 21.96N_f - 1.926N_f^2 \\
& +\tfrac{495}{2}(1+\tfrac{1}{6}N_f)(1-\tfrac{2}{33}N_f)\ln\frac{\bar{\mu}}{2\pi T}\Big](\frac{\alpha_s}{\pi})^{5/2} + \mathcal{O}(\alpha_s^3\ln\alpha_s)\Big\}.
\end{aligned}
$$

Arnold & Zhai 1995

Zhai & Kastening 1995

Braaten & Nieto 1996

No apparent convergence; steadily increasing renormalization scale ($\bar{\mu}$) dependence:



$N_f = 0$

Lattice data:

Boyd et al. (BI) 1996

Okamoto et al. (CP-PACS) 1999

$\alpha_s = \alpha_s(\bar{\mu})$

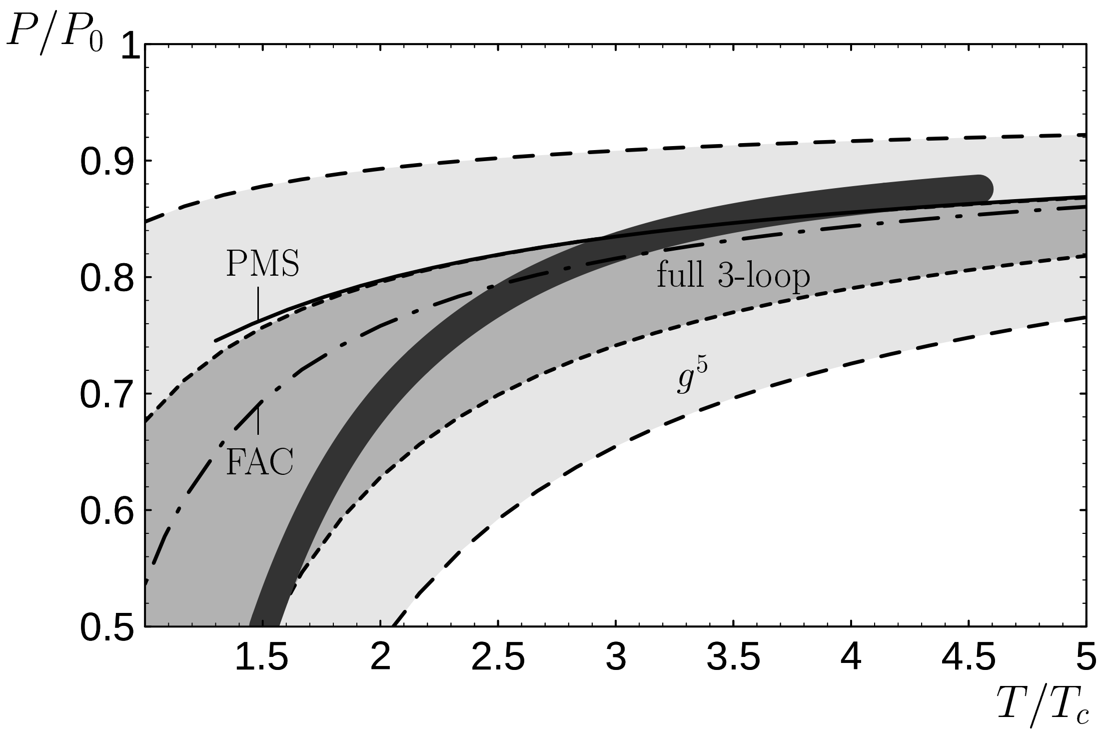
$\bar{\mu} = \pi T \ldots 4\pi T$

Improving apparent convergence in dimensional reduction

Expanding $P = P^{\rm hard} + P^{\rm soft}$ in powers (and log's) of g
$\longrightarrow$ perturbative series with bad convergence

Not expanding $m_E^2(g)$, $g_E^2(g)$, $\ldots \longrightarrow$ improved convergence for $T \gtrsim 3T_c$

J.-P. Blaizot, E. Iancu, AR, PRD68 (2003) 025011:



$\bar{\mu}_{\rm MS} = \Lambda_E = \pi T \ldots 4\pi T$

$O(g^6 \ln(g))$-contribution

Last perturbatively calculable coefficient done by

Kajantie, Laine, Rummukainen & Schröder (2003):

$$P \ni N_g \frac{(N g_E^2)^3}{(4\pi)^4} \left[\left(\frac{43}{12} - \frac{157\pi^2}{768} \right) \ln \frac{\Lambda_E}{g_E^2} + \left(\frac{43}{4} - \frac{491\pi^2}{768} \right) \ln \frac{\Lambda_E}{m_E} + \tilde{\delta} \right]$$

$\tilde{\delta}$ determined by 3-d effective field theory MQCD (magnetostatic QCD)

$\mathcal{L}_M = \frac{1}{2} \operatorname{tr} F_{ij}^2 + \ldots,$ adjoint scalar A_0 integrated out, too

- nonperturbative mass gap $\sim g^2 T$, requires lattice calculation

(and matching using 4-loop lattice perturbation theory) $\longrightarrow$ contribution $\#(g^2 T)^3 T$

steady progress: Di Renzo, Laine, Miccio, Schröder & Torrero, JHEP 07 (2006)

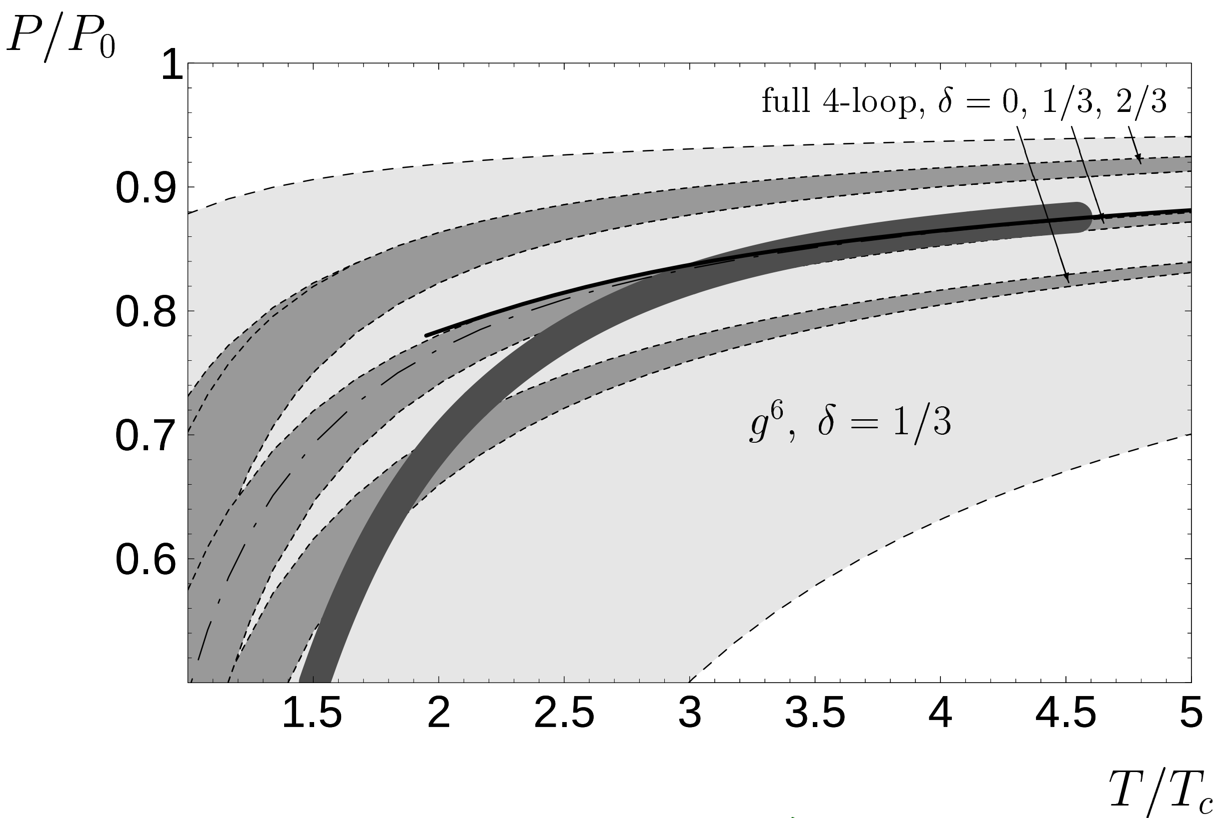
Q: How does it look for some particular value $\tilde{\delta} \sim O(1)$?

Improving apparent convergence in dimensional reduction (cont'd)

$P^{\rm hard} + P^{\rm soft}$ to order $g^6[\log(g) + \delta]$ with some $\delta \sim O(1)$:

even stronger renormalization scale dependence in strict pert.th.

even greater improvement by not expanding out $m_E^2(g)$ and $g_E^2(g)$ and truncating:

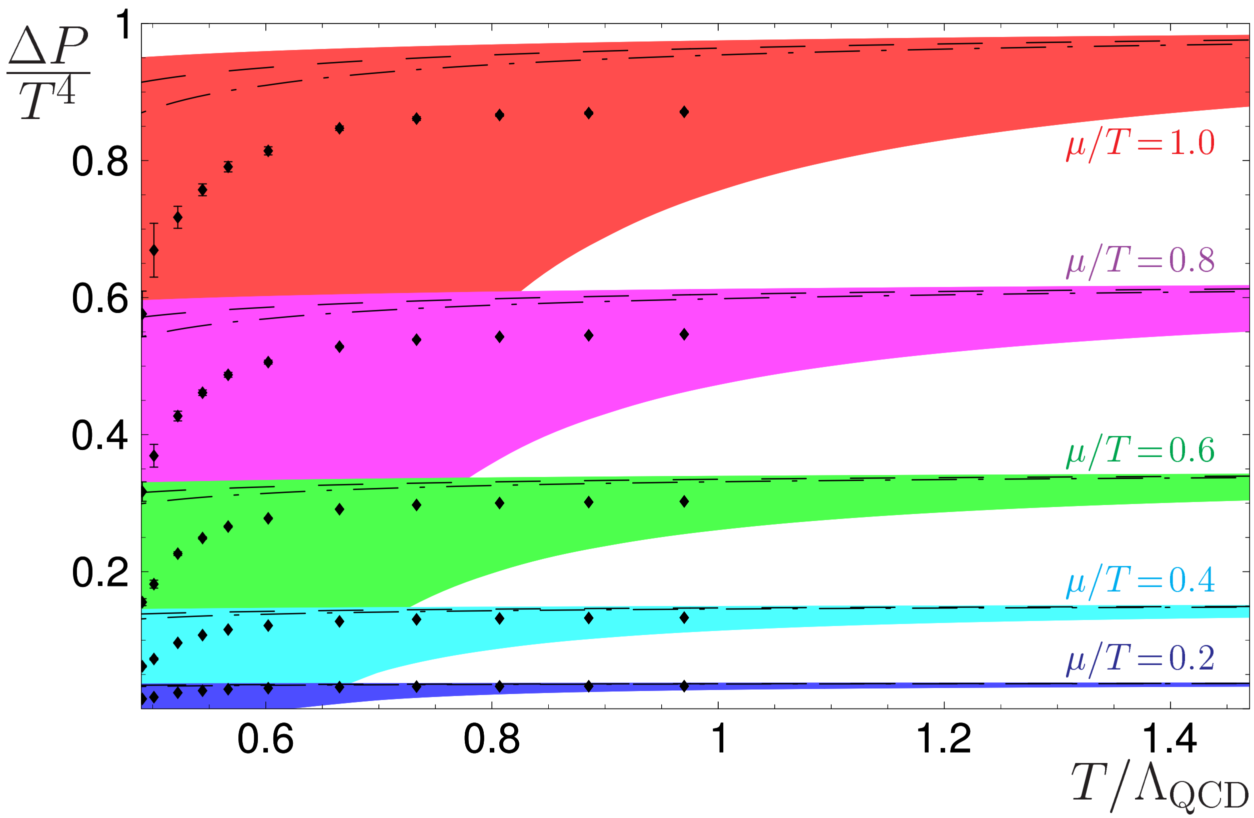


very good agreement with lattice possible for $T \gtrsim 3T_c$

Improving apparent convergence in dimensional reduction

Works also at finite chemical potential $\mu \lesssim T$:

→Vuorinen, PRD68 (2003) 054017; Ipp, AR & Vuorinen, PRD69 (2004) 077901



$\Delta P = P(T,\mu) - P(T,0)$ for $N_f = 2$,

unexpanded 3-loop results with $\bar{\mu}_{\rm MS}$ varied by a factor of 4 and two FAC schemes (dashed)

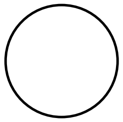
vs. lattice data from Allton et al, PRD68 (2003) 014507 (not yet continuum extrapolated!)

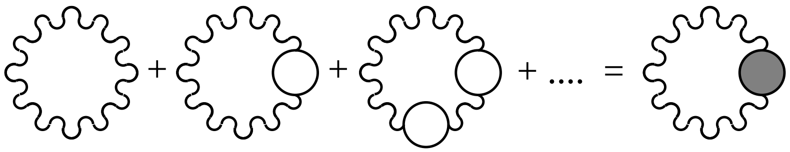
(consistent with Fodor, Katz & Szabó, PLB568 (2003))

Large N_f limit of QCD and QED

G. D. Moore, JHEP 10 (2002) 055: $N_f \to \infty$, $N_c \sim 1$, $g^2 N_f \sim 1$
as testing ground for weak-coupling techniques at high T

Much simpler than large-N_c:

order N_f^1: 

order N_f^0: 

dressed gluon propagator contains typical gauge-theory phenomena such as

- Debye screening for electrostatic modes
- unscreened magnetostatic modes
- complicated dispersion laws, Landau damping, plasmon damping

and can be solved exactly (nonperturbative w.r.t. $g_{\text{eff}}^2 \propto g^2 N_f$)

Large N_f limit of QCD and QED

Effective coupling constant $g_{\text{eff}}^2 = g^2 T_F = \begin{cases} \dfrac{g^2 N_f}{2}\,, & \text{QCD}\,, \\ e^2 N_f\,, & \text{QED}\,. \end{cases}$

One-loop beta function exact: $\dfrac{1}{g_{\text{eff}}^2(\mu)} = \dfrac{1}{g_{\text{eff}}^2(\mu')} + \dfrac{1}{6\pi^2}\ln(\mu'/\mu)\,.$

No asymptotic freedom — instead: Landau singularity at exponentially large

$$\Lambda_{\text{L}} = \bar\mu_{\text{MS}} e^{5/6} e^{6\pi^2/g_{\text{eff}}^2(\bar\mu_{\text{MS}})}\,.$$

Theory only exists as cutoff-theory with $\Lambda_{\text{Cutoff}} < \Lambda_L$

But thermodynamic potential insensitive to cutoff as long as $T, \mu \ll \Lambda_{\text{L}}$

Technicality: cutoff needs to be imposed in Euclidean invariant manner, otherwise spurious singularities

Thermodynamic potential of large-N_f QCD and QED

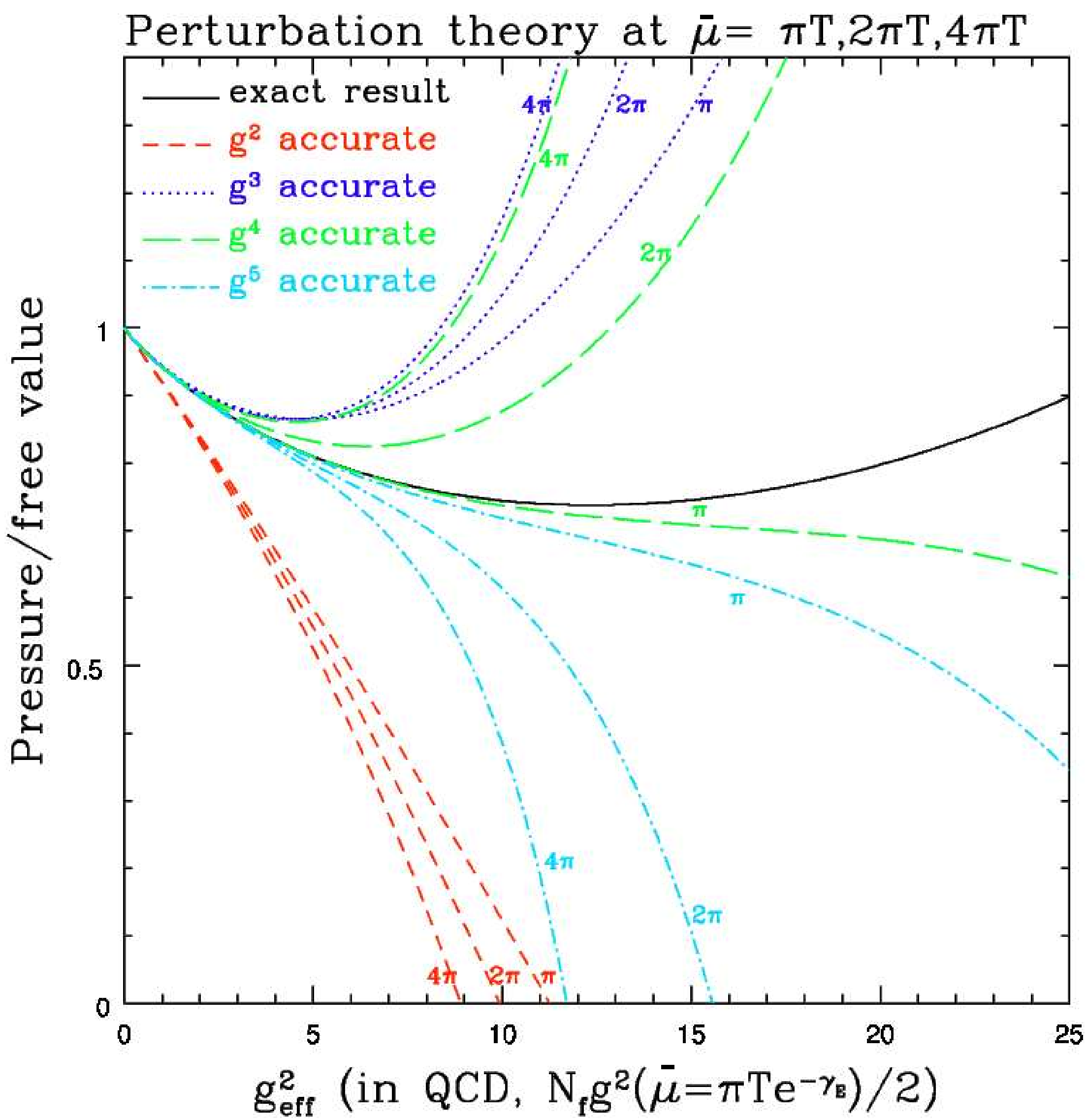
$$
\begin{aligned}
P \;=\;& N N_f \left(\frac{7\pi^2 T^4}{180} + \frac{\mu^2 T^2}{6} + \frac{\mu^4}{12\pi^2} \right) \\
+\;& N_g \int \frac{d^3q}{(2\pi)^3} \int_0^\infty \frac{d\omega}{\pi} \Bigg[2 \Big\{ \left[n_b + \tfrac{1}{2} \right] \, \mathrm{Im} \ln \left(\mathrm{q}^2 - \omega^2 + \Pi_\mathrm{T} + \Pi_\mathrm{vac} \right) \\
& \qquad - \tfrac{1}{2} \, \mathrm{Im} \ln \left(\mathrm{q}^2 - \omega^2 + \Pi_\mathrm{vac} \right) \Big\} \\
& + \Big\{ \left[n_b + \tfrac{1}{2} \right] \, \mathrm{Im} \ln \frac{\mathrm{q}^2 - \omega^2 + \Pi_\mathrm{L} + \Pi_\mathrm{vac}}{\mathrm{q}^2 - \omega^2} - \tfrac{1}{2} \, \mathrm{Im} \ln \frac{\mathrm{q}^2 - \omega^2 + \Pi_\mathrm{vac}}{\mathrm{q}^2 - \omega^2} \Big\} \Bigg] \\
& + O(N_f^{-1})
\end{aligned}
$$

with $\Pi^{\mu\nu} = \Pi^{\mu\nu}_\mathrm{vac} + \Pi^{\mu\nu}_{mat}, \quad \Pi^{\mu\nu}_{mat} \ni \Pi_T, \Pi_L$, 2 distinct structure functions

Interaction pressure $P - \underbrace{P_0}_{SB}$ finite as $N_f \to \infty$

Pressure of large-N_f QCD and QED @ $\mu = 0$

G. D. Moore, JHEP 10 (2002) 055, E: hep-ph/0209190, A. Ipp. G. D. Moore, AR, JHEP 01 (2003) 037



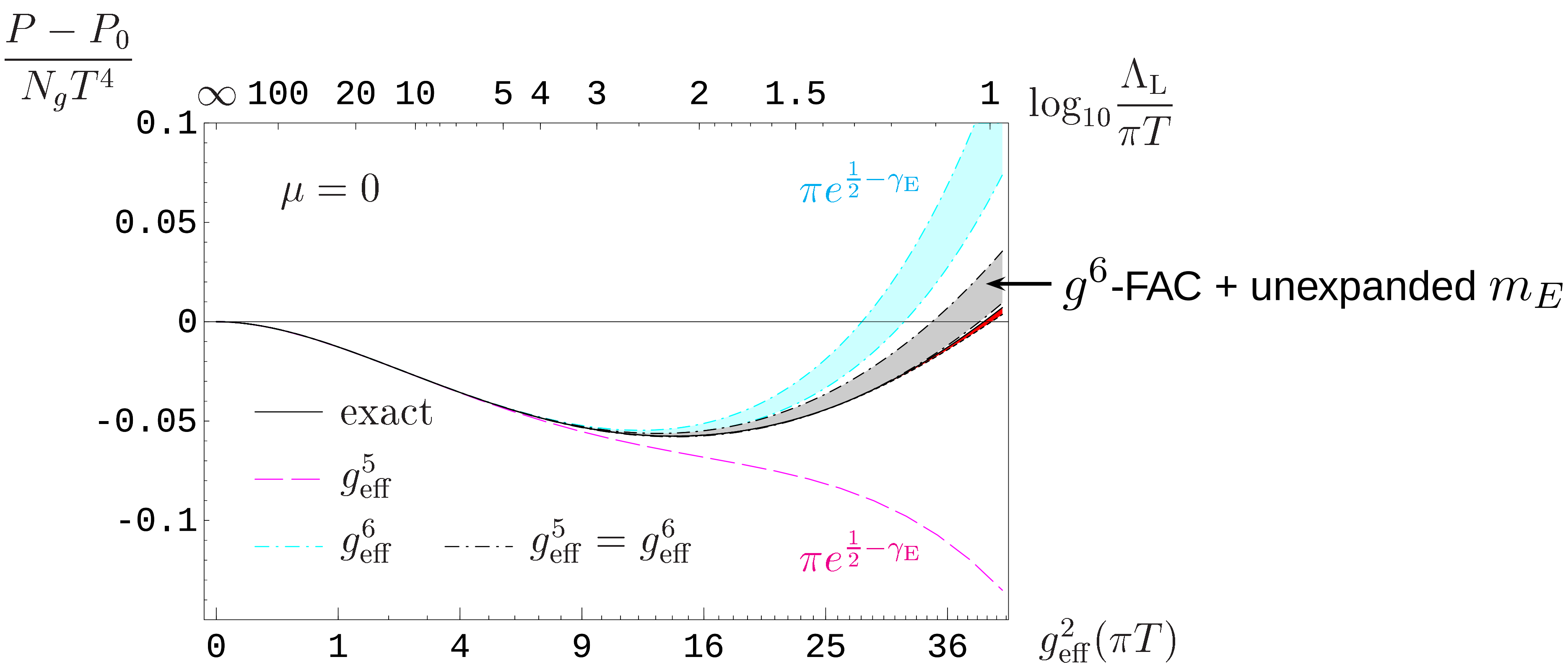
Comparison with
strict perturbation theory

Pressure of large-N_f QCD and QED @ $\mu = 0$

Numerical result sufficiently accurate to verify perturbative results through order $g_{\rm eff}^5$ and to extract $g_{\rm eff}^6$ term (no log here!)

$$P\Big|_{g_{\rm eff}^6,\mu=0,\bar\mu_{\rm MS}=\pi T} = +20(2)N_g\left(\frac{g_{\rm eff}}{4\pi}\right)^6 T^4$$

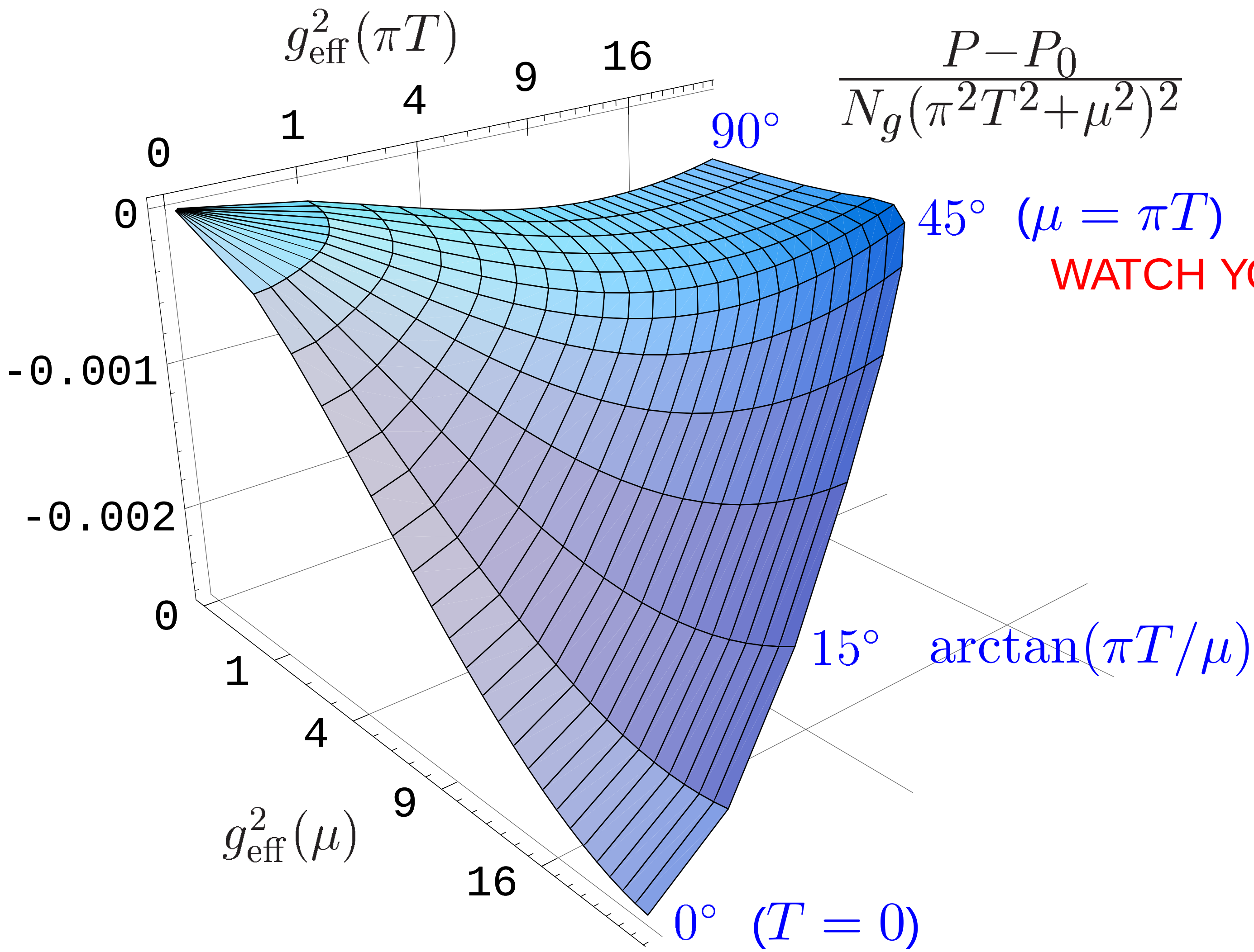
Strict perturbative $O(g^6)$-result vs. unexpanded m_E:



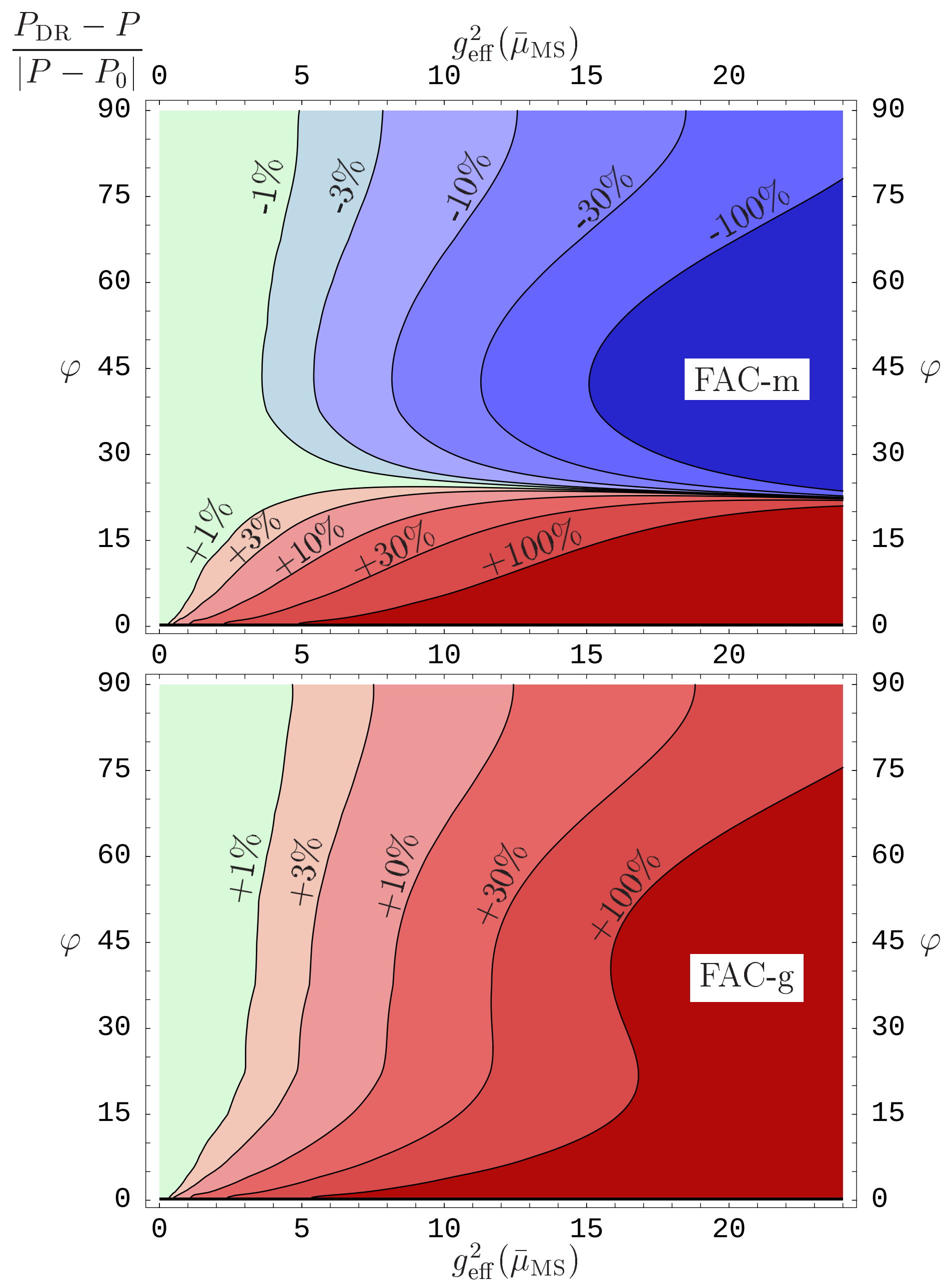
Large-N_f pressure at finite chemical potential μ

Straightforward generalization to finite chemical potential by evaluating $\Pi_{T,L}$ with

$$n_f(k,T,\mu) = \frac{1}{2}\left(\frac{1}{e^{(k-\mu)/T}+1} + \frac{1}{e^{(k+\mu)/T}+1}\right)$$



Comparison with complete dimensional reduction results for all μ, T



Ipp, AR & Vuorinen, PRD69 (2004) 077901

Comparison for two Fastest Apparent Convergence scales (for m_E^2 and g_E^2, resp.)

Breakdown of Dim.Red.

for $T \lesssim g_{\rm eff}\,\mu/\pi$

$$\varphi \equiv \arctan(\pi T/\mu)$$

Non-Fermi-Liquid Behavior at small $T \neq 0$

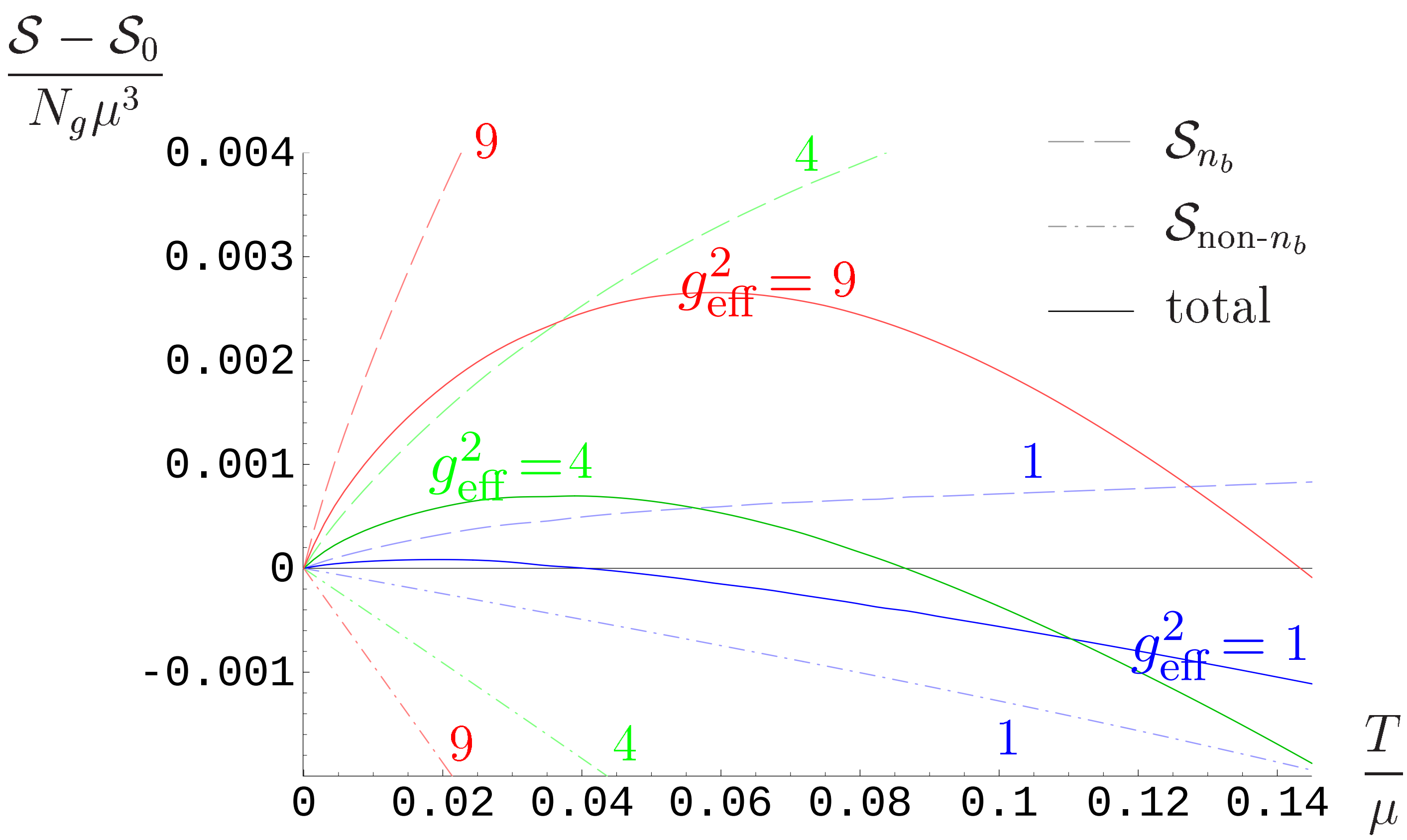
LO (2-loop) term in interaction pressure gives

$$P - P_0 = -N_g \left[\tfrac{5}{9}T^4 + \tfrac{2}{\pi^2}\mu^2 T^2 + \tfrac{1}{\pi^4}\mu^4\right] \frac{g_{\rm eff}^2}{32} + \ldots$$

Entropy $\mathcal{S} = \left(\frac{\partial P}{\partial T}\right)_\mu$ at small T should start as

$$\mathcal{S} - \mathcal{S}_0 = - N_g \frac{g_{\rm eff}^2}{8\pi^2}\mu^2 T + \ldots$$

BUT:



Hard (Thermal/Dense) Loop Effective Theory

Actual effective theory at soft scales when dimensional reduction not applicable: HTL/HDL EFT (Braaten & Pisarski, Frenkel & Taylor & Wong 1990):

$$\mathcal{L}^{\rm HTL} = \mathcal{L}_f^{\rm HTL} + \mathcal{L}_g^{\rm HTL}$$

$$= M_f^2 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \bar{\psi}\gamma^\mu \frac{v_\mu}{iv\cdot D(A)}\psi + \frac{m_D^2}{2}{\rm tr}\int \frac{d\Omega_{\mathbf{v}}}{4\pi} F^{\mu\alpha}\frac{v_\alpha v^\beta}{(v\cdot D_{adj.}(A))^2}F_{\mu\beta}$$

$v=(1,\mathbf{v})$ with $\mathbf{v}^2=1$ is direction of hard particles' momenta $p^\mu \sim T v^\mu$

$$M_f^2 = \begin{cases} \dfrac{g^2 N_c T^2}{3} + \dfrac{g^2\sum_f \mu_f^2}{2\pi^2}\,, \\[2ex] \dfrac{e^2T^2}{3} + \dfrac{e^2\mu_e^2}{\pi^2} \end{cases} \qquad m_D^2 = \begin{cases} \dfrac{g^2 N_c T^2}{3} + \dfrac{g^2\sum_f \mu_f^2}{2\pi^2}\,, & \text{QCD}\,, \\[2ex] \dfrac{e^2T^2}{3} + \dfrac{e^2\mu_e^2}{\pi^2}\,, & \text{QED}\,. \end{cases}$$

- gauge invariant also in the non-static case
- nonlocal (because modes integrated out are real rather than virtual)

HTL/HDL gauge boson self energy

Gauge invariance of HTL/HDL effective action $\longrightarrow$ transverse gauge boson self energy

2 tensors transverse w.r.t. 4-momentum in a thermal medium (rest frame velocity $u^\mu = \delta_0^\mu$)

$$A_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - B_{\mu\nu},$$

$$B_{\mu\nu} = \frac{\tilde{n}_\mu \tilde{n}_\nu}{\tilde{n}^2} \text{ with } \tilde{n}_\mu = (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})u^\nu$$

$$\Pi_A \equiv \Pi_T = \frac{1}{2} A_{\mu\nu} \Pi^{\mu\nu} = \frac{1}{2}\left(\Pi^\mu{}_\mu - \Pi_B\right)$$

$$\Pi_B \equiv \Pi_L = -\frac{k^2}{\mathbf{k}^2}\Pi_{00}$$

$$\Pi^\mu{}_\mu = m_D^2, \quad \Pi_{00} = m_D^2\left(1 - \frac{k^0}{2|\mathbf{k}|}\ln\frac{k^0+|\mathbf{k}|}{k^0-|\mathbf{k}|}\right)$$

Gauge boson propagator (Landau gauge)

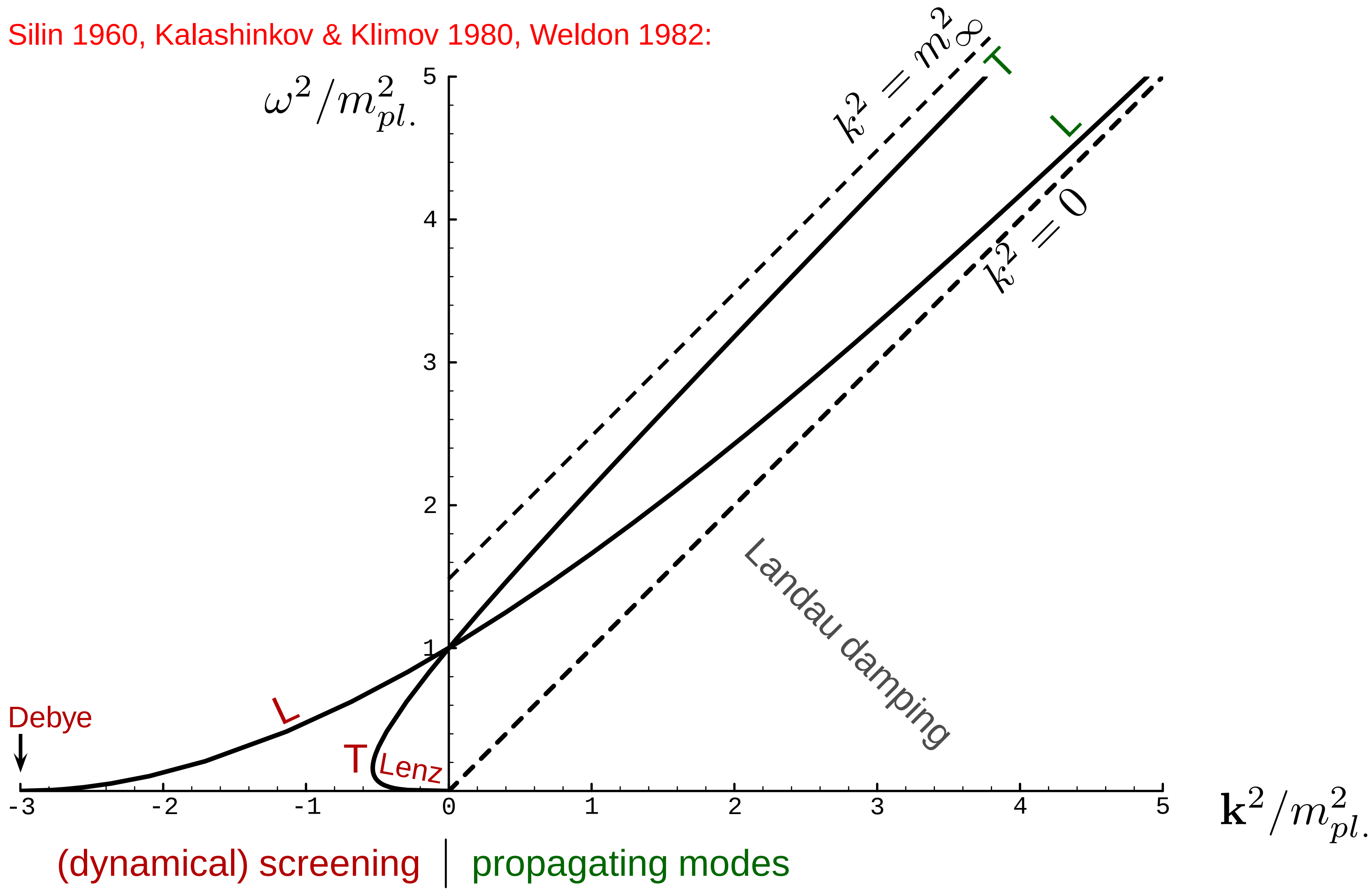
$$-G_{\mu\nu} = \Delta_T A_{\mu\nu} + \Delta_L B_{\mu\nu}$$

$$\Delta_T = [k^2 - \Pi_T]^{-1}, \quad \Delta_L = [k^2 - \Pi_L]^{-1}$$

$\longrightarrow$ 2 branches with different dispersion laws

Dispersion laws of HTL/HDL gauge bosons

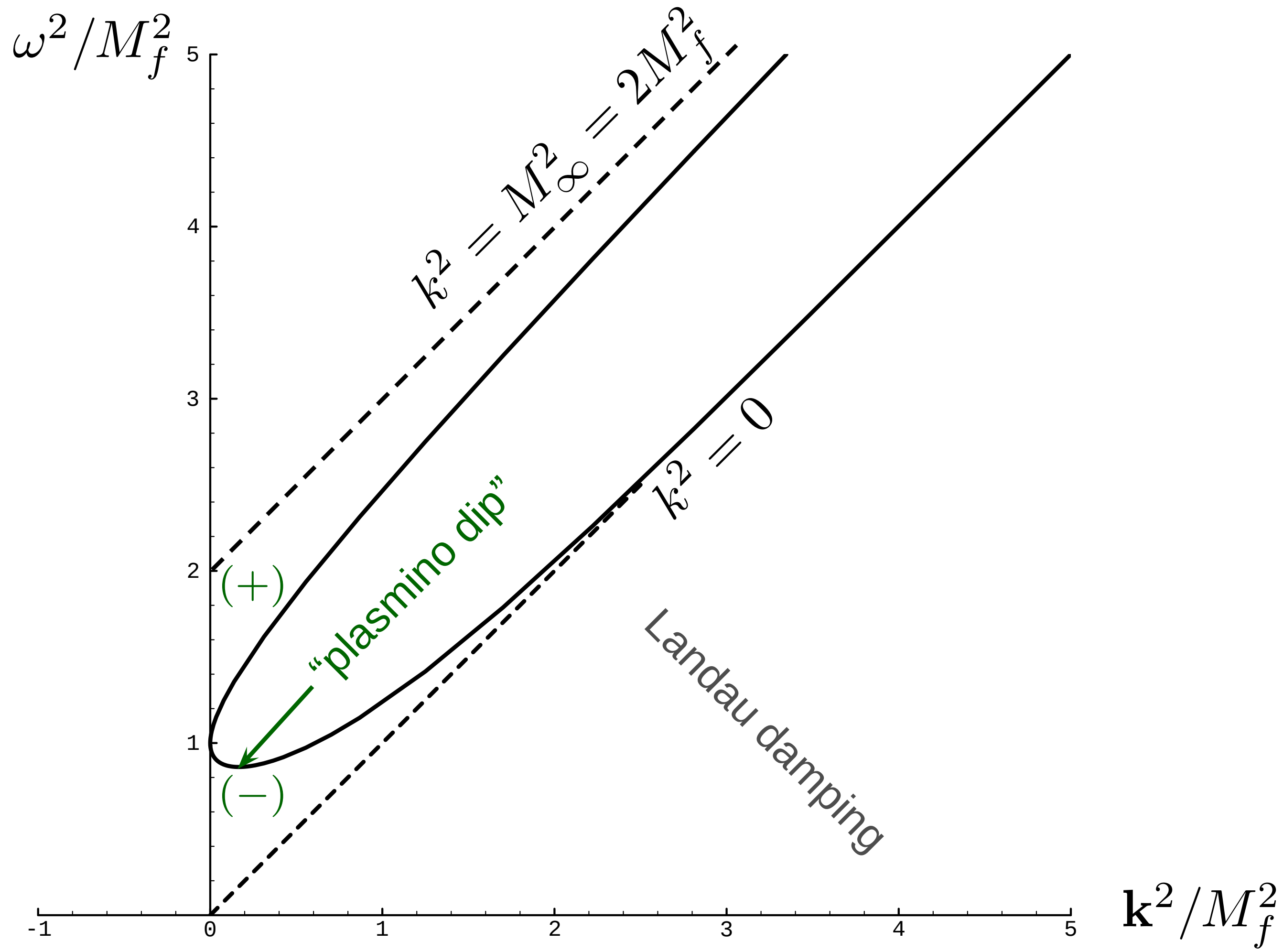
Silin 1960, Kalashinkov & Klimov 1980, Weldon 1982:



- Debye screening of electrostatic modes with $m_D^2 = 3m^2_{pl.} = 2m^2_\infty$
- Weak screening of quasistatic magnetic modes: $\kappa = \sqrt{-\mathbf{k}^2} = [\pi m_{\rm D}\omega/4]^{1/3}$

Dispersion laws of HTL/HDL fermionic excitations

Klimov 1981, Weldon 1982, 1989:



(no screening) | propagating modes

- Extra collective mode $(-)$ with negative helicity over chirality ratio

Nonanalyticity of low-T expansion

Resummed one-loop pressure from transverse gluons: ($S = \frac{\partial}{\partial T}P, c_V = \frac{\partial^2}{\partial T^2}P$)

$$\frac{P_{T,n_b}}{N_g} = -\int \frac{d^3q}{(2\pi)^3} \int_0^\infty \frac{dq_0}{\pi} 2n_b \,\mathrm{Im}\ln \Delta_T^{-1}, \quad n_b = \frac{1}{\exp(q_0/T)-1},$$

n_b restricts to $q_0 \lesssim T$, but derivatives w.r.t. T are singular because of only weakly screened low-frequency transverse gauge bosons (*dynamical screening*):

$$q_0 \to 0: \quad \mathrm{Im}\ln \Delta_T^{-1} \simeq \mathrm{Im}\ln(q^2 - \Pi_T^{\mathrm{HDL}}) \simeq \arctan \frac{-\frac{g_{\mathrm{eff}}^2}{16\pi}(4\mu^2+q^2)\frac{q_0}{q}\theta(2\mu-q)}{q^2}$$

$$\int_0^{2\mu} dq\, q^2 \arctan \frac{\alpha q_0(4\mu^2+q^2)}{q^3} \simeq \frac{4\mu^2}{3}\alpha q_0 \left(\ln \frac{2\mu}{\alpha q_0} + \frac{5}{2}\right) + O(q_0^{5/3})$$

$$\frac{S_{T,n_b}}{N_g} = \frac{g_{\mathrm{eff}}^2 \mu^2 T}{36\pi^2}\left(\ln \frac{32\pi\mu}{g_{\mathrm{eff}}^2 T} + 1 + \gamma_E - \frac{6}{\pi^2}\zeta'(2)\right) + O(T^{5/3})$$

Schäfer & Schwenzer PRD70(2004): $g^2 T \ln T^{-1}$ stable even when $g^2 \ln T^{-1} \gg 1$!

Analogous contribution from Δ_L (analytic in T, but not in g):

$$\frac{S_{L,n_b}}{N_g} = \frac{g_{\mathrm{eff}}^2 \mu^2 T}{24\pi^2}\left(2\ln \underbrace{\frac{g_{\mathrm{eff}}}{2\pi}}_{\frac{m_D}{2\mu}} + 1\right) + O(g_{\mathrm{eff}}^4) + O(T^3)$$

Low-temperature expansion of entropy

Dynamical magnetic screening scale $\kappa = [\pi m_D \omega/4]^{1/3}$
$\longrightarrow$ low-T entropy with log's and fractional powers in T:

"Anomalous specific heat"

T. Holstein, R.E. Norton & P. Pincus, PRB8 (1973) 2649; Chakravarty, Norton & Syljuasen, PRL 74 (1995) 1423

A. Ipp, A. Gerhold & AR, PRD69 (2004) 011901R; PRD70 (2004) 105015

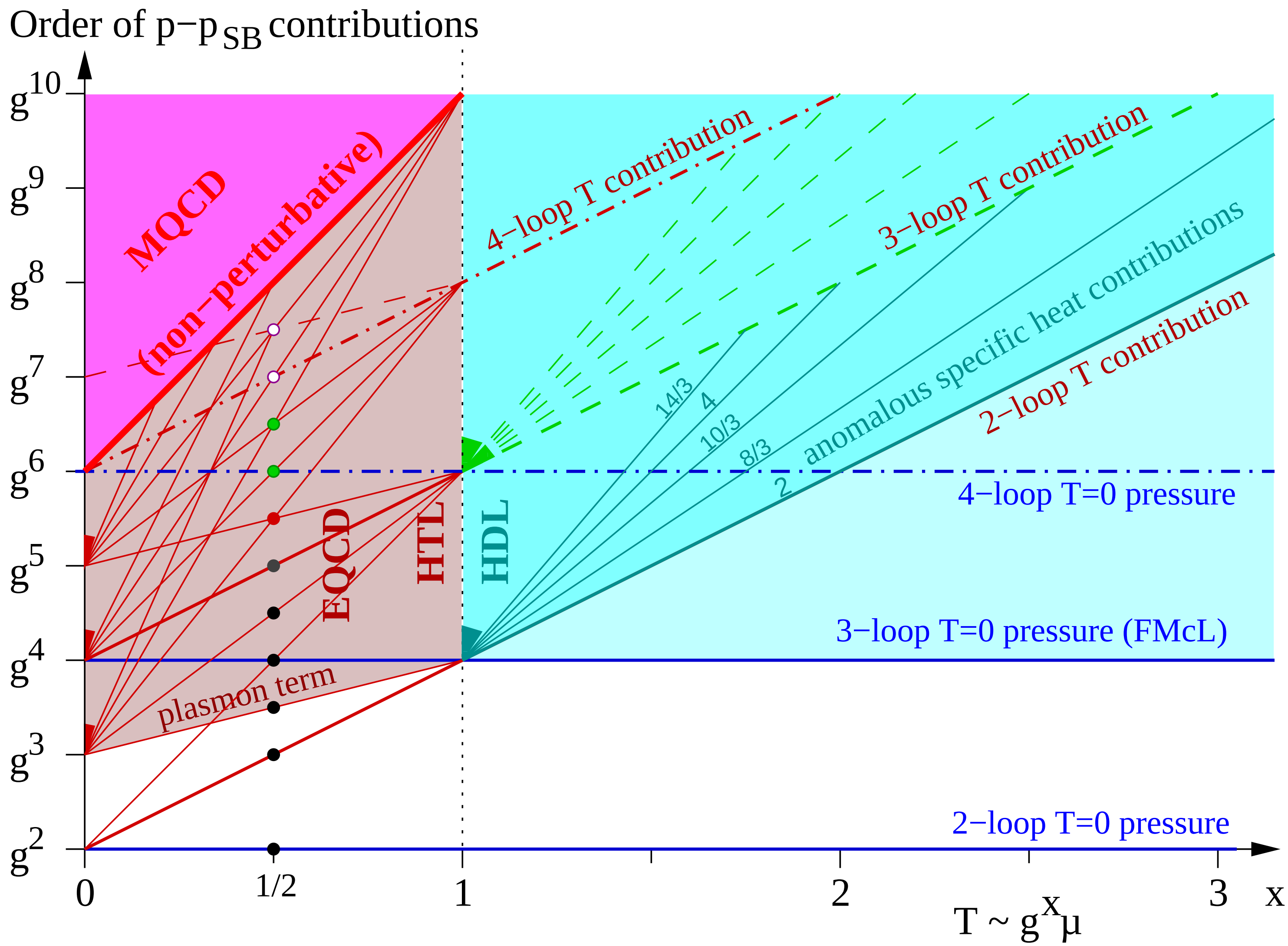
$$\frac{S-S_0}{N_g} = \frac{g_{\rm eff}^2 \mu^2 T}{36\pi^2}\left(\ln\frac{4 g_{\rm eff}\mu}{\pi^2 T} - 2 + \gamma_E - \frac{6}{\pi^2}\zeta'(2)\right)$$
$$-\frac{8\, 2^{2/3}\Gamma(\frac{8}{3})\zeta(\frac{8}{3})}{9\sqrt{3}\pi^{11/3}}(g_{\rm eff}\mu)^{4/3}T^{5/3} + \frac{80\, 2^{1/3}\Gamma(\frac{10}{3})\zeta(\frac{10}{3})}{27\sqrt{3}\pi^{13/3}}(g_{\rm eff}\mu)^{2/3}T^{7/3}$$
$$+\frac{2048-256\pi^2-36\pi^4+3\pi^6}{540\pi^2}T^3\left[\ln\frac{g_{\rm eff}\mu}{T} - 4.3493485\ldots\right] + O(T^{11/3})$$

- Systematic expansion for $T/\mu \sim g_{\rm eff}^{1+\delta}$ with $\delta > 0$:

$$\frac{S-S_0}{N_g\mu^3} \sim g_{\rm eff}^{3+\delta}\ln\frac{c}{g_{\rm eff}} + g_{\rm eff}^{3+(5/3)\delta} + g_{\rm eff}^{3+(7/3)\delta} + g_{\rm eff}^{3+3\delta}\ln\frac{c}{g_{\rm eff}} + g_{\rm eff}^{3+(11/3)\delta} + \ldots$$

Structure of perturbation theory at parametrically small T/μ

Individual terms in coupling expansion for given $x \geq 0$ in $T \sim g^x \mu$



Leading term of interaction entropy for $T \sim g\mu$

Anomalous low-temperature series is applicable only for $T \ll g_{\rm eff}\,\mu$

complete infinite low-temperature series is contained in

HDL-resummed expression Gerhold, Ipp & AR, PRD70 (2004)

$$\frac{1}{N_g}(\mathcal{S}-\mathcal{S}^0) = -\frac{g_{\rm eff}^2\mu^2 T}{24\pi^2} - \frac{1}{2\pi^3}\int_0^\infty dq_0\, \frac{\partial n_b(q_0)}{\partial T}\int_0^\infty dq\, q^2 \Bigg[2\,{\rm Im}\,\ln\left(\frac{q^2-q_0^2+\Pi_T^{\rm HDL}}{q^2-q_0^2}\right) + {\rm Im}\,\ln\left(\frac{q^2-q_0^2+\Pi_L^{\rm HDL}}{q^2-q_0^2}\right)\Bigg] + O(g_{\rm eff}^4\mu^2 T)$$

- full leading-order result $\forall\, T \ll \mu$

$g_{\rm eff}\,\mu \ll T \ll \mu$:

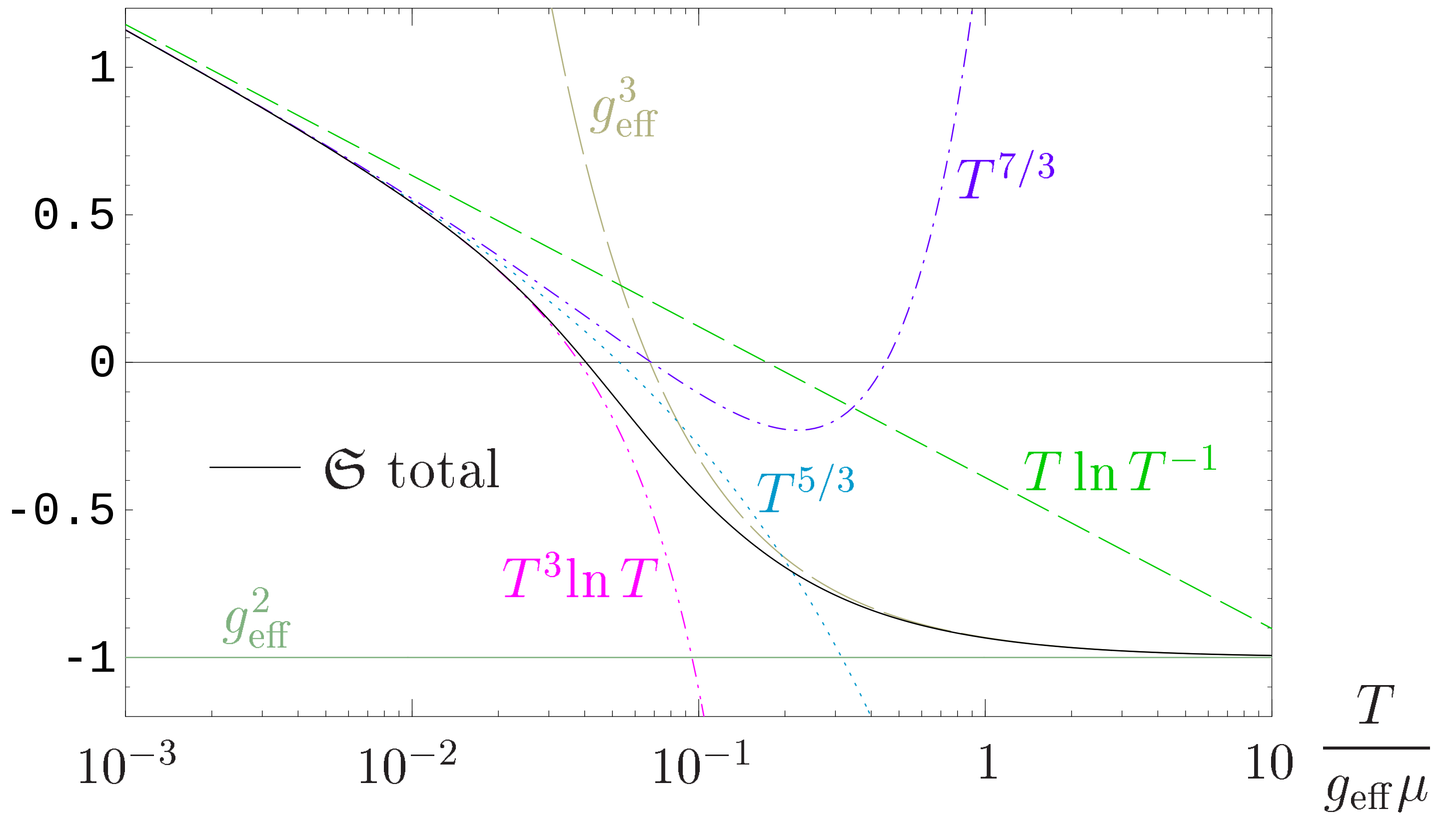
dominant resummation effect now *longitudinal plasmon effect* (Debye screening)

$$\frac{1}{N_g}(\mathcal{S}-\mathcal{S}_0) \simeq -\frac{g_{\rm eff}^2\mu^2 T}{8\pi^2} + \frac{g_{\rm eff}^3\mu^3}{12\pi^4}$$

$\longleftarrow$ also from dimensional reduction

HDL-resummed low-T entropy

$$\mathfrak{S} = \frac{\mathcal{S} - \mathcal{S}_0}{N_g (g_{\rm eff}\,\mu)^2 T/(8\pi^2)}$$



low-temperature expansion to order $T \ln T$, $T^{5/3}$, $T^{7/3}$, $T^3 \ln T$, resp.

$g_{\rm eff}^2$, $g_{\rm eff}^3$: perturbative result for $g_{\rm eff}\,\mu \ll T \ll \mu$

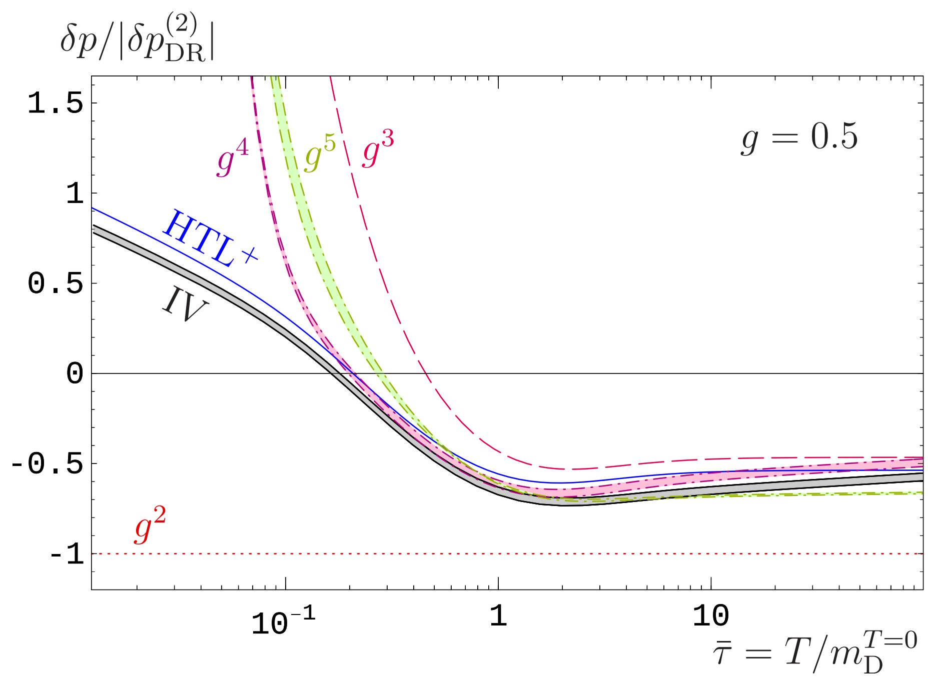
Unified calculation of QCD pressure

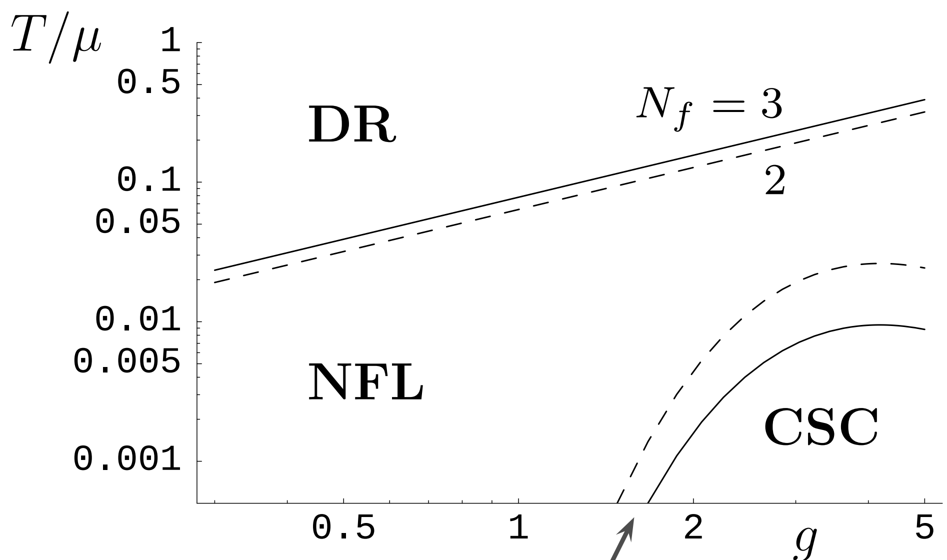
Ipp, Kajantie, AR & Vuorinen, PRD74 (2006)

Full fourth-order calculation (IV): Perturbative calculation of IR-safe diagrams
+ full one-loop resummation of 2GR (2-gluon-reducible) diagrams

- Confirmation of both HDL/HTL resummation and dimensional reduction results and their range of applicability:

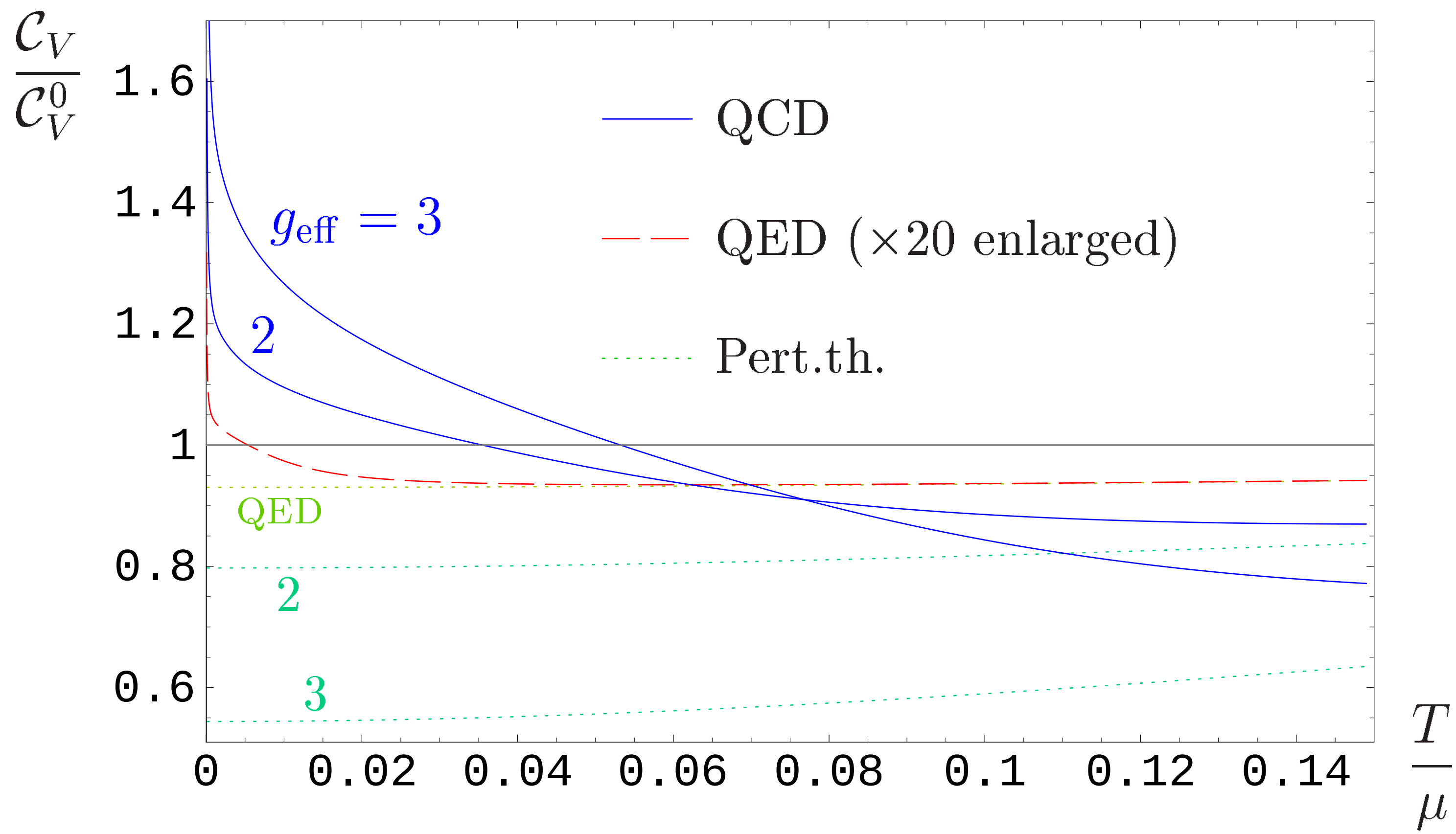
$$\delta p \equiv p - p_{\rm SB} - (p - p_{\rm SB})|_{T=0}$$





$$\frac{T_c^{\rm 2SC}}{\mu} \simeq \frac{T_c^{\rm CFL}}{\mu} \simeq 2\frac{e^\gamma}{\pi} e^{-(\pi^2+4)/8} (4\pi)^4 \left(\frac{2}{N_f}\right)^{5/2} g^{-5} e^{-3\pi^2/\sqrt{2}g}$$

HDL-resummed result for the specific heat



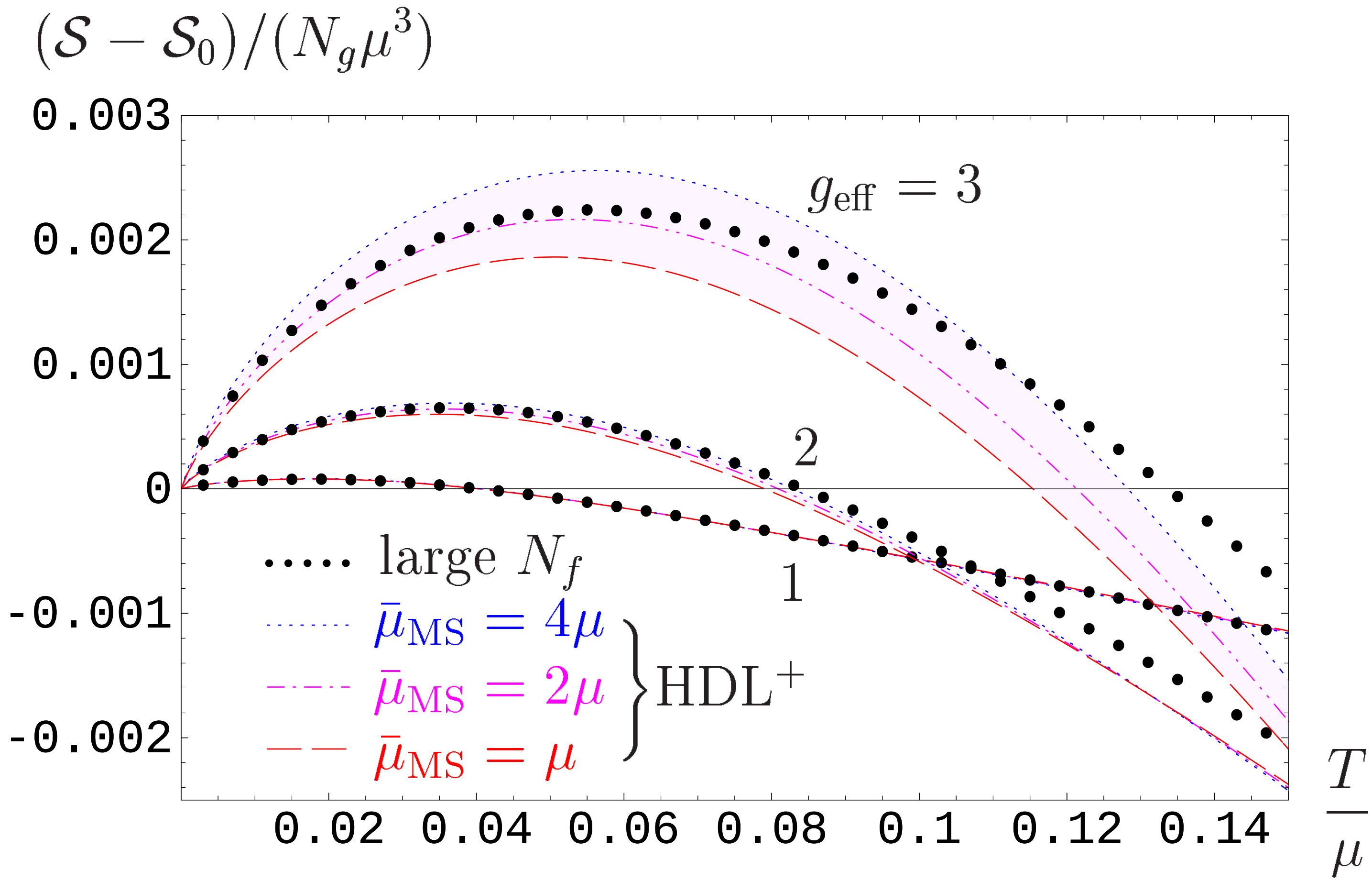
$N_f = 2$ QCD: $g_{\text{eff}} = 2$ and 3 correspond to $\alpha_s \approx 0.32$ and 0.72

significant deviations from naive perturbative result for low-T specific heat
in QCD for $T/\mu \lesssim 0.05$

$\longrightarrow$ cooling of (proto-)neutron stars with normal quark matter component

Non-Fermi-liquid effects also in neutrino emissivity (T. Schäfer & K. Schwenzer, PRD70 (2004) 114037)

HDL-resummed entropy vs. nonperturbative large-N_f resul



. . . $\bar\mu_{\rm MS}$-dependence displays uncertainties due to contributions suppressed by powers of $\frac{g_{\rm eff}^2}{4\pi}$

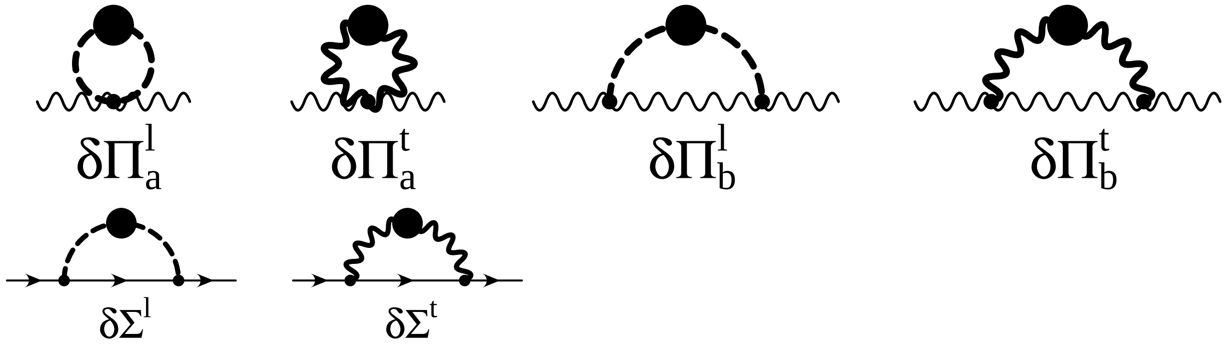
Very good agreement for small T; less good at higher T, $g_{\rm eff}$

HTL-resummed entropy at high T, $\mu = 0$

Blaizot, Iancu & AR (1999):
HTL resummation through 2-loop Φ-derivable (2PI) entropy expression:

$$\mathcal{S} = -\,\mathrm{tr}\int_K \frac{\partial n(k_0)}{\partial T}\left[\Im m \log G^{-1} - \Im m \Pi\, \Re e G\right]$$
$$-2\,\mathrm{tr}\int_K \frac{\partial f(k_0)}{\partial T}\left[\Im m \log S^{-1} - \Im m \Sigma\, \Re e S\right] + \mathcal{S}_{3-\mathrm{loop}},$$

- nontrivial reorganization of perturbation theory:
  convergence-spoiling g^3 contribution kept in nonpolynomial form;
  3/4 of g^3 contribution contained in NLO correction to m_∞^2, M_∞^2 at $k \sim T$:



(momentum-dependent even for $k \gg gT$)

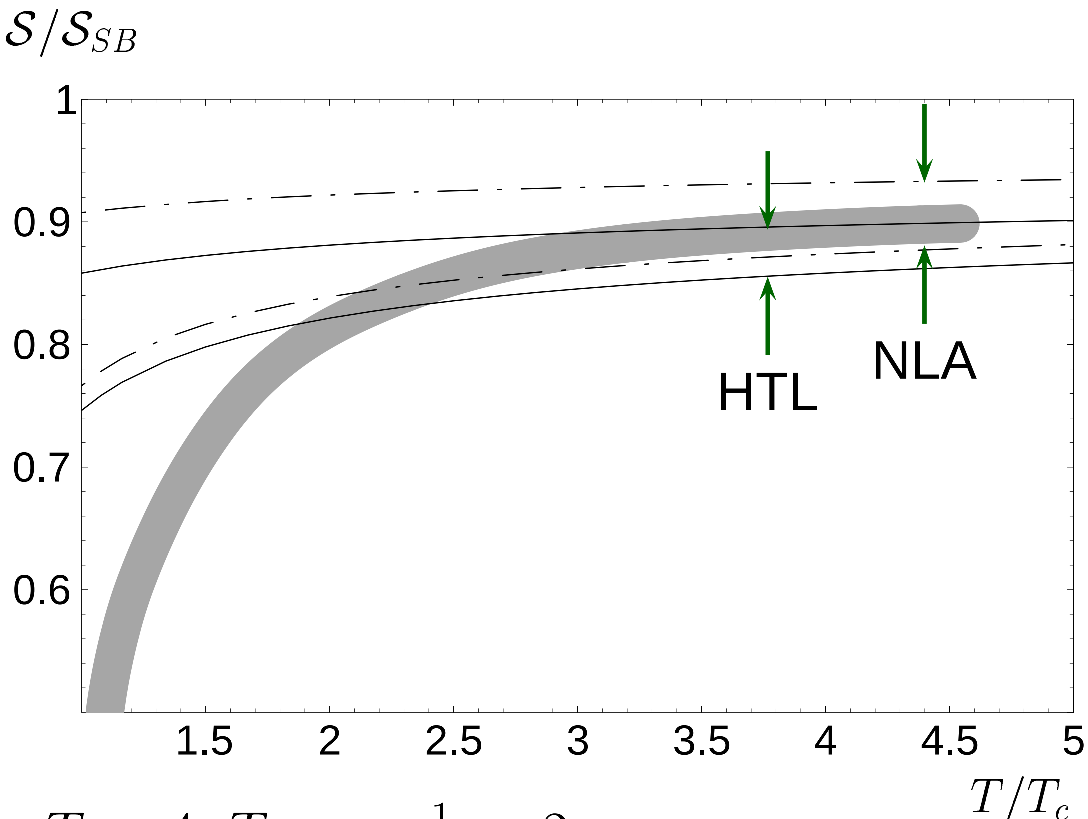
Application to QCD

Approximately self-consistent evaluations:

1) HTL self energies and propagators (full lines)

2) NLA (dash-dotted): momentum averaged NLO corrections to m_∞^2, M_∞^2, included through quadratic gap equation for hard momenta $p > \sqrt{c_\Lambda 2\pi T m_D}$

pure glue, $N_f = 0$:



$\bar{\mu}_{MS} = \pi T \ldots 4\pi T, c_\Lambda = \frac{1}{2} \ldots 2$

Application to $\mathcal{N} = 4$ super-Yang-Mills

Weak-coupling: $\mathcal{S}/\mathcal{S}_0 = 1 - \frac{3}{2\pi^2}\lambda + \frac{\sqrt{2}+3}{\pi^3}\lambda^{3/2} + \ldots$

even more poorly convergent: $\mathcal{S}/\mathcal{S}_0 \geq 1$ for $\lambda \equiv g^2 N \geq 1.14$

(1.85 for pure glue QCD)

Strong coupling (AdS/CFT): $S/\mathcal{S}_0 = \frac{3}{4}\left(1 + \frac{15\zeta(3)}{8}\lambda^{-3/2} + \ldots\right)$

Possible interpolation: Padé

Just enough information to fix uniquely all coefficients of [4,4] Padé approximant:

$$R_{[4,4]} = \frac{1+\alpha\lambda^{1/2}+\beta\lambda+\gamma\lambda^{3/2}+\delta\lambda^2}{1+\bar{\alpha}\lambda^{1/2}+\bar{\beta}\lambda+\bar{\gamma}\lambda^{3/2}+\bar{\delta}\lambda^2}$$

$$\bar{\alpha} = \alpha, \quad \bar{\beta} = \frac{4}{3}\beta, \quad \bar{\gamma} = \frac{4}{3}\gamma, \quad \bar{\delta} = \frac{4}{3}\delta,$$

$$\alpha = \frac{2(9+3\sqrt{2}+\gamma\pi^3)}{9\pi}, \quad \beta = \frac{9}{2\pi^2}, \quad \gamma = \frac{2}{15\zeta(3)}, \quad \delta = \frac{2}{15\zeta(3)}\alpha$$

All coefficients positive: no poles anywhere, smooth monotonic interpolation

Blaizot, Iancu, Kraemmer & AR, JHEP 0706,35

Application to $\mathcal{N}=4$ super-Yang-Mills

(Blaizot, Iancu, Kraemmer & AR, JHEP 0706,35

Compare to HTL/NLA resummation of weak-coupling result:

HTL energies

scalar: $\Pi_s \equiv m^2_{\infty(s)}$,

gluons: $\Pi_T = m^2_{\infty(g)} + \frac{\omega^2-k^2}{2k^2}\Pi_L, \Pi_L = 2m^2_{\infty(g)}\left(1-\frac{\omega}{2k}\log\frac{\omega+k}{\omega-k}\right)$,

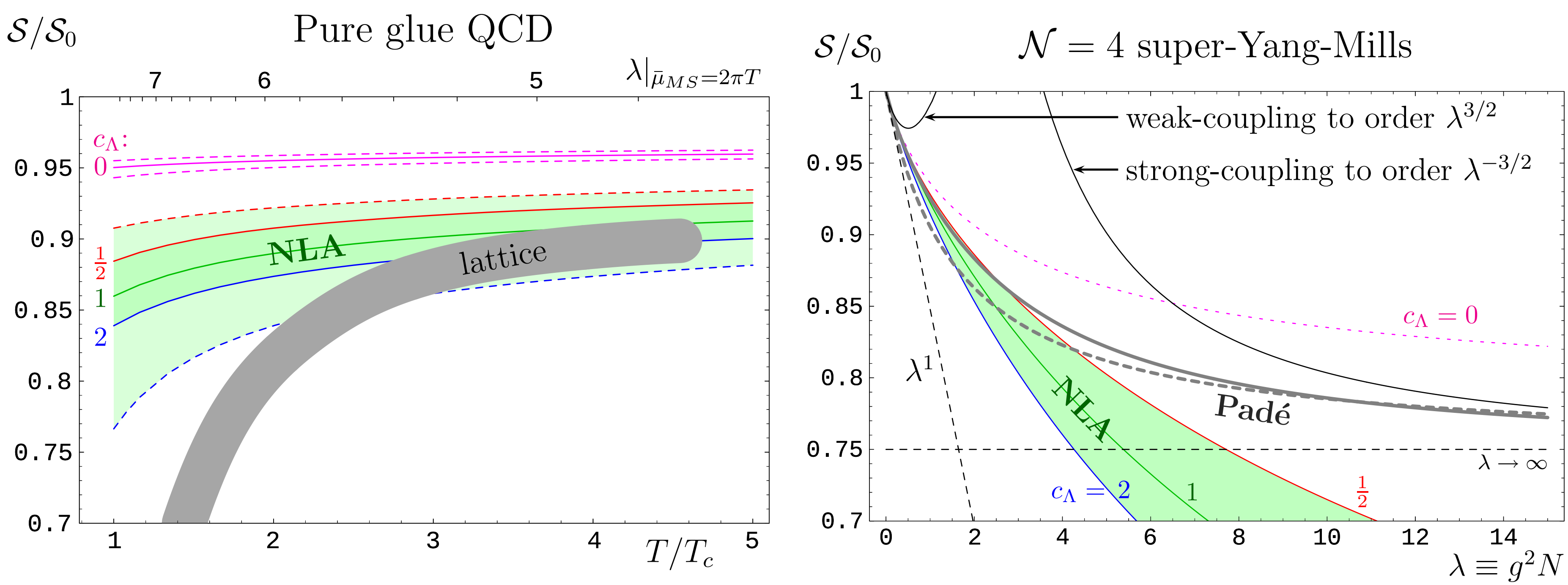
gluinos: $\Sigma_\pm = \frac{m^2_{\infty(f)}}{2k}\left(1-\frac{\omega\mp k}{2k}\log\frac{\omega+k}{\omega-k}\right)$

with $$m^2_{\infty(s)} = m^2_{\infty(g)} = \frac{2+n_s+n_f/2}{12}\lambda T^2 = \lambda T^2,$$
$$m^2_{\infty(f)} = \frac{2+n_s}{8}\lambda T^2 = \lambda T^2$$

weighted NLO correction of (hard) thermal masses for all excitations

$$\bar\delta m^2_\infty = \frac{\int dk\, k\, n'(k)\Re e\delta\Pi(\omega{=}k)}{\int dk\, k\, n'(k)} = -\lambda T m_\infty \frac{2\sqrt{2}+n_s}{4\pi} = -\lambda T m_\infty \frac{\sqrt{2}+3}{2\pi}$$

Comparison pure-glue QCD and $\mathcal{N} = 4$ super-Yang-Mills



- roughly $\lambda_{\rm SYM} \leftrightarrow \frac{1}{2}\lambda_{\rm QCD}$
- QCD at $T \gtrsim 3T_c$ corresponds to $\lambda_{\rm SYM} \lesssim 2.5$
  where unresummed perturbative result fails,
  but simple HTL/NLA resummation agrees well with Padé extrapolation
- (numerically) important additional nonpert. physics in QCD for $T \lesssim 2.5T_c$
  [in SYM for $\lambda_{\rm SYM} \gtrsim 4$]
  (but there behavior of entropy no longer comparable)

Conclusions Part I

- Perturbative results at finite T and μ often *very* poorly convergent
- Suitable resummation of soft contributions seem to improve applicability down to $T \sim 2.5T_c$ in entropy of thermal QCD
  - Works well also for chemical potential $\mu \sim T$
- sQGP for $T \lesssim 2.5T_c$, wQGP for $T \gtrsim 2.5T_c$?
- Breakdown of dimensional reduction at $\pi T \lesssim m_D$
  - Non-Fermi-liquid behavior at $T \ll g\mu$:

    (analytically calculable by HDL resummation)

Part II: Hard Anisotropic Loops and Nonabelian Plasma Instabilities

Scales of wQGP

- T: energy of hard particles
- gT: thermal masses, Debye screening mass,
  Landau damping, **plasma instabilities** [Mrówczyński 1988, 1993, ...]
- g^2T: magnetic confinement, color relaxation, rate for small angle scattering
- g^4T: rate for large angle scattering; inverse shear viscosity $\eta^{-1}T^4$

Effective theory at scale gT: Hard-(~~Thermal~~)Loop Effective Action

[Frenkel, Taylor & Wong; Braaten & Pisarski 1991]

equivalent to: gauge-covariant Boltzmann-Vlasov

[Blaizot & Iancu 1993, Kelly, Liu, Lucchesi & Manuel 1994]

in particular required (to leading order!) for:

- Bottom-up thermalization [Baier, Mueller, Schiff & Son 2000]

$$t_{eq} \propto g^{-13/5} \longrightarrow g^{-?}$$

[Arnold, Lenaghan, Moore, JHEP 08 ('03) 002]

- Shear viscosity [Arnold, Moore & Yaffe]

$$(\eta/s)^{-1} = g^4 \ln(1/g) f(\ln(1/g)) + (\eta/s)^{-1}_{\text{anomalous}} \text{ (!)}$$

[Asakawa, Bass & Müller, PRL 96 ('06) 252301]

Hard (Thermal) Loops — Boltzmann-Vlasov

With color-neutral background distribution $v \cdot \partial\, f_0(\mathbf{p}, \mathbf{x}, t) = 0$, $v^\mu = p^\mu / p^0$

gauge covariant Boltzmann-Vlasov:

$$v \cdot D\, \delta f_a(\mathbf{p}, \mathbf{x}, t) = g v_\mu F_a^{\mu\nu} \partial_\nu^{(p)} f_0(\mathbf{p}, \mathbf{x}, t) = -g(\mathbf{E}_a + \mathbf{v} \times \mathbf{B}_a) \cdot \nabla_{\mathbf{p}} f_0,$$

$$D_\mu F_a^{\mu\nu} = j_a^\nu = g \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu}{2p^0} \delta f_a(\mathbf{p}, \mathbf{x}, t).$$

- isotropic: $f_0(\mathbf{p}) = f_0(|\mathbf{p}|)$, $\nabla_{\mathbf{p}} f_0 \propto \mathbf{v}$

$$v \cdot D\, \delta f_a(\mathbf{p}, \mathbf{x}, t) = -g \mathbf{E}_a \cdot \nabla_{\mathbf{p}} f_0$$

- anisotropic $f_0(\mathbf{p})$, $\nabla_{\mathbf{p}} f_0 \not\propto \mathbf{v}$

$$v \cdot D\, \delta f_a(\mathbf{p}, \mathbf{x}, t) = -g(\mathbf{E}_a + \mathbf{v} \times \mathbf{B}_a) \cdot \nabla_{\mathbf{p}} f_0$$

- anisotropic expansion (part III): $f_0(\mathbf{p}; \mathbf{x}, t)$

Hard loop gauge boson self energy

Linearize in A^μ and Fourier transform

$$j^\mu(k) = g^2 \int \frac{d^3p}{(2\pi)^3} v^\mu \underbrace{\partial^\beta_{(p)} f(\mathbf{p})}_{0 \text{ for } \beta=0} \left(g_{\gamma\beta} - \frac{v_\gamma k_\beta}{k\cdot v + i\epsilon}\right) A^\gamma(k) = \Pi^{\mu\nu}(k) A_\nu(k)$$

$i\epsilon \leftrightarrow$ retarded boundary condition

Isotropic case: $\partial^\beta_{(p)} f(|\mathbf{p}|) = f'(|\mathbf{p}|)\left(0, p^b/|\mathbf{p}|\right)$

$$\rightarrow \Pi_T(k_0/|\mathbf{k}|),\ \Pi_L(k_0/|\mathbf{k}|) \propto m^2 = g^2 p^2_{\text{hard}}$$

Generic case:

$10 - 4 = 6$ structure functions, each depending on 3 variables k_i/k_0

Recap: Isotropic gauge boson self energy

Gauge invariance of HTL/HDL effective action $\longrightarrow$ transverse gauge boson self energy

2 tensors transverse w.r.t. 4-momentum in a thermal medium (rest frame velocity $u^\mu = \delta_0^\mu$)

$$A_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - B_{\mu\nu},$$

$$B_{\mu\nu} = \frac{\tilde{n}_\mu \tilde{n}_\nu}{\tilde{n}^2} \text{ with } \tilde{n}_\mu = (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})u^\nu$$

$$\Pi_A \equiv \Pi_T = \tfrac{1}{2} A_{\mu\nu}\Pi^{\mu\nu} = \tfrac{1}{2}\left(\Pi^\mu{}_\mu - \Pi_B\right)$$

$$\Pi_B \equiv \Pi_L = -\frac{k^2}{\mathbf{k}^2}\Pi_{00}$$

$$\Pi^\mu{}_\mu = m_D^2, \quad \Pi_{00} = m_D^2 \left(1 - \frac{k^0}{2|\mathbf{k}|} \ln \frac{k^0 + |\mathbf{k}|}{k^0 - |\mathbf{k}|}\right)$$

Gauge boson propagator (Landau gauge)

$$-G_{\mu\nu} = \Delta_T A_{\mu\nu} + \Delta_L B_{\mu\nu}$$

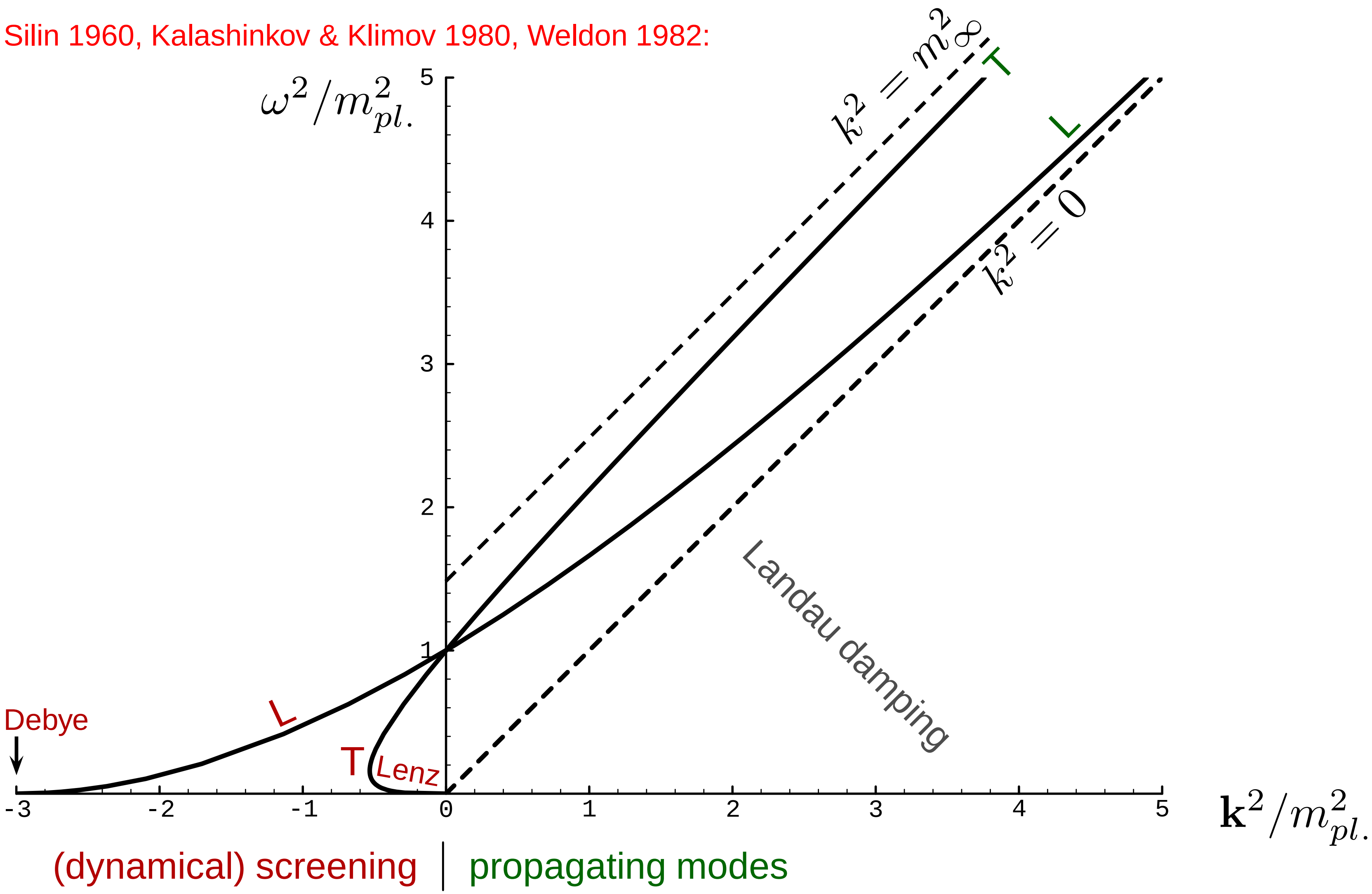
$$\Delta_T = [k^2 - \Pi_T]^{-1}, \quad \Delta_L = [k^2 - \Pi_L]^{-1}$$

$\longrightarrow$ 2 branches with different dispersion laws

Dispersion laws of HTL/HDL gauge bosons

Isotropic case (not necessarily thermal)

Silin 1960, Kalashinkov & Klimov 1980, Weldon 1982:



Hard anisotropic loop gauge boson self energy

$$\Pi^{\mu\nu}(k) = g^2 \int \frac{d^3p}{(2\pi)^3} v^\mu \partial_\beta^{(p)} f(\mathbf{p}) \left(g^{\nu\beta} - \frac{v^\nu k^\beta}{k \cdot v + i\epsilon} \right), \quad v^\mu \equiv \frac{p^\mu}{p^0}, \quad p^0 = |\mathbf{p}|$$

$\Pi^{\mu\nu}$ symmetric, $\Pi^{0\nu}$ fixed by transversality $k_\mu \Pi^{\mu\nu} = 0 \longrightarrow$ 6 structure functions in general

Assume just one direction of anisotropy (axisymmetry): $\mathbf{n} = (0, 0, 1)$

$\longrightarrow$ 4 symmetric tensors for Π^{ij}, 4 independent structure functions

$$A^{ij} = \delta^{ij} - k^i k^j / k^2, \quad B^{ij} = k^i k^j / k^2,$$

$$C^{ij} = \tilde{n}^i \tilde{n}^j / \tilde{n}^2, \qquad D^{ij} = k^i \tilde{n}^j + k^j \tilde{n}^i, \qquad \tilde{n}^i = A^{ij} n^j$$

$$\Pi^{ij} = \alpha A^{ij} + \beta B^{ij} + \gamma C^{ij} + \delta D^{ij}$$

Propagator (temporal axial gauge $A^0 = 0$ for simplicity)

$$\mathbf{\Delta}(K) = \Delta_T \mathbf{A} + \cancel{(k^2 - \omega^2 + \alpha + \gamma)} \Delta_{\mathcal{L}} \mathbf{B} + [(\beta - \omega^2)\Delta_{\mathcal{L}} - \Delta_T] \mathbf{C} - \delta \Delta_{\mathcal{L}} \mathbf{D}$$

$$\Delta_T(k) = [k^2 - \omega^2 + \alpha]^{-1}$$

$$\Delta_{\mathcal{L}}(k) = [\cancel{(k^2 - \omega^2 + \alpha + \gamma)}(\beta - \omega^2) \cancel{- k^2 \tilde{n}^2 \delta^2}]^{-1}$$

generally: 2 branches from $\Delta_{\mathcal{L}}$; only 1 from $\Delta_{\mathcal{L}}$ when $\mathbf{k} \parallel \mathbf{n} \Rightarrow \tilde{n} = 0$

Hard anisotropic loop gauge boson self energy

$$\Pi^{\mu\nu}(k) = g^2 \int \frac{d^3p}{(2\pi)^3} v^\mu \partial_\beta^{(p)} f(\mathbf{p}) \left(g^{\nu\beta} - \frac{v^\nu k^\beta}{k \cdot v + i\epsilon} \right), \quad v^\mu \equiv \frac{p^\mu}{p^0}, \quad p^0 = |\mathbf{p}|$$

Special important case: $f(\mathbf{p}) = f_{\rm iso}\left(\mathbf{p}^2 + \xi(\mathbf{p}\cdot\mathbf{n})^2\right)$

$\xi = 0$: isotropic; $-1 < \xi < 0$: prolate (cigar-shaped); $0 < \xi < \infty$: oblate (squashed)

Can be evaluated in closed form: [Romatschke & Strickland 2003]

Change variables $\mathbf{p}^2 + \xi(\mathbf{p}\cdot\mathbf{n})^2 = \bar{\mathbf{p}}^2$

$$\Pi^{ij}(k) = m^2 \int \frac{d\Omega}{4\pi} v^i \frac{v^l + \xi(\mathbf{v}.\mathbf{n})n^l}{(1+\xi(\mathbf{v}.\mathbf{n})^2)^2} \left(\delta^{jl} + \frac{v^j k^l}{k \cdot v + i\epsilon} \right)$$

$$m^2 \equiv -\frac{g^2}{2\pi^2} \int_0^\infty d\bar{p}\, \bar{p}^2 \frac{df_{\rm iso}(\bar{p}^2)}{d\bar{p}}$$

Magnetostatic polarization function for k||n

Static limit: $\alpha(k) \equiv \Pi_T \to \frac{1}{2}\Pi^{ii}(\omega = 0, \mathbf{k}.\mathbf{n}/k)$ because then $k^i \Pi^{ij} \to 0$

Easy exercise: calculate $\Pi^{ii}(\omega = 0)$ for $\mathbf{k} \parallel \mathbf{n}$!

Solution: $\alpha/m^2 = \frac{1}{2}\Pi^{ii}(\omega = 0, \mathbf{k})/m^2 = \int \frac{d\Omega}{8\pi} \frac{v^l + \xi(\mathbf{v}.\mathbf{n})n^l}{(1+\xi(\mathbf{v}.\mathbf{n})^2)^2}\left(v^l + \frac{k^l}{-\mathbf{k}.\mathbf{v} + i\epsilon}\right),$

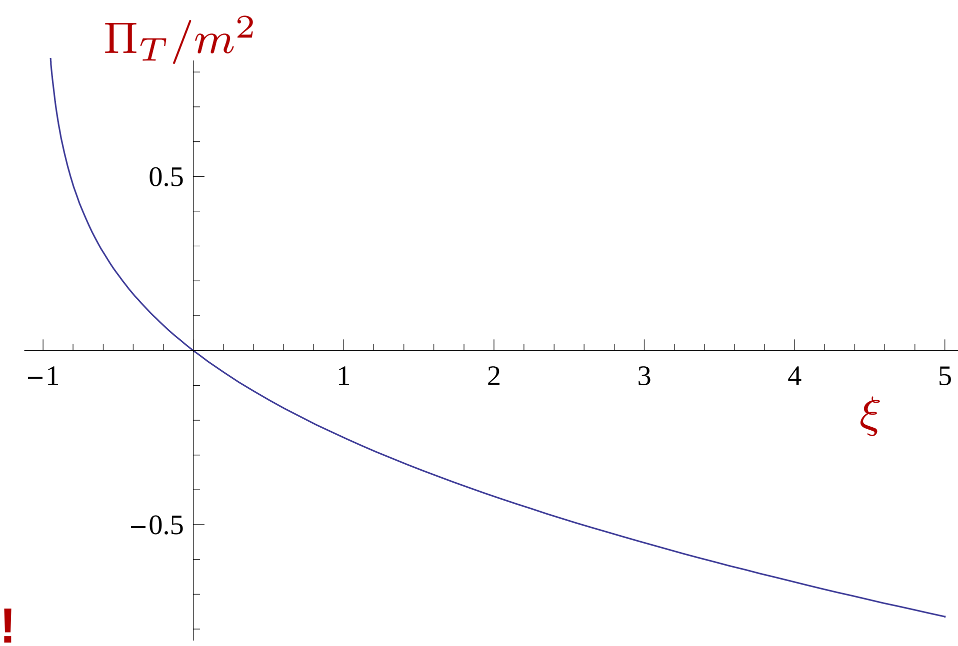
with $k^l = k\delta^l_z$, $n^l = \delta^l_z$:

$$\alpha/m^2 = \int \frac{d\Omega}{8\pi} \frac{\xi v_z}{(1+\xi v_z^2)^2}\left(v_z + \frac{1}{-v_z + i\epsilon}\right) = \frac{\xi}{4}\int_{-1}^{1} dz \frac{z^2 - 1}{(1+\xi z^2)^2} = \frac{1}{4}[(1-\xi)\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}} - 1]$$

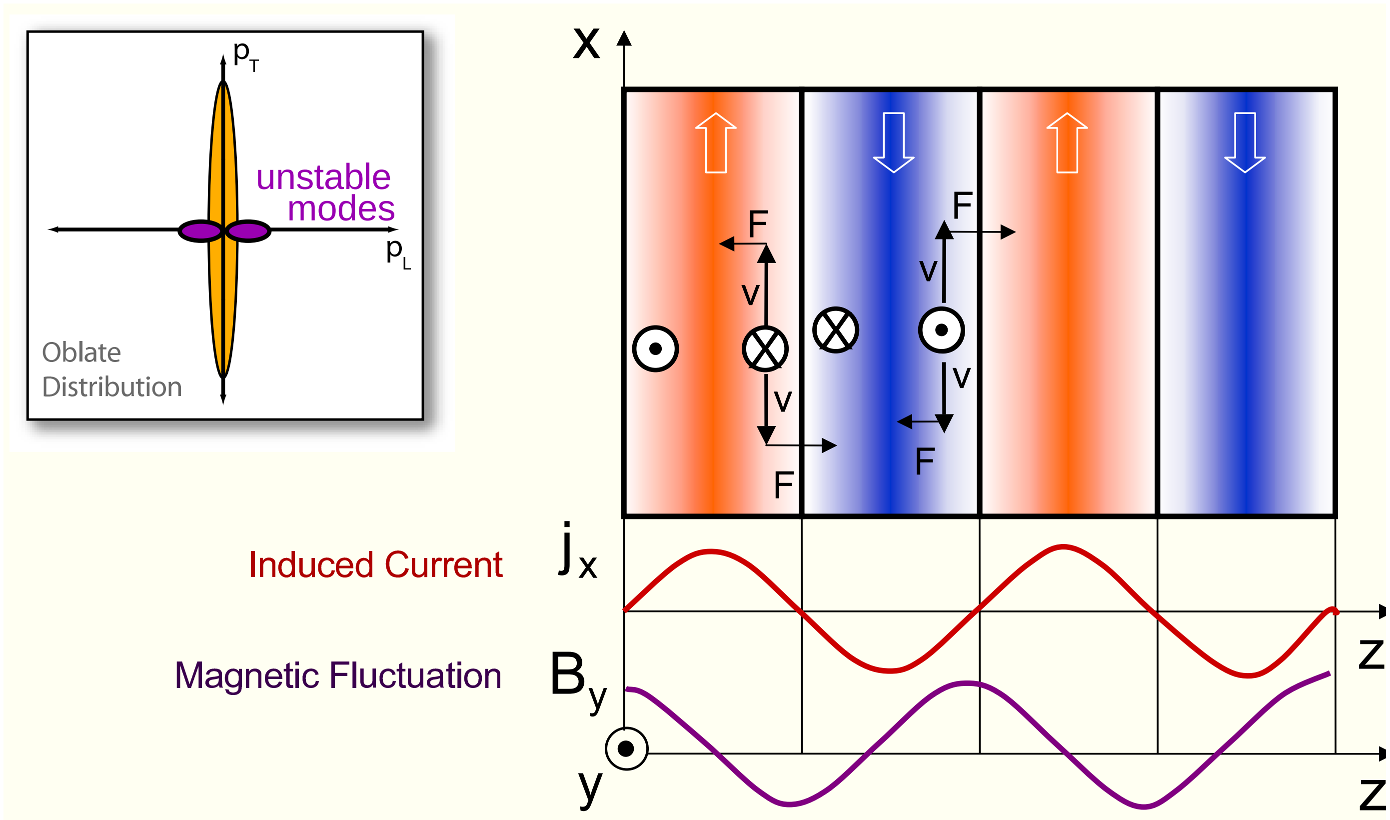
$[\xi < 0]$: $\quad = \frac{1}{4}[(1-\xi)\frac{\mathrm{atanh}\sqrt{-\xi}}{\sqrt{-\xi}} - 1]$

$\alpha = \Pi_T$ is magnetic screening mass

- $\xi = 0$ (isotropic):
  no magnetic screening mass
- $\xi < 0$ (prolate):
  magnetostatic screening!
- $\xi > 0$ (oblate):
  "tachyonic" magnetic mass — **instability!**



Filamentation (Weibel) instabilities



Unstable modes add (small) cigar to (large) squashed sphere in momentum space

Full anisotropic polarization tensor for $\mathrm{k}||\mathrm{n}$

For full dispersion laws (for $\mathbf{k}||\mathbf{n}$ which contains the most unstable modes) need complete frequency dependences ($\eta \equiv \omega/k$) [Romatschke & Strickland 2004]

$$\begin{aligned}
\alpha &= \frac{m^2}{4\sqrt{\xi}(1+\xi\eta^2)^2}\Big[\big(1+\eta^2+\xi(-1+(6+\xi)\eta^2-(1-\xi)\eta^4)\big)\arctan\sqrt{\xi} \\
&\qquad +\sqrt{\xi}\,(\eta^2-1)\left(1+\xi\eta^2-(1+\xi)\eta\ln\frac{\eta+1+i\epsilon}{\eta-1+i\epsilon}\right)\Big], \\
\beta &= -\frac{\eta^2 m^2}{2\sqrt{\xi}(1+\xi\eta^2)^2}\Big[(1+\xi)(1-\xi\eta^2)\arctan\sqrt{\xi} \\
&\qquad +\sqrt{\xi}\left((1+\xi\eta^2)-(1+\xi)\eta\ln\frac{\eta+1+i\epsilon}{\eta-1+i\epsilon}\right)\Big]
\end{aligned}$$

more complicated: $\mathbf{k} \not\parallel \mathbf{n}$

- second branch of poles in $\Delta_{\mathcal{L}}$ which can contain *electric* (Buneman) instability

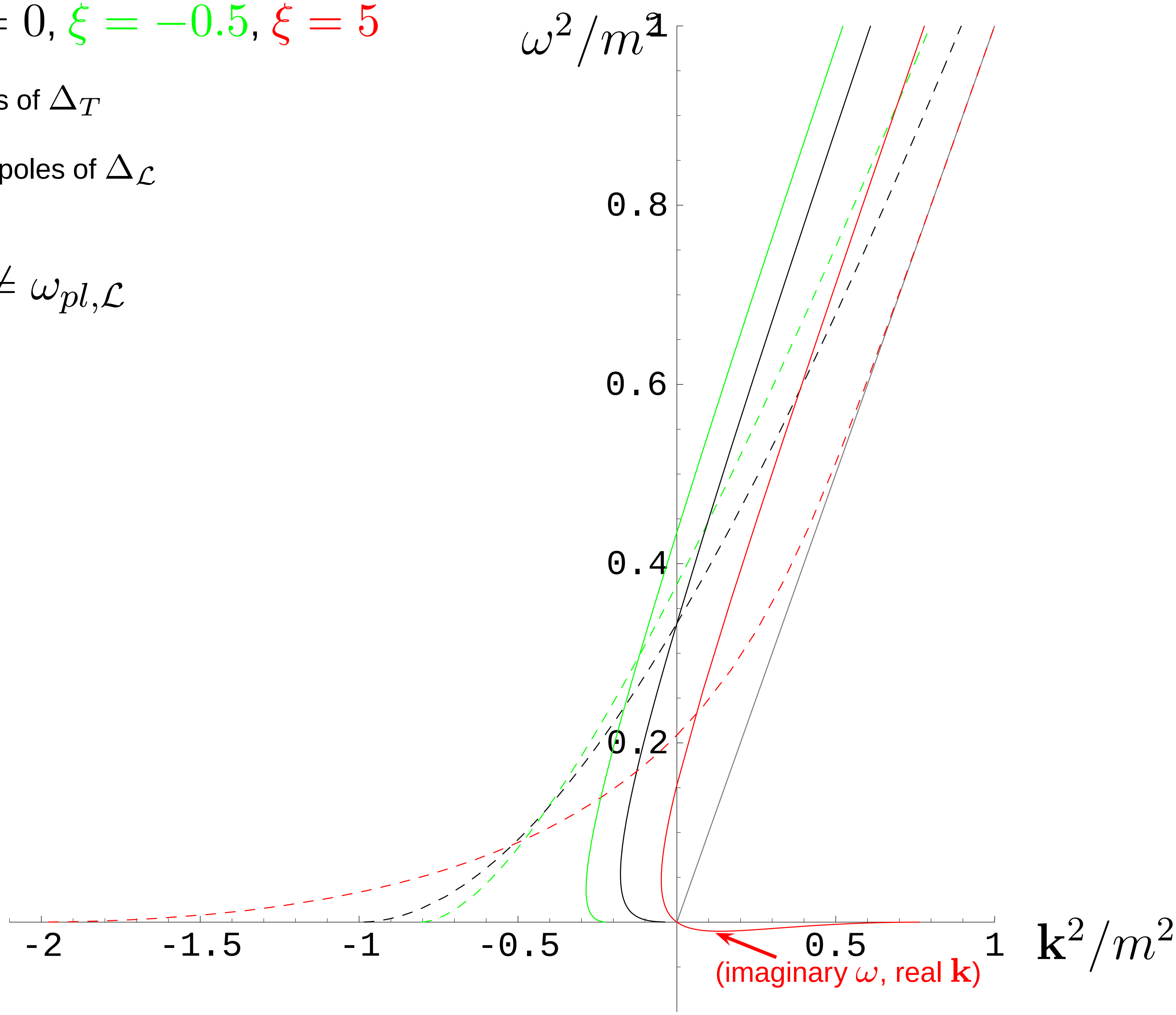
Dispersion laws for $\mathbf{k}||\mathbf{n}$

Comparing $\xi = 0$, $\xi = -0.5$, $\xi = 5$

full lines: poles of Δ_T

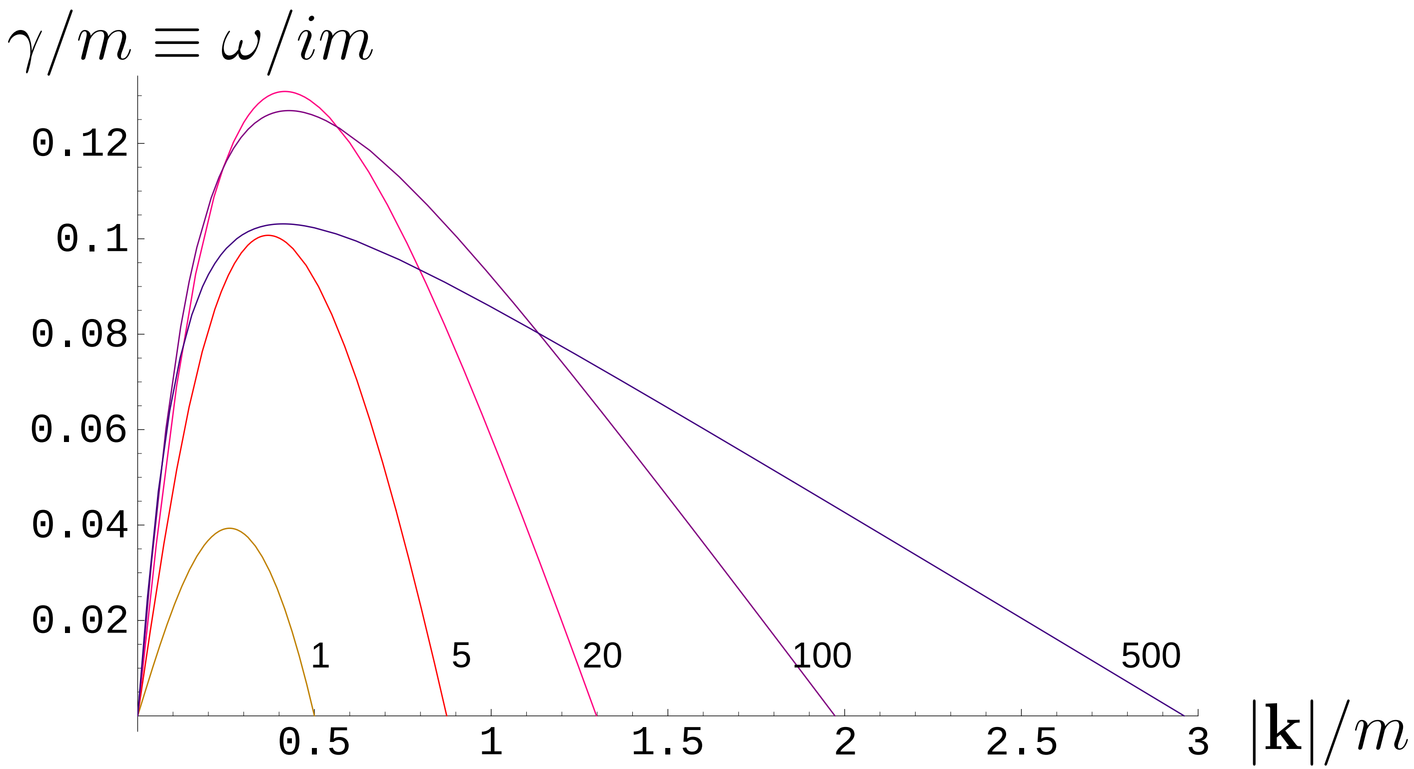
dashed lines: poles of $\Delta_{\mathcal{L}}$

$\xi \neq 0$: $\omega_{pl,T} \neq \omega_{pl,\mathcal{L}}$



Growth rates for Weibel instabilities (k||n)

Anisotropy parameter $\xi = 1,\ 5,\ 20,\ 100,\ 500$ (increasing oblateness)



large ξ behavior: $k_{\rm max}/m \sim \xi^{1/4}, \quad k/m|_{\gamma=\gamma_{\rm max}} \sim 1$

compared to asymptotic gluon mass m_∞: $k_{\rm max}/m_\infty \sim \sqrt{\xi}$

$\gamma_{\rm max}/m_\infty \to 1/\sqrt{2}$

Hard Loops from Boltzmann-Vlasov

With color-neutral background distribution $v \cdot \partial\, f_0(\mathbf{p}, \mathbf{x}, t) = 0$, $v^\mu = p^\mu / p^0$

gauge covariant Boltzmann-Vlasov:

$$v \cdot D\, \delta f_a(\mathbf{p}, \mathbf{x}, t) = g v_\mu F_a^{\mu\nu} \partial_\nu^{(p)} f_0(\mathbf{p}, \mathbf{x}, t) = -g(\mathbf{E}_a + \mathbf{v} \times \mathbf{B}_a) \cdot \nabla_\mathbf{p} f_0,$$

$$D_\mu F_a^{\mu\nu} = j_a^\nu = g \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu}{2p^0} \delta f_a(\mathbf{p}, \mathbf{x}, t).$$

Linear response: **Hard loop gauge boson self energy**

$$j^\mu(k) = g^2 \int \frac{d^3p}{(2\pi)^3} v^\mu \partial_{(p)}^\beta f(\mathbf{p}) \left(g_{\gamma\beta} - \frac{v_\gamma k_\beta}{k \cdot v + i\epsilon} \right) A^\gamma(k) = \Pi^{\mu\nu}(k) A_\nu(k)$$

- Instabilities for *any* amount of anisotropy

Special case: magnetic Weibel instabilities for oblate momentum distribution

Beyond linear response: **Full hard-loop effective theory**

Discretized Hard Loop Effective Theory

Useful:

auxiliary field formulation: [Nair; Blaizot & Iancu 1994; Mrówczyński, AR & Strickland 2004]

$$\delta f^a(x;p) = -g W^a_\mu(t,\mathbf{x};\mathbf{v})\partial^\mu_{(p)} f_0(\mathbf{p})$$

$$\boxed{[v\cdot D(A)]W_\mu(x;\mathbf{v}) = F_{\mu\gamma}(A)v^\gamma}$$

$$v^\mu \equiv p^\mu/|\mathbf{p}| = (1,\mathbf{v})$$

$$\boxed{D_\rho(A)F^{\rho\mu} = j^\mu(x) = -g^2\int \frac{d^3p}{(2\pi)^3}\frac{1}{2|\mathbf{p}|}\,p^\mu\,\frac{\partial f(\mathbf{p})}{\partial p^\nu}W^\nu(x;\mathbf{v})}$$

Hard Loop effective theory: (hard) scale $|\mathbf{p}|$ can be integrated out

Auxiliary field version: local in terms of field living also on velocity space S_2

Nonlinear response $\longrightarrow$ real-time lattice simulation

$\longrightarrow$ discretize also velocity space

$$D_\rho(A)F^{\rho\mu} = j^\mu(x) = \frac{1}{\mathcal{N}}\sum_{\mathbf{v}} v^\mu \mathcal{W}_{\mathbf{v}}(x)$$

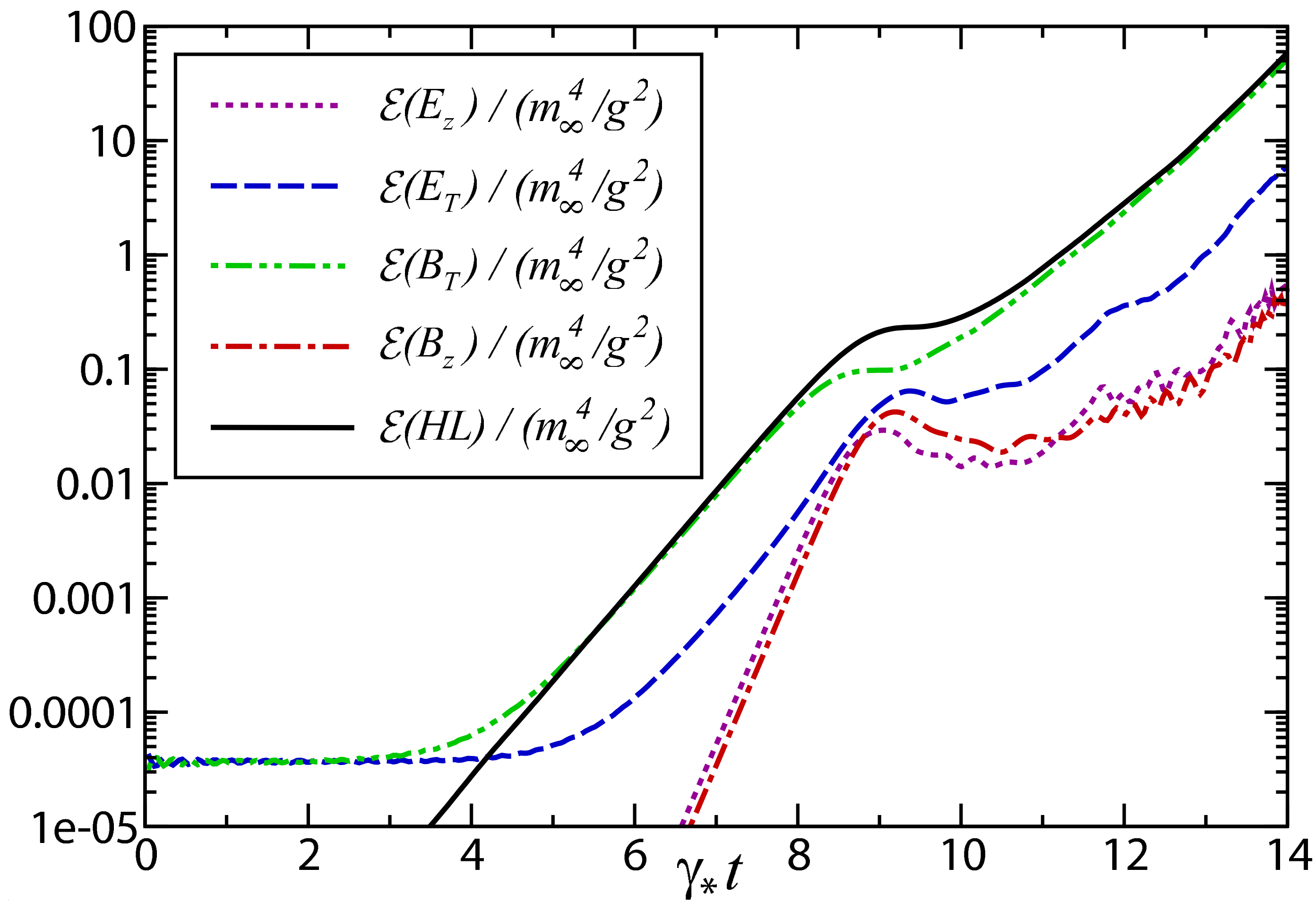
“disco balls”

Transversely constant modes: 1D+3V

Most unstable modes in linear response: $\mathbf{k} \parallel \mathbf{n}$

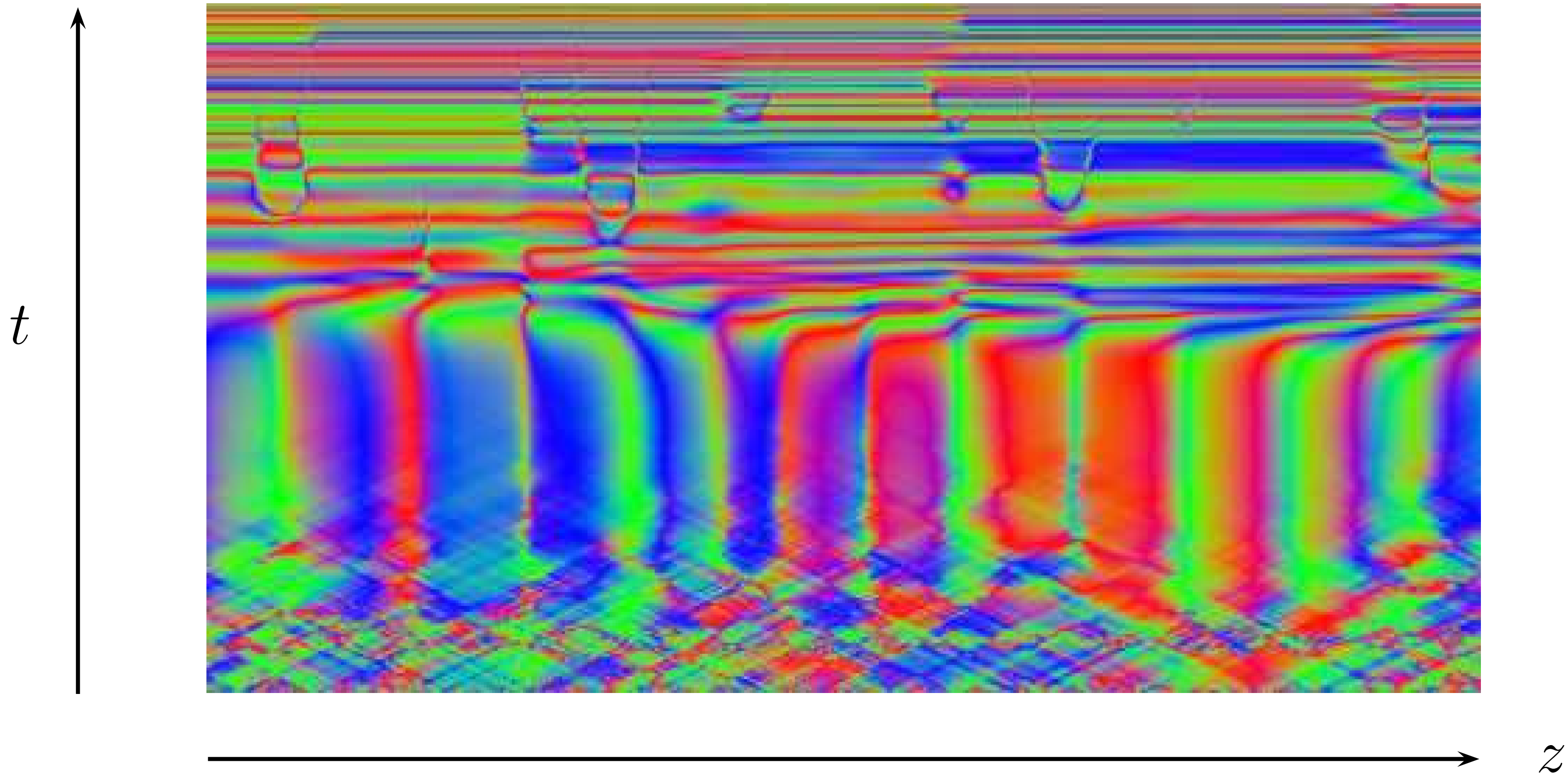
$\Longrightarrow$ no dependence on transverse coordinates;
dimensional reduction to 1 spatial dimension

[AR, Romatschke & Strickland, PRL 94 ('05) 102303]



Transversely constant modes: 1D+3V

Evolution of color degrees of freedom:
(parallel-transported color from fixed spatial point)



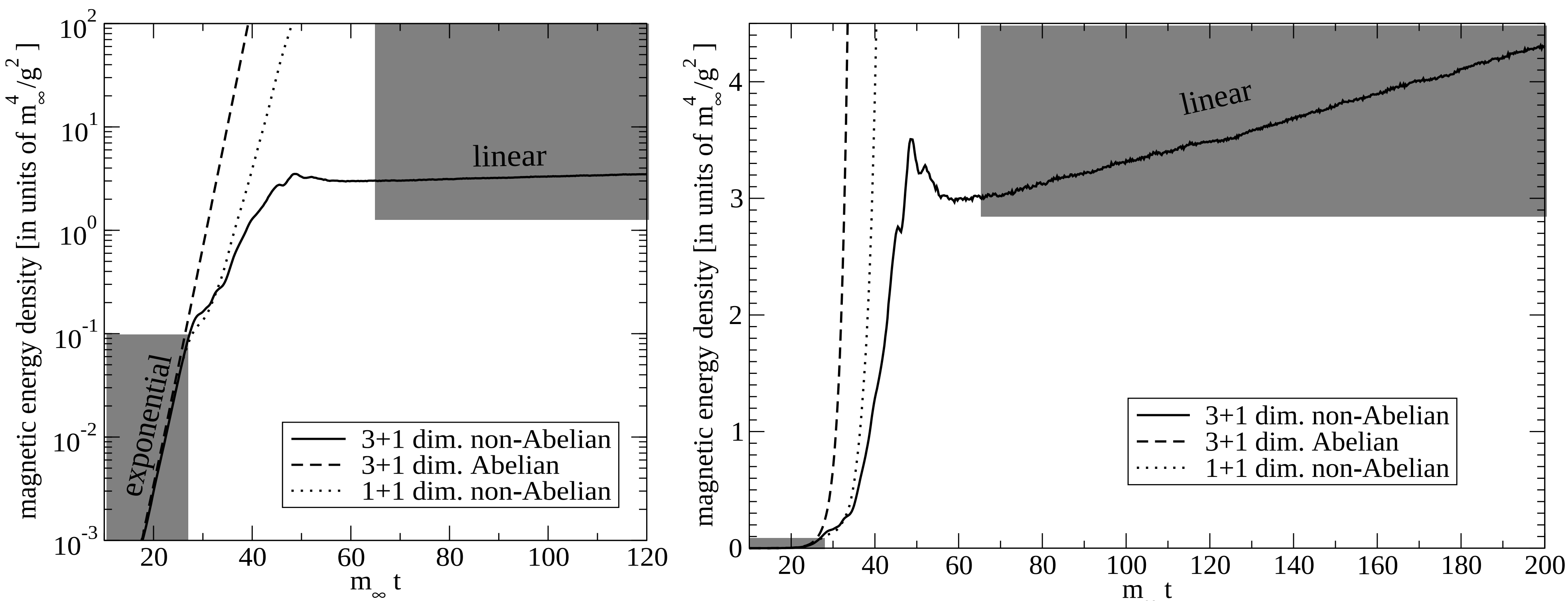
Late-time (non-linear) regime: Abelianization over extended spatial domains
– responsible for continued Abelian-like growth in non-linear regime

3D+3V

Can local Abelianization can be destroyed by interactions with not perfectly transversely constant modes?

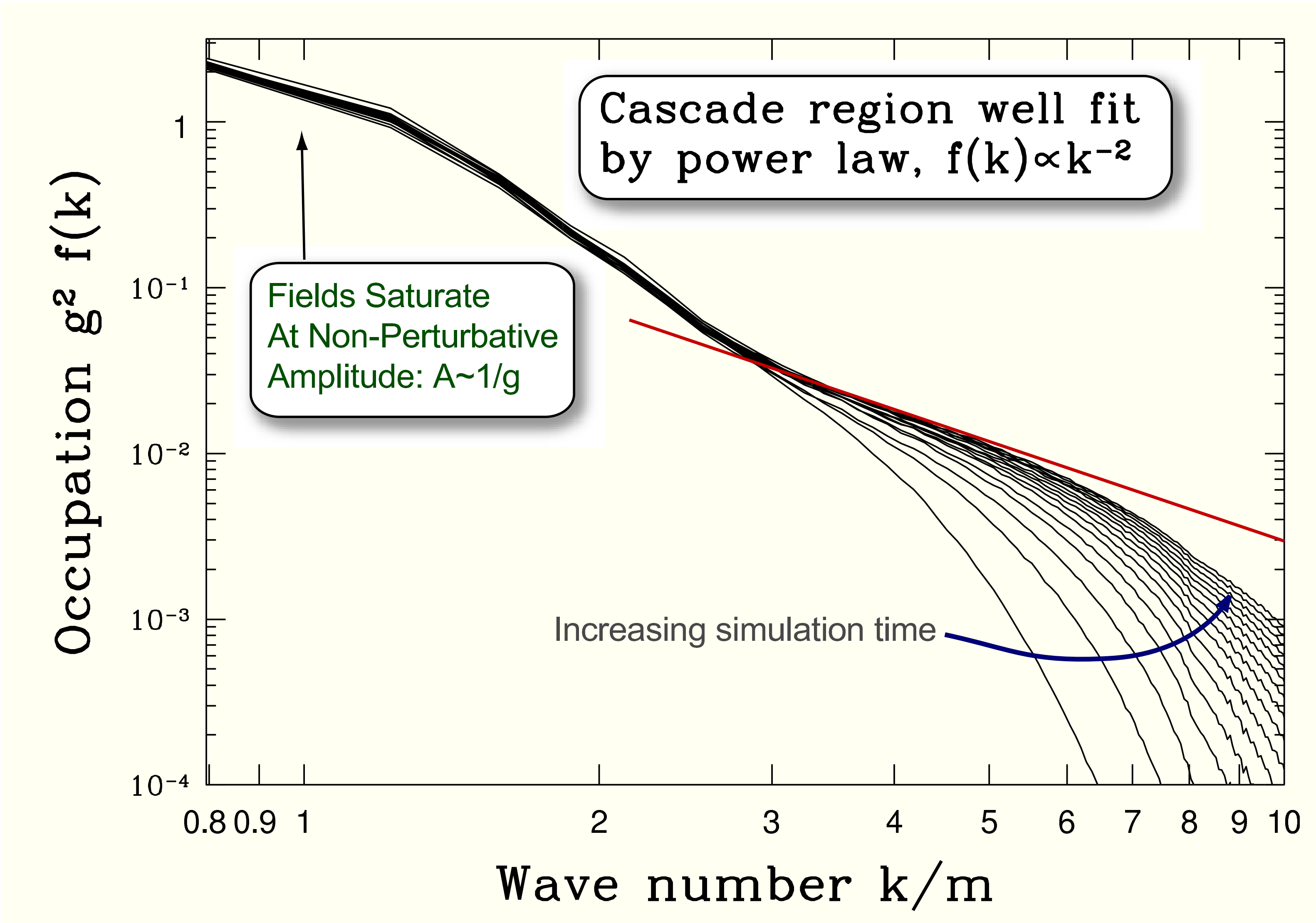
Yes: with saturation of exponential growth to only linear one:

[Arnold, Moore & Yaffe, PRD72 ('05) 054003]



(btw different discretization method: finite number of spherical harmonics W_{lm})

Cascade

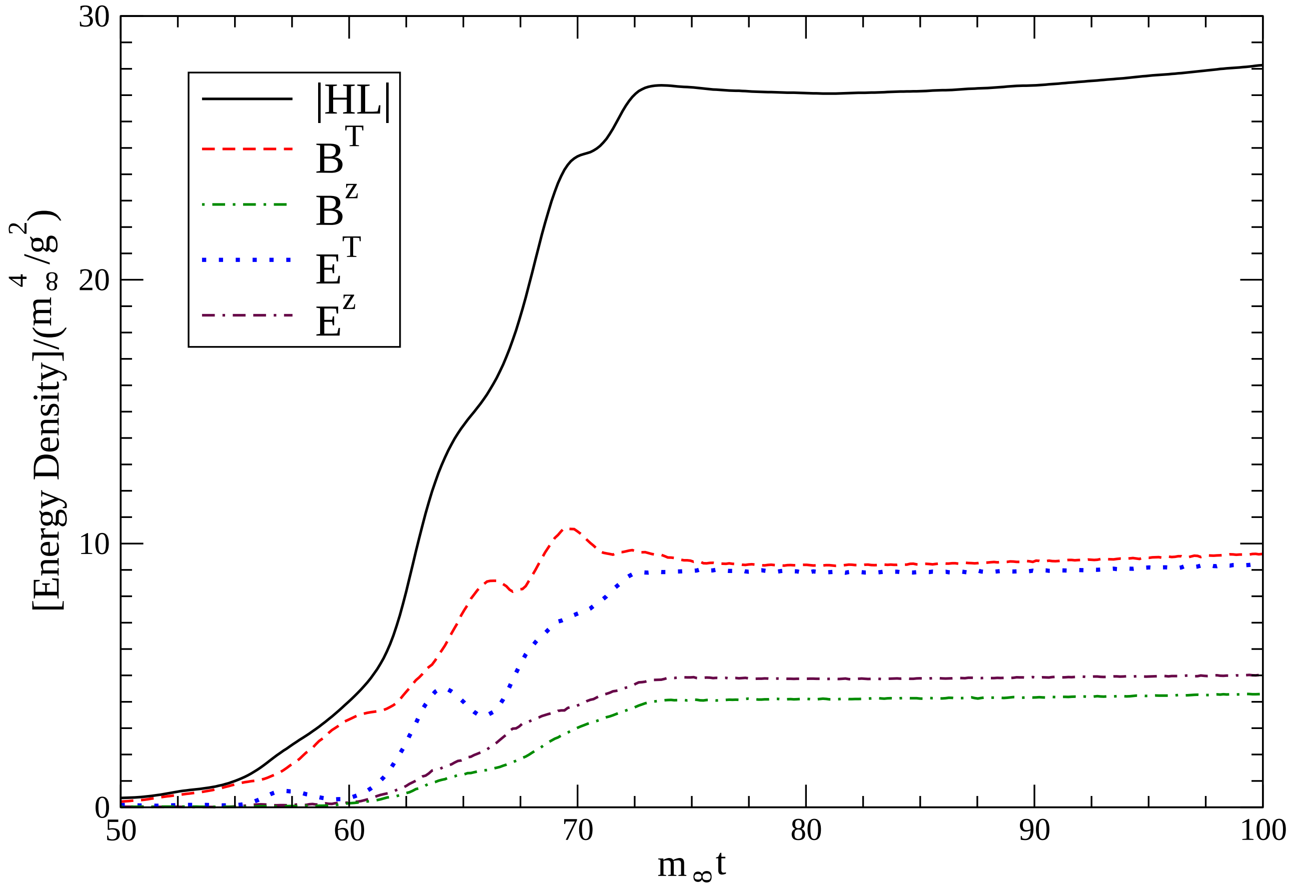


Cascade region well fit by power law, $f(k) \propto k^{-2}$
Fields Saturate At Non-Perturbative Amplitude: A~1/g
Increasing simulation time
Occupation $g^2 f(k)$
Wave number k/m
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0.8 0.9 1 2 3 4 5 6 7 8 9 10

3D+3V

Similar results with discoball discretization (using somewhat larger anisotropy)

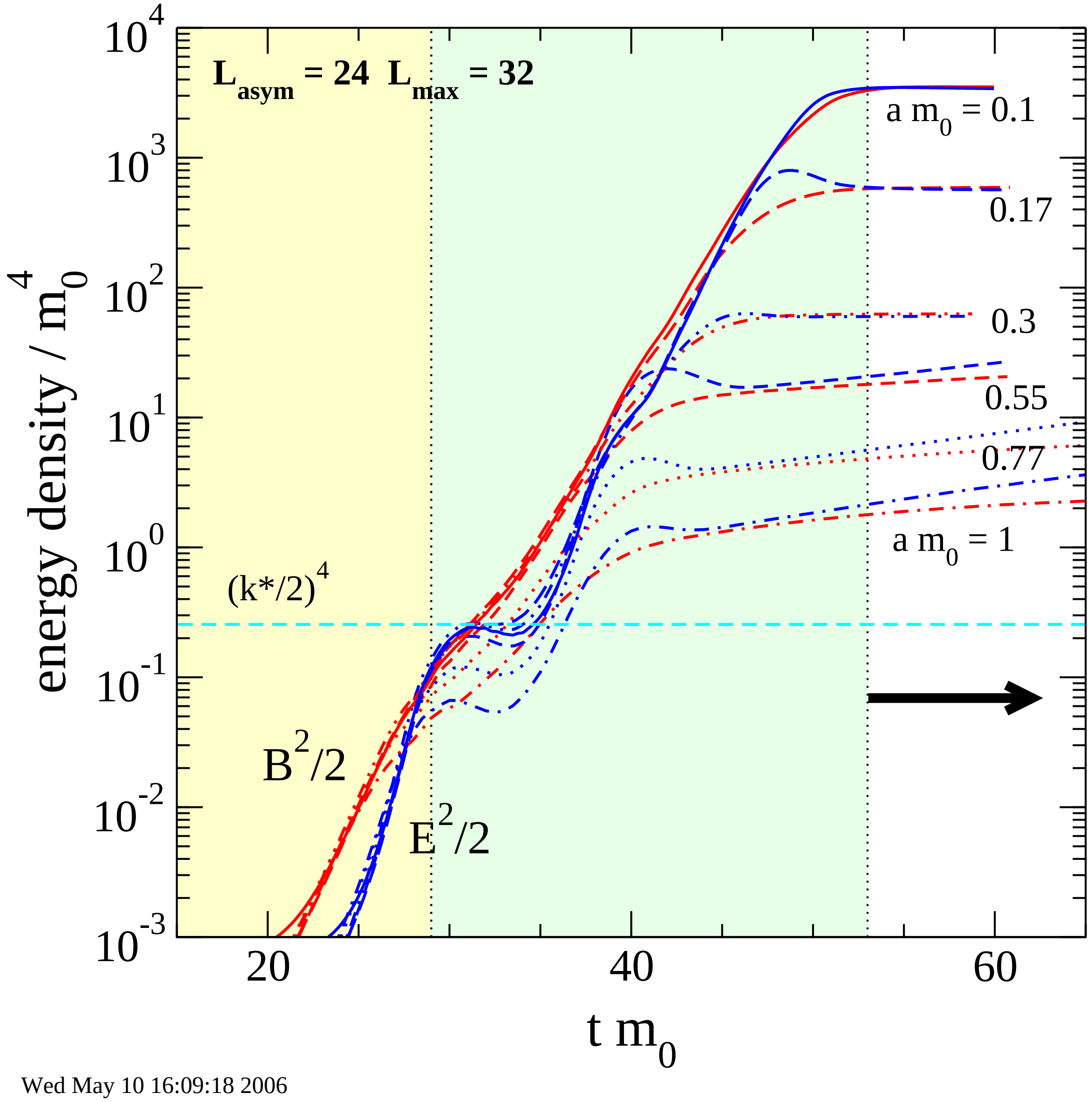
[AR. Romatschke & Strickland. JHEP 09 (2005) 041]



3D+3V - Strong anisotropy

More recently – Strong anisotropy: no saturation before lattice saturation

[Bödeker & Rummukainen, JHEP 07 (2007) 022]



PART III: Anisotropic (longitudinal) expansion

Notation: proper time $\tau = \sqrt{t^2 - z^2}$ and space-time rapidity $\eta = \mathrm{atanh}\frac{z}{t}$

$$x^\mu \to x^\alpha = (\tau, x^i, \eta) \text{ with } g_{\alpha\beta} = (1, -1, -1, -\tau^2)$$

momentum rapidity $y = \mathrm{atanh}\frac{p^0}{p^z}$:

$$p^\mu \to p^\alpha = |\mathbf{p}_\perp|(\cosh(y-\eta), \cos\phi, \sin\phi, \tau^{-1} \underbrace{\sinh(y-\eta)}_{p'^z/|\mathbf{p}_\perp|})$$

Boost invariant and transversely isotropic $f_0(\mathbf{p}, x) = f_0(p_\perp, p'^z, \tau)$

$$\boxed{p^\mu \partial_\mu f_0(x,p) = p^\alpha \partial_\alpha f_0\Big|_{fixed\ p^\mu} = 0}$$

solved by $f_0(\mathbf{p}, \mathbf{x}, t) = f_0(\mathbf{p}_\perp, p_\eta(x))$ because $(p^\alpha \partial_\alpha) p_\eta(x)|_{fixed\ p^\mu} = 0$

$$p^\tau \partial_\tau p_\eta(x)\Big|_{y,\mathbf{p}_\perp} = -p_\perp^2 \sinh(y-\eta)\cosh(y-\eta) = -p^\eta \partial_\eta p_\eta(x)\Big|_{y,\mathbf{p}_\perp}$$

Boost-invariant free-streaming background

Will use:

$$f_0(\mathbf{p},x) = f_{\rm iso}\left(\sqrt{p_\perp^2 + p_\eta^2/\tau_{\rm iso}^2}\right) = f_{iso}\left(\sqrt{p_\perp^2 + (p'^z\tau/\tau_{iso})^2}\right)$$

space-time dependent anisotropy parameter $\xi(\tau) = (\tau/\tau_{iso})^2 - 1$

increasingly oblate momentum space anisotropy at $\tau > \tau_{iso}$

(but prolate anisotropy for $\tau < \tau_{iso}$)

Will start at finite τ_0 (mostly $\gg \tau_{iso}$)

as motivated by CGC initial conditions at $\tau_0 \sim Q_s^{-1}$

$n \propto \alpha_s^{-1}$ — particle interpretation/kinetic theory actually only appropriate for $\tau \gg \tau_0$

strong initial anisotropy which gets even stronger, $\xi \sim \tau^2$

(bottom-up scenario: $\xi \sim \tau^{(<2/3)}$

Bödeker: $\tau^{1/2}$; Arnold & Moore: $\tau^{1/4}$)

Hard-Expanding-Loop formalism

Romatschke & AR, PRL 97 (2006) 252301

Since $p^\beta \partial_\beta \left[\partial^\alpha_{(p)} f_0(\mathbf{p}_\perp, p_\eta)\right]|_{p^\mu = const.} = 0$ (with index α upstairs!) can solve

$$p \cdot D\, \delta f_a(\mathbf{p}, \mathbf{x}, t)|_{p^\mu = const.} = g p^\beta F^a_{\beta\alpha} \partial^\alpha_{(p)} f_0(\mathbf{p}, \mathbf{x}, t),$$

by introducing auxiliary fields

$$\delta f^a(x;p) = -g W^a_\alpha(\tau, x^i, \eta; \phi, y) \partial^\alpha_{(p)} f_0(p_\perp, p_\eta)$$

that obey

$$\boxed{v \cdot D\, W_\alpha(\tau, x^i, \eta; \phi, y)|_{\phi, y} = v^\beta F_{\alpha\beta},}$$

where $v^\alpha \equiv \frac{p^\alpha}{|\mathbf{p}_\perp|} = (\cosh(y-\eta), \cos\phi, \sin\phi, \frac{\sinh(y-\eta)}{\tau})$.

Discretized HEL

For $f_0(\mathbf{p},x) = f_{\rm iso}\left(\sqrt{p_\perp^2 + p_\eta^2/\tau_{\rm iso}^2}\right)$

$$j^\alpha(\tau,x^i,\eta) = -\frac{m_D^2(\tau=\tau_{iso})}{2}\int_0^{2\pi}\frac{d\phi}{2\pi}\int_{-\infty}^{\infty} dy\, v^\alpha \left(1+\frac{\tau^2}{\tau_{iso}^2}\sinh^2(y-\eta)\right)^{-}$$

$$\times \underbrace{\left\{\cos\phi\, W_1 + \sin\phi\, W_2 - \frac{\tau}{\tau_{iso}^2}\sinh(y-\eta)\, W_\eta\right\}}_{\mathcal{W}(\tau,x^i,\eta;\phi,y)}$$

instead of discoballs [$\mathcal{W}(t,\mathbf{x};\phi_n,\theta_m)$ with equally spaced ϕ_n, $\cos\theta_m$]

now *disco cylinders*: $\mathcal{W}(\tau,x^i,\eta;\phi,y)$ with equally spaced ϕ_n, y_m

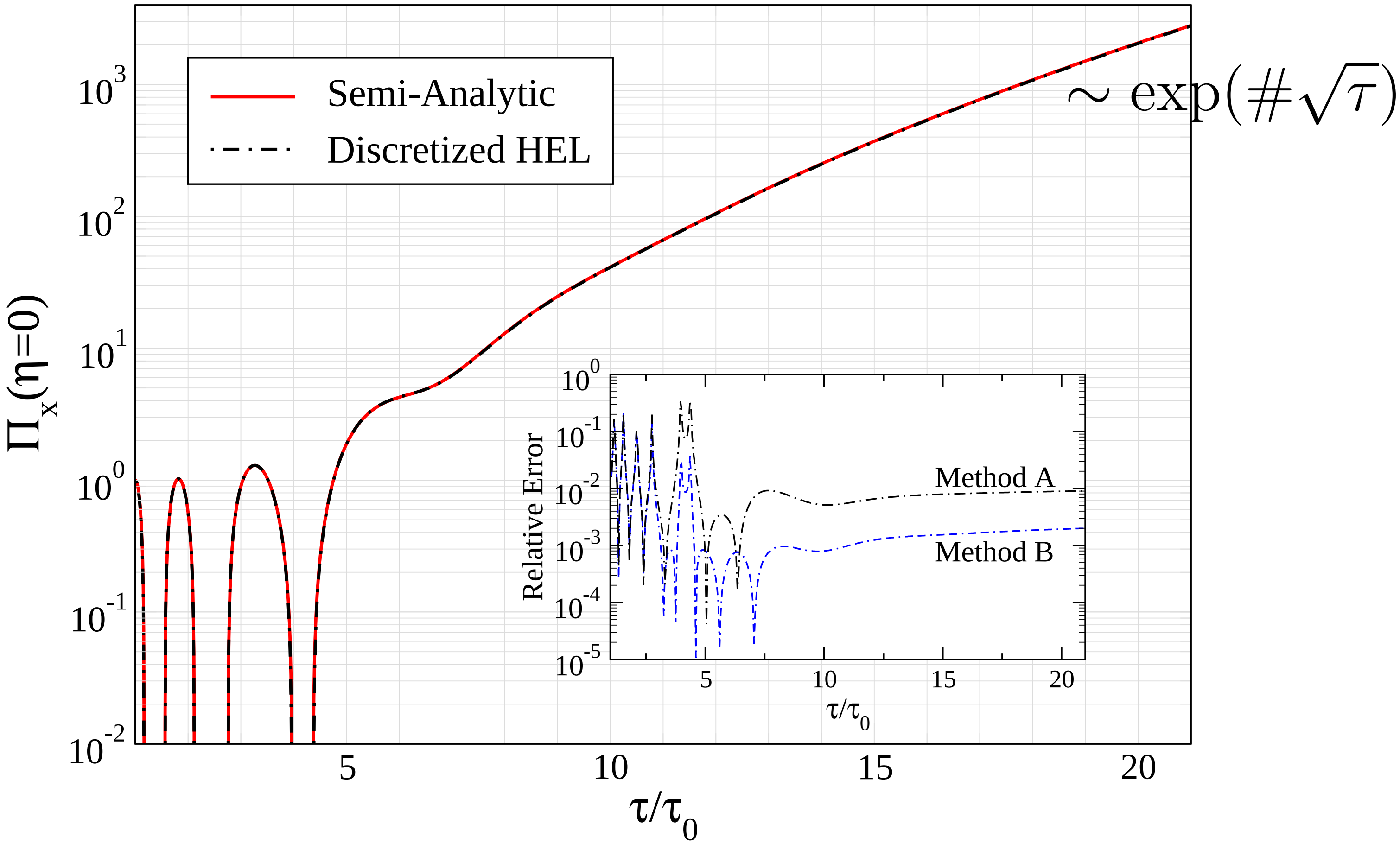
finite rapidity interval for $y-\eta$ because of exponential suppression

$\longrightarrow$ numerical simulation on space-time & $\phi, y-\eta$ grid

AR, M. Strickland, M. Attems: arXiv:0802.1714

Discretized HEL - Abelian checks

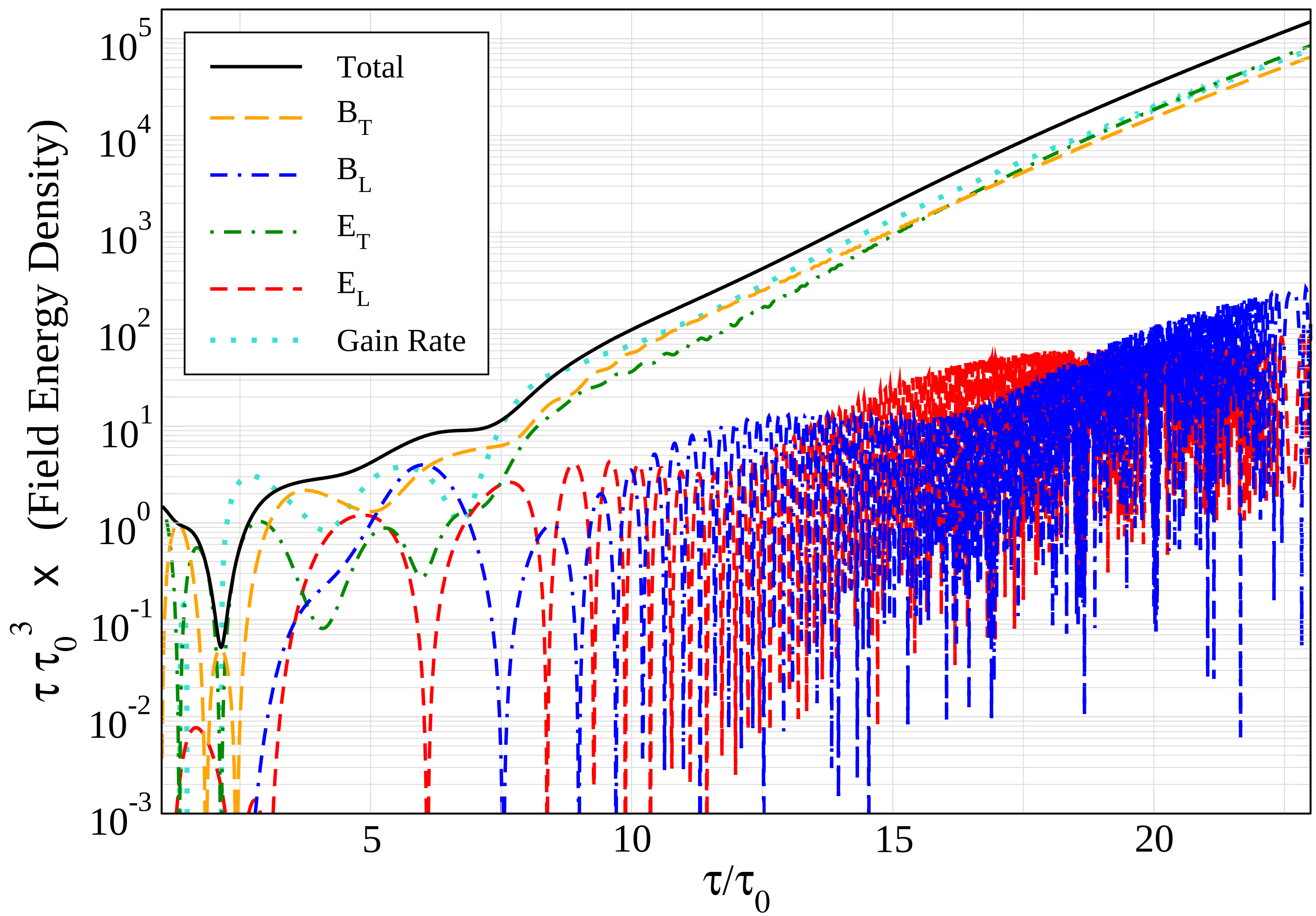
Abelian: can solve e.o.m. for $\mathcal{W}$ to give 1D integro-differential equation ("Semi-Analytic")



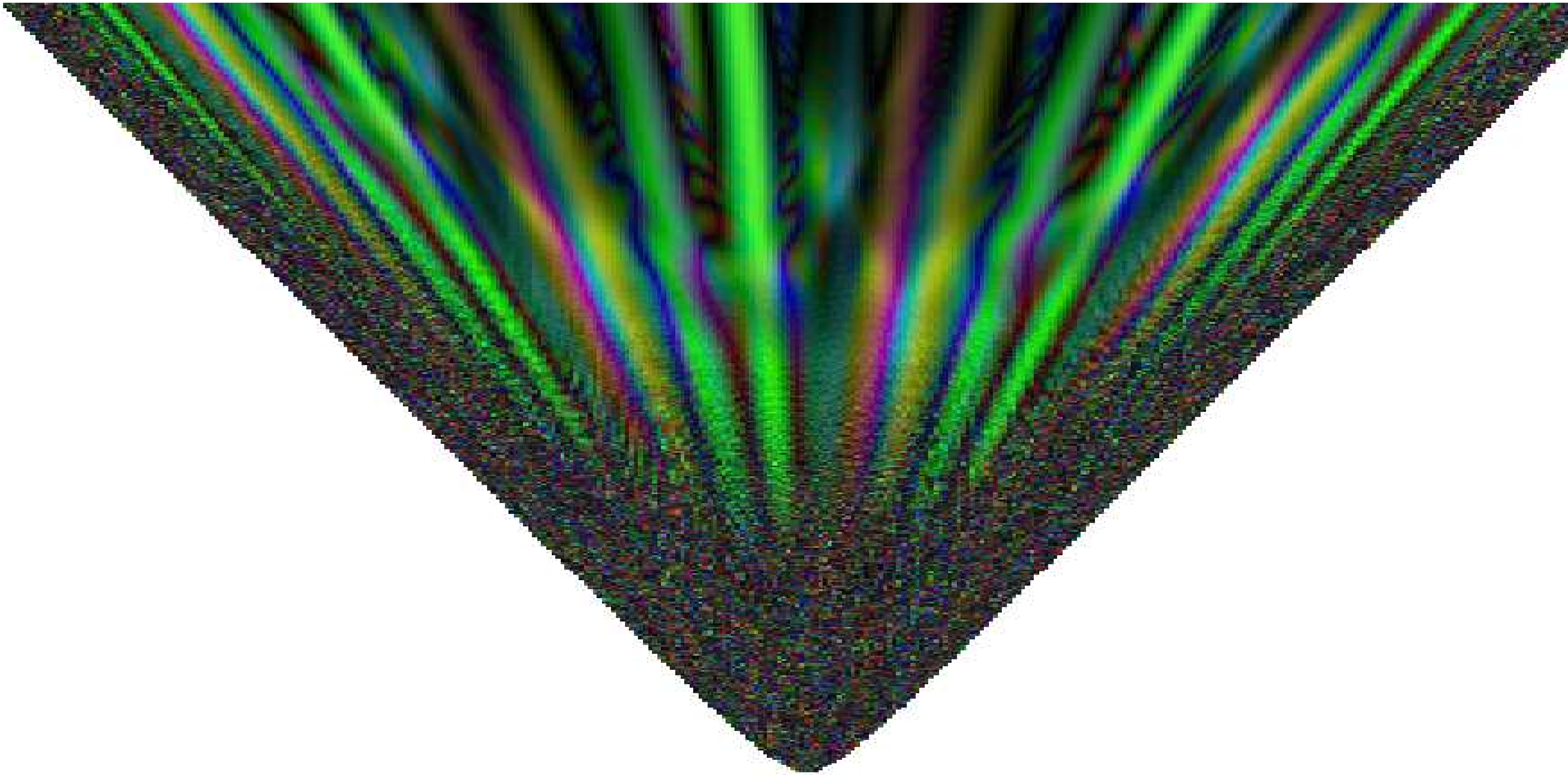
$\tau_{\rm iso} = 0.1$, $\tau_0 = 1.0$, $m_D = 10$, $a = 0.0025$, $\epsilon = 0.001$, $N_\eta = 250$, $N_\phi = 8$,

and (Method A) $N_y = 1000$, (Method B: $x = \tanh(y - \eta)$) $N_x = 1000 \ldots$ $\boxed{N_{\mathcal{W}} = 8000}$

Non-Abelian Discretized HEL

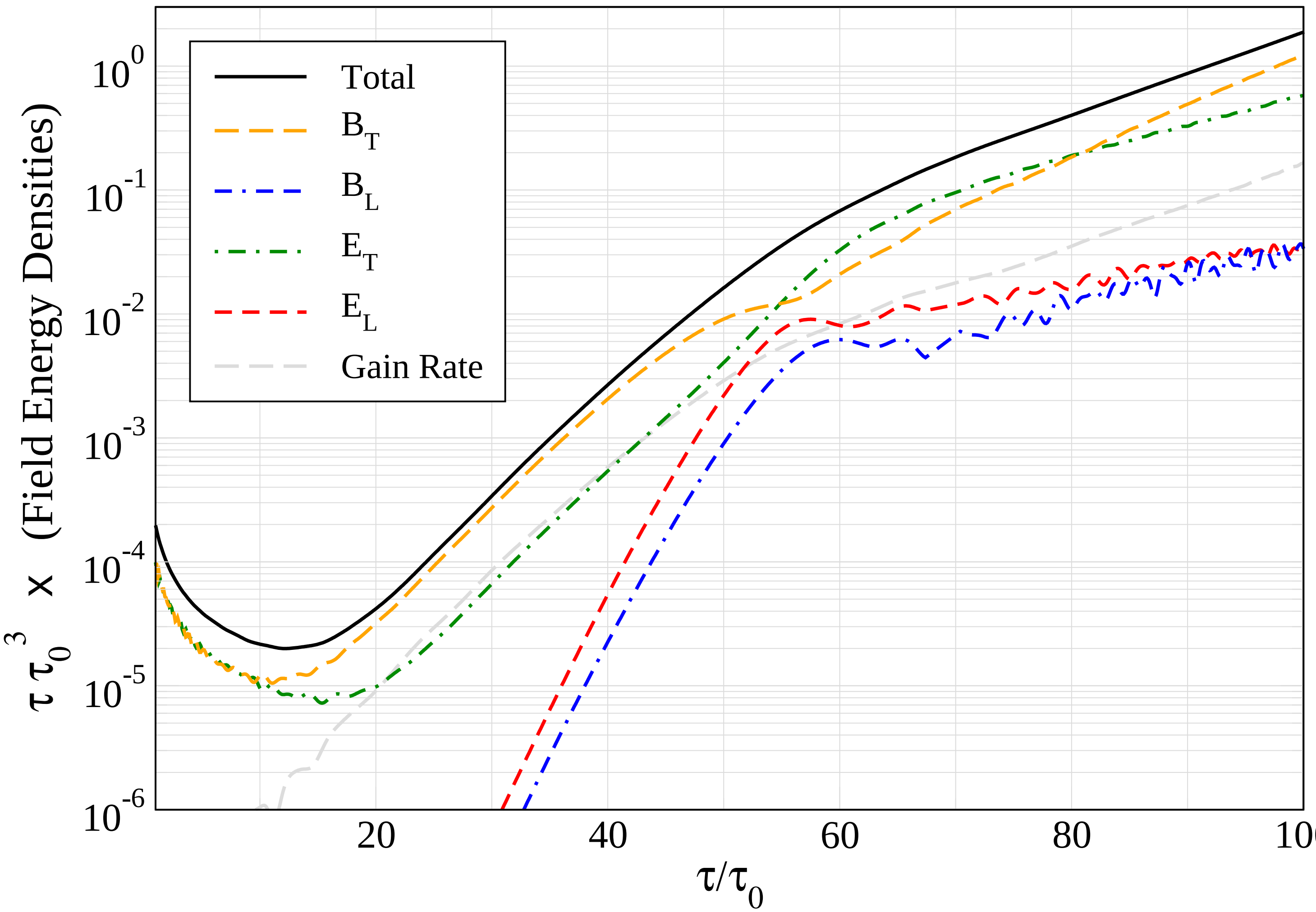


Non-Abelian Discretized HEL — Visualization in Lab Frame



Non-Abelian Discretized HEL

Hard gluon number density and initial fluctuation spectrum from CGC ⟶

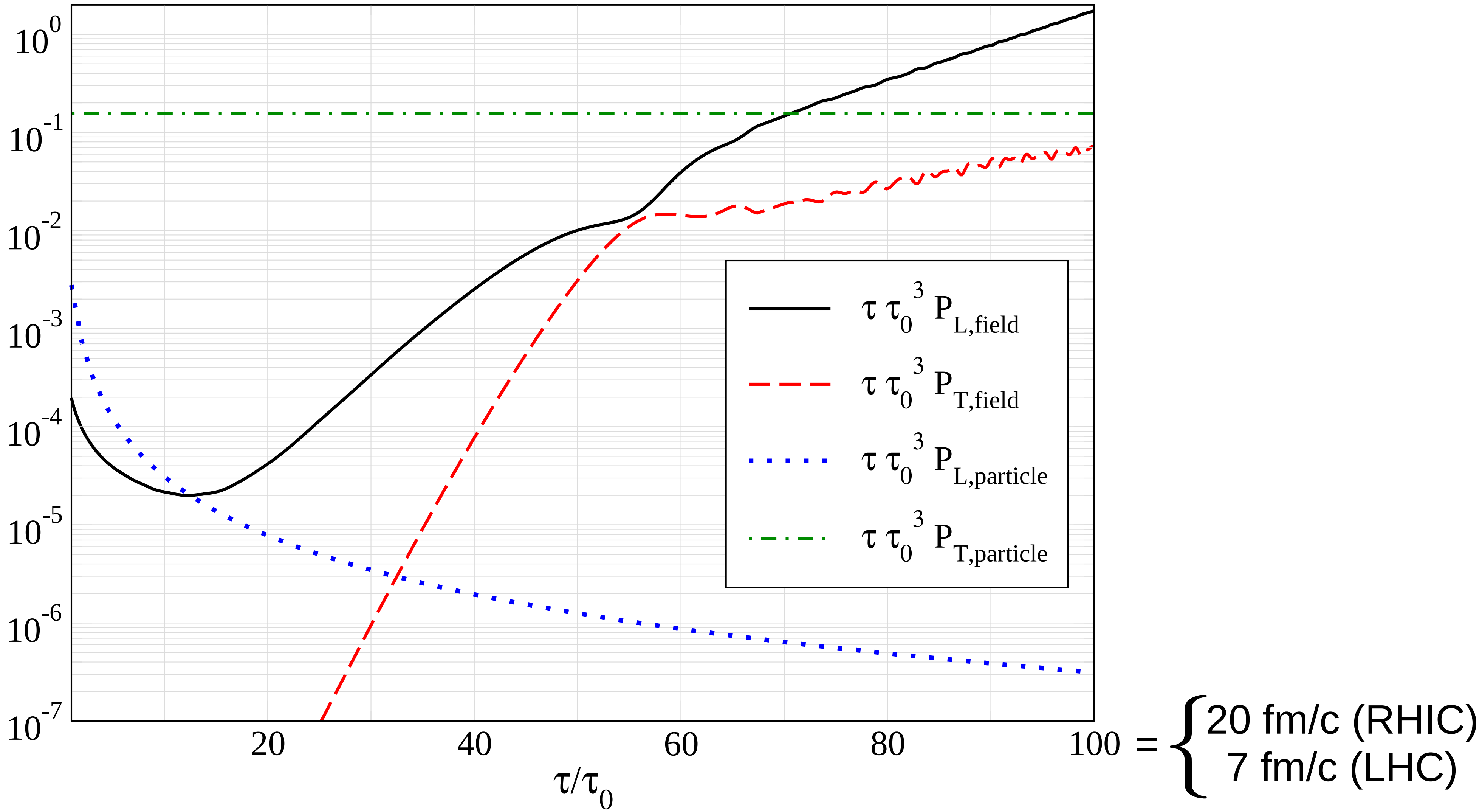


2)

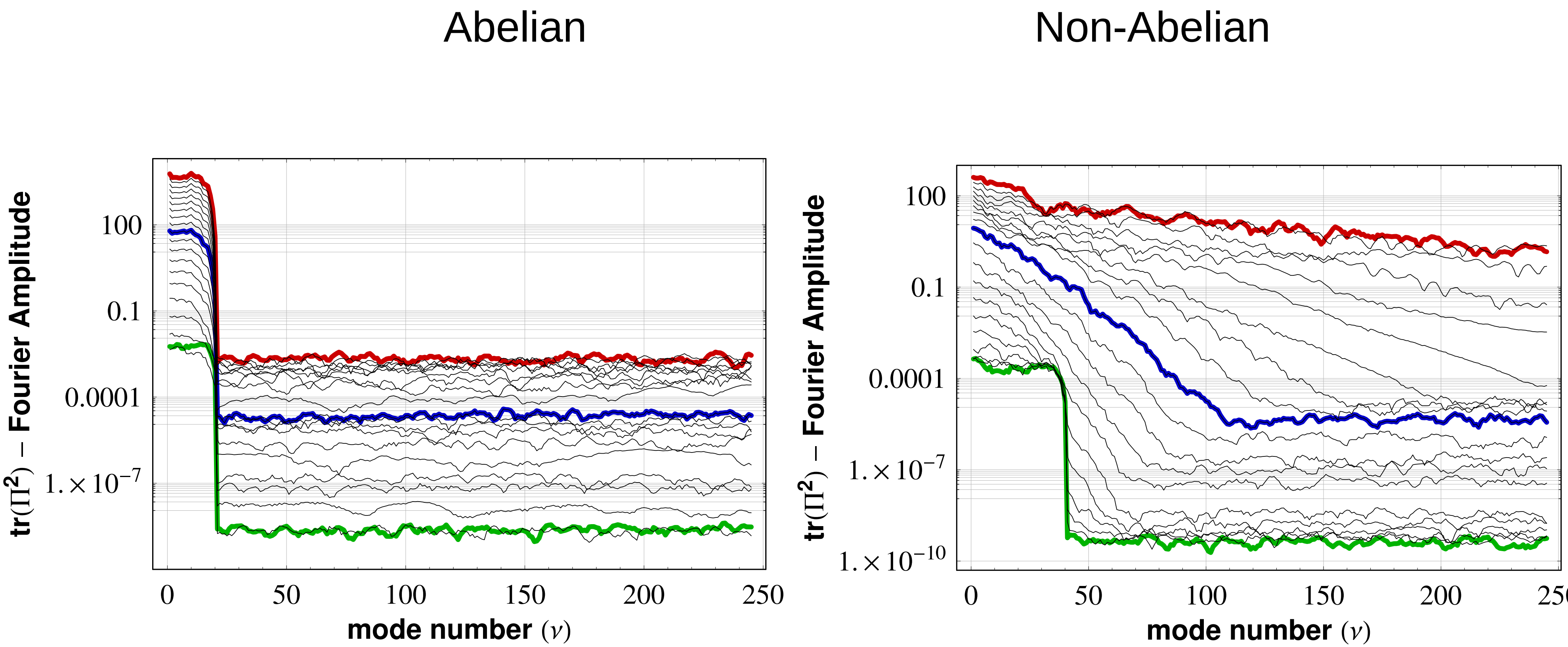
IC) . . . 3 GeV (LHC) ?

= { 20 fm/c (RHIC)
7 fm/c (LHC)

Non-Abelian Discretized HEL



Non-Abelian Discretized HEL – Cascade



Nonabelian dynamics $\longrightarrow$ quasithermal spectrum of soft modes

Conclusions and Outlook

Conclusions:

- Uncomfortably long delay of onset of PI for early thermalization
- Important role for PI possible for LHC

Upcoming:

- Full 3D+3V
- Generic large initial fields

To do:

- Impact on bottom-up thermalization
- Generalization of HEL to non-free streaming?

Supplement: Transversely constant modes in linear (Abelian) regime

Most unstable modes for $\tau > \tau_{\rm iso}$ have $\boxed{\partial_i A^\alpha \equiv 0}$

Linearize ($A^\tau = 0$): $\boxed{\left[\frac{1}{\tau}\partial_\tau \tau \partial_\tau - \frac{1}{\tau^2}\partial_\eta^2\right] A^i(\tau,\eta) = j^i,}$ $\boxed{\partial_\tau \frac{1}{\tau}\partial_\tau A_\eta = \frac{j_\eta}{\tau},}$

Solving $v \cdot \partial W = v^\beta F_{\alpha\beta}$:

$$W_\alpha(\tau,\eta;\phi,y) = \int_{\tau_0}^{\tau} d\tau' \frac{v^\beta F_{\alpha\beta}\big|_{\tau',\eta(\tau')}}{\cosh(y-\eta(\tau'))}\,, \qquad y-\eta(\tau') = \mathrm{asinh}\left(\tfrac{\tau}{\tau'}\sinh\left(y-\eta\right)\right),$$

$$\longrightarrow \quad j^i[W] = -\frac{m_D^2}{4}\int_{-\infty}^{\infty} dy \left(1+\frac{v_\eta^2}{\tau_{\rm iso}^2}\right)^{-2} \int_{\tau_0}^{\tau} d\tau'$$

$$\times \left[\left(\partial_\tau' - \frac{\tanh\bar\eta'}{\tau'}\partial_{\eta'}\right) A^i(\tau',\eta') + \frac{v_\eta}{\tau_{\rm iso}^2}\frac{\partial_{\eta'}A^i(\tau',\eta')}{\cosh\bar\eta'}\right],$$

$$j^\eta[W] = -\frac{m_D^2}{2\tau_{\rm iso}^2}\int \frac{dy\, v^\eta v_\eta}{\left(1+\frac{v_\eta^2}{\tau_{\rm iso}^2}\right)^2}\int_{\tau_0}^{\tau} d\tau' \partial_{\tau'} A_\eta(\tau',\eta'),$$

where $\eta' = \eta(\tau')$ and $\bar\eta' = \eta(\tau') - y$.

Transversely constant modes in linear (Abelian) regime

Fourier transform in space-time rapidity $(\nu \sim k_z \tau$ at $\eta \sim 0)$

$$A^i(\tau,\eta) = \int \frac{d\nu}{2\pi} \exp(i\nu\eta) \widetilde{A}^i(\tau,\nu),$$

$\Rightarrow$

$$\widetilde{j}^i(\tau,\nu) = -\frac{m_D^2}{4} \int \frac{dy}{\left(1+\frac{\tau^2 \sinh^2 y}{\tau_{\text{iso}}^2}\right)^2} \left\{ \widetilde{A}^i(\tau,\nu) - \int_{\tau_0}^{\tau} d\tau' \frac{\widetilde{A}^i(\tau',\nu)\tau'^2}{\tau_{\text{iso}}^2} \partial_{\tau'} e^{i\nu\left[y - \text{asinh}\left(\frac{\tau}{\tau'} \sinh y\right)\right]}$$

(similar equation for $\widetilde{j}^\eta(\tau,\nu)$)

Integro-differential equations, solved by numerical leap-frog algorithm

$$\tau \partial_\tau \widetilde{A}^i(\tau,\nu) = \widetilde{\Pi}^i(\tau,\nu) \quad \text{and}$$

$$\partial_\tau \widetilde{\Pi}^i(\tau,\nu) = -\nu^2 \tau^{-1} \widetilde{A}^i(\tau,\nu) + \tau \widetilde{j}^i(\tau,\nu)$$

Transversely constant modes in linear regime: *Analytical results*

Late-time behavior: approximate 4th order ODE

$\tau \gg \tau_0 \gtrsim \tau_{\rm iso}$: $\left[\partial_\tau^2 \tau \partial_\tau \tau \partial_\tau + \nu^2 \partial_\tau^2 + \mu \partial_\tau^2 \tau - \mu\nu^2 \frac{1}{\tau}\right] \widetilde{A}^i(\tau,\nu) \approx 0,$

$\left[\partial_\tau \frac{1}{\tau} \partial_\tau + \mu \frac{2}{\tau^2}\right] \widetilde{A}_\eta(\tau,\nu) \approx 0$, where $\boxed{\mu = \frac{1}{8} m_D^2 \pi \tau_{\rm iso}}$.

Stable plasma oscillations for $\nu \ll 1$:

$\widetilde{A}^i(\tau,\nu) = c_1 J_0\left(2\sqrt{\mu\tau}\right) + c_2 Y_0\left(2\sqrt{\mu\tau}\right)$,

$\tau^{-1}\widetilde{A}_\eta(\tau,\nu) = c_1 J_2\left(2\sqrt{2\mu\tau}\right) + c_2 Y_2\left(2\sqrt{2\mu\tau}\right)$, indeed: $\lim_{\xi\to\infty} \omega_{\rm pl}^\ell / \omega_{\rm pl}^t = \sqrt{2}$

[Romatschke & Strickland, PRD68]

Unstable transverse modes for $\nu \gtrsim 1$:

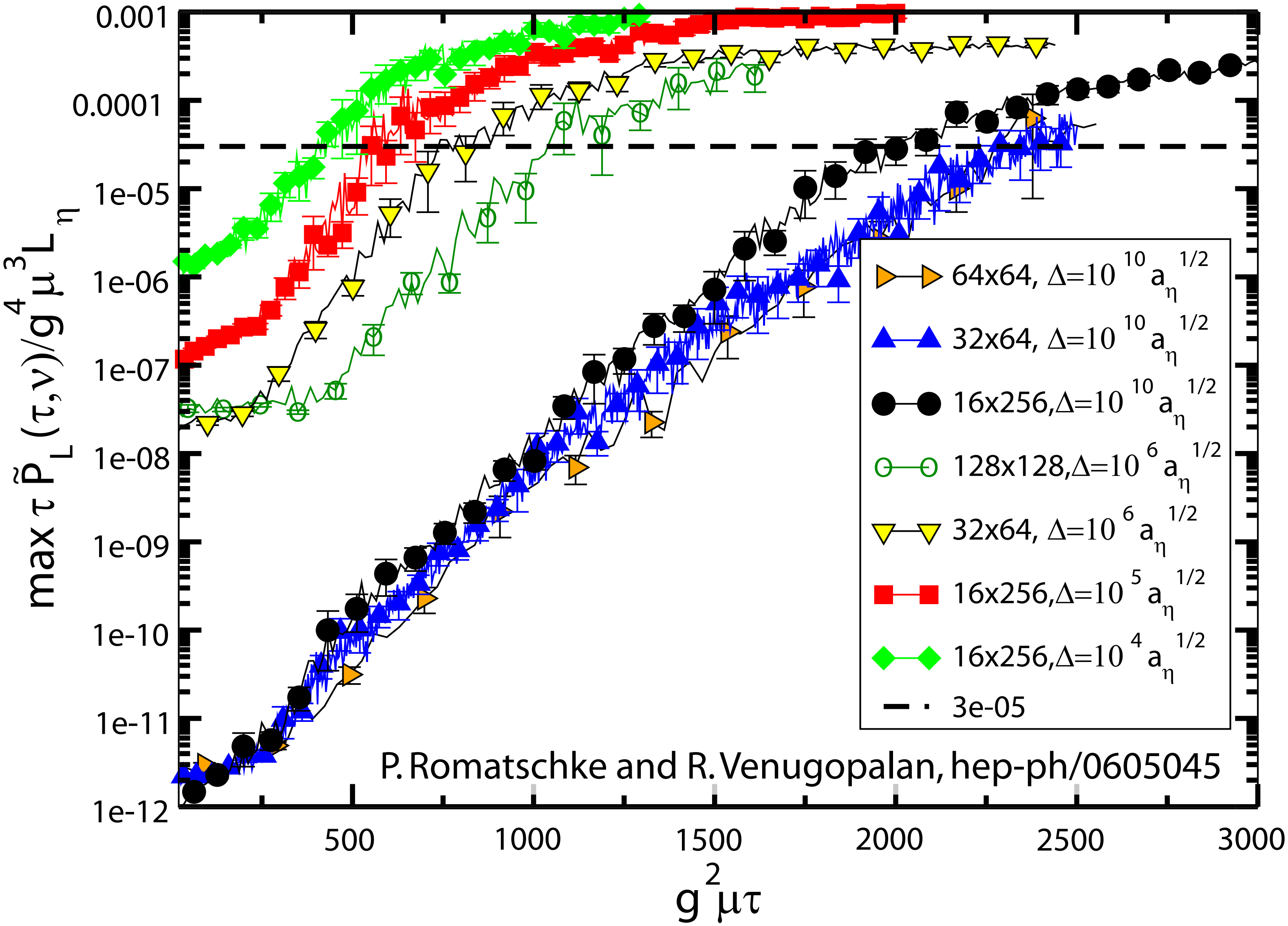
$$\widetilde{A}^i(\tau,\nu) \quad \sim \quad \tau \; {}_2F_3\left(\frac{3-\sqrt{1+4\nu^2}}{2}, \frac{3+\sqrt{1+4\nu^2}}{2}; 2, 2-i\nu, 2+i\nu; -\mu\tau\right)$$

$$\rightarrow \tau^{1/4} \exp\left(2\sqrt{\mu\tau}\right) \quad \text{for } \nu \gg 1$$

qualitative agreement with unstable melting color glass-condensate
of [P. Romatsche & R. Venugopalan, PRL96(2006)062302; hep-ph/0605045]

Unstable glasma

P. Romatschke and R. Venugopalan, PRL 96, PRD 74 (2006)



Transversely constant modes in linear regime: *Numerical results*

Numerical result *vs.* asymptotic ${}_2F_3$ behavior (thin bright lines) $(c = 0.5)$